Finite Family Developments

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Abstract. Associate to a rewrite system $\mathcal{R}$ having rules $l \to r$, its labelled version $\mathcal{R}^\omega$ having rules $l_{\underset{m+1}{\circ}} \to r_{\underset{m}{\bullet}}$, for any natural number $m \in \omega$. These rules roughly express that a left-hand side l carrying labels all larger than m can be replaced by its right-hand side r carrying labels all smaller than or equal to m. A rewrite system $\mathcal{R}$ enjoys *finite family developments* (FFD) if $\mathcal{R}^\omega$ is terminating. We show that the class of higher order pattern rewrite systems enjoys FFD, extending earlier results for the lambda calculus and first order term rewrite systems.

1 Preliminaries

The abstract property we are interested in is termination of a binary relation.

Definition 1. An *abstract rewrite system* (ARS) is a binary relation on a set. We use (sub/superscripted) arrows to range over ARSs. An ARS $\to$ is *terminating* if the transitive closure of its inverse $\leftarrow^+$ is a well-founded order.

Except for the notation $\twoheadrightarrow$ denoting the reflexive-transitive closure of an ARS $\to$, ordinary notations for operations on relations are employed, e.g. $\to^+$ denotes the transitive closure of $\to$. See [Klo92] for these and for standard rewrite concepts such as rewrite sequence, confluence, normal form, etc.. In general we assume knowledge of both first order term rewriting (TRS [Klo92]) and the lambda calculus (Λ [Bar84]).

The ARSs we are interested in are the rewrite relations on terms induced by higher order pattern rewrite systems [Nip]. Higher order rewriting in this paper stands for rewriting modulo the simply typed λ-calculus [Chu40]. Concretely, the objects to be rewritten are simply typed λ-terms modulo the theory $\boldsymbol{\lambda\eta}$ [Bar84]. Our presentation only considers λ-terms in expanded η-normal form [Pra71], which is no restriction [Aka93], and has the advantage of being close to first order rewriting. We quickly recapitulate their syntax.

A *higher order pattern rewrite system* (PRS [MN94]) is a structure $(\mathcal{F},\mathcal{R})$ consisting of an alphabet $\mathcal{F}$ and a set $\mathcal{R}$ of rewrite rules over $\mathcal{F}$.

Definition 2 (Term). *Types* $(\theta,\ \iota,\ \kappa \in)\mathbb{T}$ are defined by the grammar $\mathbb{T} ::= \boldsymbol{\mathbb{T}} \to \mathbb{B}$, where $\boldsymbol{\mathbb{T}}$ stands for vectors of types[1] and $(b \in)\mathbb{B}$ is a set of base types. $\varepsilon \to b$ is abbreviated to b and the *order* function Ord on types is defined by

[1] In this paper boldface letters stand for vectors, ε for the empty vector, and juxtaposition or , is used for concatenation of vectors.

$Ord(\boldsymbol{\theta} \to b) = 1 + max(Ord(\boldsymbol{\theta}))$. To each symbol a among the disjoint sets of *function symbols* $(f, g, h \in)\mathcal{F}$ and *variables* $(x, X, y, Y \in)\mathcal{X}$ a unique type $\mathbb{T}(a)$ is associated. *Preterms* $(M, N, O \in)\mathcal{T}(\mathcal{F}, \mathcal{X})$ over $\mathcal{F} \cup \mathcal{X}$ are objects M such that $M : \theta$ for some type θ can be inferred from:

(head)	$a(\mathbf{M}) : b$	$\Longleftarrow$	a is a symbol in $\mathcal{F} \cup \mathcal{X}$ of type $\boldsymbol{\theta} \to b$ and $\mathbf{M} : \boldsymbol{\theta}$
(abs)	$x.M : \iota\boldsymbol{\theta} \to b$	$\Longleftarrow$	x is a variable in $\mathcal{X}$ of type ι and $M : \boldsymbol{\theta} \to b$
(beta)	$N(\mathbf{M}) : b$	$\Longleftarrow$	$N : \boldsymbol{\theta} \to b$ and $\mathbf{M} : \boldsymbol{\theta}$

where in (beta) the type vector $\boldsymbol{\theta}$ (and hence the preterm vector $\mathbf{M}$) is assumed to be nonempty. Identity on inferences is denoted by $\equiv$ which coincides with identity on preterms by the condition on (beta). Provable equality in the theory $\boldsymbol{\lambda\eta}$ [Bar84, Def. 2.1.28] is denoted by $=$. *Terms* are $=$-equivalence classes of preterms.

Preterms are just simply typed λ-terms (a bit in disguise)[2], hence we may employ some standard no(ta)tions and results for these. In particular, we will now sketch the usual way to construct unique representatives of terms.

Note that in (beta) N is necessarily of the form $\mathbf{x}.N'$ for some nonempty variable vector $\mathbf{x}$ of type $\boldsymbol{\theta}$ and preterm N' of base type b, hence it makes sense to define the following *macro* β-rule:

$$(\mathbf{x}.M)(\mathbf{N}) \to M^{[\mathbf{x} \mapsto \mathbf{N}]}$$

with M of base type, and where $[\mathbf{x} \mapsto \mathbf{N}]$ denotes the (simultaneous) substitution defined as the homomorphic extension of the map $\mathbf{x} \mapsto \mathbf{N}$. *Substitutions* ($\sigma$, τ) are automorphisms on the preterms.[3] The relation $\to_\beta$ on preterms is the compatible closure [Bar84, p. 50] of the macro β-rule. Each term $[M]_=$ may be represented by the $\to_\beta$-normal form of M, which exists uniquely [Aka93]. Note that a preterm M is in $\to_\beta$-normal form iff (beta) was not used in its construction, and M can then be written in a unique way as:

$$\mathbf{x}.a(\mathbf{M})$$

for some variable vector $\mathbf{x}$ (the *binder*), symbol a (the *head*), and vector $\mathbf{M}$ (the *matrix*) of preterms (the *arguments*) in the same format [Wol93, Notation 2.29]. This preterm is taken as representative of the term $[M]_=$ and terms are frequently identified with their representatives.

After this intermezzo on representing terms, we continue with our presentation of PRSs.

Definition 3 (Rule). A *rewrite rule* over alphabet $\mathcal{F}$ is a pair (l,r) such that:

(plug) l and r are closed terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ of the same type,

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[2] More precisely, one may check that the set of preterms coincides with the set of simply typed λ-terms in expanded η-normal form [Aka93].

[3] Strictly speaking, the macro β-rule only makes semantic sense when preterms have been identified already up to renaming of bound variables [Bar84, Appendix C].

(head) the head of l is a function symbol, and

(pat) l is a *pattern* [Mil91], i.e. if $l = \mathbf{x}.a(\mathbf{M})$ then $a(\mathbf{M}) \in \mathcal{A}(\mathbf{x}, \varepsilon)$, where the collection of sets $\mathcal{A}(\mathbf{X}, \mathbf{Y})$ is the least such that $\mathbf{y}.b(\mathbf{N}) \in \mathcal{A}(\mathbf{X}, \mathbf{Y})$, if either
($\notin$) $b \notin \mathbf{X}$ and $\mathbf{N} \in \mathcal{A}(\mathbf{X}, \mathbf{yY})$, or
($\in$) $b \in \mathbf{X}$ and $\mathbf{N}$ is a vector of mutually distinct variables among $\mathbf{Y}$.
If moreover the latter case occurs precisely once for each variable among $\mathbf{X}$ and such that $\mathbf{N} = \mathbf{Y}$, the pattern is called *local*[4] [Mel96]. We use P, Q to range over the set $\mathcal{A}$ of local patterns.

In order to stay close to conventional notation in rewriting [Klo92], we write $l \to r$ for (l,r) and sometimes omit the binder of l and r, and in general of patterns. In that case, the free variables will be denoted by capitals (X, Y, Z). The restriction to patterns on left-hand sides of rewrite rules serves to make the rewrite relation generated by the rules (known to be) decidable [Mil91, Nip].

Define the order of a PRS to be the maximum order of the variables occurring in the binders of its rules. TRSs form a subclass of first order PRSs, and Λ, combinatory reduction systems (CRSs [Klo80]), interaction systems (ISs [AL94]), and expression reduction systems (ERSs [Kha90]) all are subclasses of second order PRSs [Kah93, Raa96].

Example 1 (Classical Rewrite Systems). 1. The prime example of a first order PRS (TRS) is Combinatory Logic (CL [Cur30]). Its alphabet consists of S, K, I all of base type o, and @ of type $o, o \to o$. It has three rewrite rules (all variables being of type o):

$$x.@(I, x) \to x.x$$
$$x.y.@(@(K, x), y) \to x.y.x$$
$$x.y.z.@(@(@(S, x), y), z) \to x.y.z.@(@(x, z), @(y, z))$$

All left-hand sides are local patterns. This is pretty unreadable, but using the convention above and turning @ into an implicit infix operator associating to the left, the rules get the following more pleasant format:

$$IX \to X$$
$$KXY \to X$$
$$SXYZ \to XZ(YZ)$$

2. The prime example of a second order PRS (CRS) is the lambda calculus [Chu33]. Its alphabet consists of λ of type $(o \to o) \to o$ and @ of type $o, o \to o$. Using the same conventions as in the previous item, and writing λM instead of $\lambda(M)$, the beta rule is rendered as:

$$(\lambda x.X(x))Y \to X(Y)$$

where X is a second order variable of type $o \to o$, and x,Y are first order variables of type o. Note that the left-hand side of the beta rule is indeed a local pattern since the (free) variable X only has the variable x as argument.

[4] Local occurs in literature also as ‘linear and fully extended’.

After the statics of PRSs, we now turn to their dynamics as embodied by their induced rewrite relation.

Definition 4 (Rewrite). The ARS $\rightarrow_{\mathcal{P}}$ on terms over $\mathcal{F}$ of a PRS $\mathcal{P} =^{\text{def}} (\mathcal{F},\mathcal{R})$ is called its *rewrite* relation and is induced by the rules in $\mathcal{R}$ via the following inference system:

$$\begin{array}{lll}
(\mathsf{beta}) & l(\mathbf{M}) \rightarrow_{\mathcal{P}} r(\mathbf{M}) & \Longleftarrow\ l \rightarrow r \in \mathcal{R} \\
(\mathsf{head}) & a(M_1,\dots,M,\dots,M_m) \rightarrow_{\mathcal{P}} a(M_1,\dots,N,\dots,M_m) & \Longleftarrow\ M \rightarrow_{\mathcal{P}} N \\
(\mathsf{abs}) & x.M \rightarrow_{\mathcal{P}} x.N & \Longleftarrow\ M \rightarrow_{\mathcal{P}} N
\end{array}$$

Note that substitution is missing from our inference system, or rather it is brought about in an implicit way via the (beta)-inference rule. In general, applying a substitution σ to a term M containing free variables $\mathbf{X}$ can be effectuated by writing $\mathbf{X}.M(\mathbf{X}^\sigma)$ and vice versa.

Example 2. In CL as above we have the rewrite step $KIS \rightarrow_{CL} I$ by (beta), since $KXY \rightarrow X$ is a rule, $KIS = (x.y.Kxy)(I,S)$, and $(x.y.x)(I,S) = I$.

Basic facts on PRSs. No(ta)tions for terms are pointwise extended to corresponding ones for substitutions, e.g. for substitutions σ and τ we write $\sigma \twoheadrightarrow_{\mathcal{P}} \tau$ if $X^\sigma \twoheadrightarrow_{\mathcal{P}} X^\tau$ for each variable X. The following lemma states compatibility of rewriting with term formation.

Lemma 5 (Substitution). *Let $\mathcal{P}$ be a PRS.*

1. *If $M \twoheadrightarrow_{\mathcal{P}} N$ and $\sigma \twoheadrightarrow_{\mathcal{P}} \tau$, then $M^\sigma \twoheadrightarrow_{\mathcal{P}} N^\tau$ [MN94, Theorem 3.9].*
2. *Let $P \in \mathcal{A}$ be a local pattern containing a free variable X (after removing the binder). If σ and τ only differ on X and $X^\sigma \rightarrow_{\mathcal{P}} X^\tau$, then $P^\sigma \rightarrow_{\mathcal{P}} P^\tau$.*

In the sequel we assume all patterns to be local. Since left-hand sides of PRSs are patterns, this restricts the class of PRSs to the class of local PRSs, but entails no loss in generality since any PRS can be 'localised' (cf. [OR94, Definition 25]). Localisation essentially consists in removing some of the conditions on rules, hence termination of the localised PRS carries over to the original PRS.

Definition 6. The *(proper) subterm* relation $\lhd$ on terms is defined as $\lhd_1^+$, where the *direct subterm* relation $\lhd_1$ defined by

$$\begin{aligned}
M &\lhd_1 M^\sigma, \text{ for } \sigma \text{ a non-bijective variable renaming} \\
M &\lhd_1 x.M \\
\mathbf{x}.M &\lhd_1 \mathbf{x}.a(\mathbf{M})
\end{aligned}$$

where $M \in \mathbf{M}$. The *substep* order $\ll_{\mathcal{P}}$ is defined as $(\lhd \cup \leftarrow_{\mathcal{P}})^+$.

Patterns have the nice property that terms substituted for variables in the binder are subterms of the resulting term. The induction of our main theorem relies on the property that termination of a rewrite relation is preserved by adding 'subterm steps'.

Lemma 7 (Substep). *Let M be a term and $P =^{\text{def}} \mathbf{x}.f(\mathbf{P})$ a pattern.*

1. *If $x^\sigma =^{\text{def}} \mathbf{y}.g(\mathbf{N})$ for $x \in \mathbf{x}$, then $g(\mathbf{N}) \lhd f(\mathbf{P})^\sigma$.*
2. *M is terminating iff $x.M$ is terminating.*
3. *M is $\to_{\mathcal{P}}$-terminating iff M is $\gg_{\mathcal{P}}$-terminating.*

Proof. 1. By induction on P, using that the head of P is a function symbol to get proper subterms.
2. Because (abs) is the last inference step, for any rewrite step from $x.M$.
3. One direction being trivial the claim follows from the observation that $\rhd_1$-steps are terminating and can be postponed (w.r.t. $\to_{\mathcal{P}}$-steps). ☐

Termination of PRSs is characterised by termination of the terminating instances of their right-hand sides.

Lemma 8 (RHS). *For a PRS $\mathcal{P} =^{\text{def}} (\mathcal{F},\mathcal{R})$, $\to_{\mathcal{P}}$ is terminating iff $a(\mathbf{M})^\sigma$ is terminating for every $l \to \mathbf{x}.a(\mathbf{M}) \in \mathcal{R}$ and terminating substitution σ.*

Proof. One direction is trivial. The other follows by considering a $\lhd$-minimal non-terminating term $\mathbf{x}.f(\mathbf{M})$. By minimality and Lemma 7 the binder $\mathbf{x}$ is empty, hence an infinite rewrite sequence issuing from the term is of the form $f(\mathbf{M}) \twoheadrightarrow f(\mathbf{M'}) = l^\sigma \to r^\sigma \to \dots$, with $\mathbf{M} \twoheadrightarrow \mathbf{M'}$. By definition of PRS l is a pattern having a function symbol as head, hence σ is terminating by minimality and Lemma 7, so r^σ is terminating by assumption. ☐

Remark. For the lemma it is essential that left-hand sides are patterns, as witnessed by the non-pattern rewrite rule $f(X(Y)) \to Y$. If σ is terminating, $\sigma(Y)$ is terminating by definition. Nevertheless, the rewrite relation is not terminating as witnessed by the rewrite step $f(\mathsf{a}) \to f(\mathsf{a})$ with $X^\sigma = x.\mathsf{a}$, $Y^\sigma = f(\mathsf{a})$.

2 Finiteness of Family Developments

We start by giving a definition of labelled PRS, analogous to the definition of the labelled lambda calculus due to Hyland and Wadsworth [Hyl76, Wad76]. That is, the terms and rules of a PRS are supplied with labels from the set of natural numbers and these labels may restrict applicability of a rewrite rule to a term (restrict with respect to its unlabelled version, see the example).

Definition 9. Let $(m, n \in)$ ω be a set of natural number *labels* ordered by $<$ in the usual way. The *labelled* version $\mathcal{P}^\omega =^{\text{def}} (\mathcal{F}^\omega,\mathcal{R}^\omega)$ of a PRS $\mathcal{P} =^{\text{def}} (\mathcal{F},\mathcal{R})$ is defined as follows.

1. $\mathcal{F}^\omega$ is $\mathcal{F}$ extended with a function symbol m_b of type $b \to b$, for every label $m \in \omega$ and base type b. Since the types of the labels do not play a rôle in the sequel, we will not write them henceforth.
2. $\mathcal{R}^\omega$ consists of labelled versions rules in $\mathcal{R}$ together with rewrite rules managing merger of labels. Their definition employs some auxiliary definitions. Let m be a label, $P =^{\text{def}} \mathbf{X}.a(\mathbf{P})$ a pattern, and $M =^{\text{def}} \mathbf{X}.a(\mathbf{M})$ a term.

(a) The *internally m-labelled* pattern $P_{\underset{m}{\circ}}$ is defined as $\mathbf{X}.a(\mathbf{P}_{\underset{m}{\circ}}\langle\mathbf{X}\rangle)$, where

$$Q_{\underset{m}{\circ}}\langle\mathbf{X}\rangle =^{\mathrm{def}} \mathbf{Y}.m(b(\mathbf{Q}_{\underset{m}{\circ}}\langle\mathbf{X}\rangle)), \text{ if } Q =^{\mathrm{def}} \mathbf{Y}.b(\mathbf{Q}) \text{ and } b \notin \mathbf{X},$$
$$=^{\mathrm{def}} Q, \text{ otherwise.}$$

(b) The *fully m-labelled* term $M_{\underset{m}{\bullet}}$ is defined as $\mathbf{X}.m(a(\mathbf{M}_{\underset{m}{\bullet}}))$.

Finally, we can specify the rewrite rules of $\mathcal{P}^\omega$.

(a) For every rule $l \to r \in \mathcal{R}$, its *labelled versions* are of the form

$$l_{\underset{m+1}{\circ}} \to r_{\underset{m}{\bullet}}$$

for every $m \in \omega$.

(b) For all labels $n < m \in \omega$, there are *merger* rules (*contract* and *decrease*)

$$x.n(n(x)) \to_c x.n(x)$$
$$x.m(x) \to_d x.n(x)$$

Example 3. Consider the PRS $\mathcal{P}$ having the rule $f(g) \to g$. An example of a rewrite sequence in $\mathcal{P}^\omega$ starting from the term $f(1(f(2(f(2(g))))))$ is:

$$f1f2\underline{f2g} \to_{\mathcal{P}^\omega} f1f\underline{21}g \to_d f1f\underline{11}g \to_c f1\underline{f1g} \to_{\mathcal{P}^\omega} f\underline{10}g \to_d f\underline{00}g \to_c f0g$$

where parentheses have been removed and the actions are underlined. The rewrite sequence ends in the $\mathcal{P}^\omega$-normal form $f(0(g))$. Remark that its 'unlabelling' $f(g)$ is not a $\mathcal{P}$-normal form!

Our constructions will work for any ordinal, not just ω after the following adaptation: in a labelled rule $l \to r$, r may contain only labels smaller than the least label $\bigwedge l$ occurring in l.

Remark. Since 0-labels can be inserted if needed, we may assume that every symbol $a \in \mathcal{F} \cup \mathcal{X}$ occurring in a labelled term carries a label, i.e. it occurs as $m(a(\ldots))$ for some label m. Since internally labelling is the identity on a left-hand side l of the form $\mathbf{x}.f(\mathbf{x})$, the labelled version of a rule such as $l \to l$ is non-terminating. This technical problem can be resolved by (before labelling) extending the alphabet with some fresh symbol v and expanding terms and rules via the homomorphic extension of the map $f \mapsto x.v(f(x))$.

A labelled PRS can be mapped to its underlying PRS simply by forgetting the labels and this map enjoys natural properties, allowing for the definition of the main notion of this paper, that of a family development.

Definition 10. The *unlabelling* function $|\cdot| : \mathcal{T}(\mathcal{F}^\omega, \mathcal{X}) \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ is defined as the homomorphic extension of the mapping $m \mapsto x.x$, for every $m \in \omega$.

Lemma 11. *1.* $|P_{\underset{m}{\circ}}| = P$ *and* $|M_{\underset{m}{\bullet}}| = M$*, for pattern* P *and term* M*.*
2. If $M \to_{\mathcal{P}^\omega} N$*, then* $|M| \to_{\mathcal{P}} |N|$ *in case a labelled version of a rule in* $\mathcal{P}$ *was applied, and* $|M| = |N|$ *in case a merger rule was applied.*

Proof. 1. By induction on patterns and terms.
2. By induction on the generation of $\to_{\mathcal{P}^\omega}$, using 1 in the base case. □

Definition 12. A *family development* is the projection $|M_1| \to_{\mathcal{P}}^{=} |M_2| \to_{\mathcal{P}}^{=} |M_3| \to_{\mathcal{P}}^{=} \ldots$ of some labelled rewrite sequence $M_1 \to_{\mathcal{P}^\omega} M_2 \to_{\mathcal{P}^\omega} M_3 \to_{\mathcal{P}^\omega} \ldots$.

We start the investigation of the correspondence between a PRS and its labelled version with generalising to PRSs an observation for Λ due to Lévy (cf. [Bar84, Sec. 14.2]).

Theorem 13 (Lift). *Every finite rewrite sequence is a family development.*

Proof. One shows that *any rewrite sequence $M \twoheadrightarrow_{\mathcal{P}} N$ of length m is the projection of a labelled rewrite sequence $M_{\underset{m}{\bullet}} \twoheadrightarrow_{\mathcal{P}^\omega} N_{\underset{0}{\bullet}}$* by induction on m. □

We now show the converse of this theorem, i.e. that all family developments are finite which is equivalent to all labelled rewrite sequences being finite, since by well-foundedness of $<$ the merger rules on their own are terminating.

Lemma 14. *A labelled term $m(M)$ is terminating iff M is terminating.*

Proof. The head label of a term can only be involved in merger steps, which can be postponed (at the head). □

The first part of the following lemma expresses that for a step $M \to_{\mathcal{P}} N$, the labels in a given prefix of N are determined by some prefix of M. The second part extends this to rewrite sequences of arbitrary length and corresponds to the *square bracket lemma* for Λ [NGV94].

Lemma 15 (Invert). *Let M be a term, Q a pattern, and τ be a substitution.*

1. *If $M \to_{\mathcal{P}^\omega} Q^\tau$, then there exist a pattern P and substitutions σ, υ such that $M = P^\sigma$, $\bigwedge Q \leq \bigwedge P$, and either*
   (trm) $P \to_{\mathcal{P}^\omega} Q^\upsilon$ and $\upsilon^\sigma = \tau$, or
   (sub) $P = Q$ and $\sigma \to_{\mathcal{P}^\omega} \tau$.
2. *If σ is a substitution with head labels all m and $M^\sigma \twoheadrightarrow_{\mathcal{P}^\omega} Q^\tau$, then*
   (int) $\bigwedge Q \leq m$, or
   (ext) $M \twoheadrightarrow_{\mathcal{P}^\omega} Q^\upsilon$ and $\upsilon^\sigma \twoheadrightarrow_{\mathcal{P}^\omega} \tau$ for some substitution υ.

Proof. 1. By induction on $\to_{\mathcal{P}^\omega}$ and on Q in the base case.
2. By induction on the length of the rewrite sequence $M^\sigma \twoheadrightarrow_{\mathcal{P}^\omega} Q^\tau$
   (0) If a substitution υ exists such that $M = Q^\upsilon$, the conditions of (ext) are trivially satisfied. Otherwise $\bigwedge Q \leq m$ by induction on Q.
   $(m+1)$ Suppose $M^\sigma \twoheadrightarrow_{\mathcal{P}^\omega} M' \to_{\mathcal{P}^\omega} Q^\tau$. By 1 there exists a pattern P and substitutions σ', υ' such that $M' = P^{\sigma'}$, $\bigwedge Q \leq \bigwedge P$, and either
   (trm) $P \to_{\mathcal{P}^\omega} Q^{\upsilon'}$ and $\upsilon'^{\sigma'} = \tau$, or
   (sub) $P = Q$ and $\sigma' \to_{\mathcal{P}^\omega} \tau$.
   By the induction hypothesis, either
   (int) $\bigwedge P \leq m$ but then $\bigwedge Q \leq m$ by transitivity of $<$, or
   (ext) $M \twoheadrightarrow_{\mathcal{P}^\omega} P^\upsilon$ and $\upsilon^\sigma \twoheadrightarrow_{\mathcal{P}^\omega} \sigma'$ for some substitution υ, then
      (trm) , $M \twoheadrightarrow_{\mathcal{P}^\omega} P^\upsilon \to_{\mathcal{P}^\omega} Q^{\upsilon'^\upsilon}$, and $\upsilon'^{\upsilon^\sigma} \twoheadrightarrow_{\mathcal{P}^\omega} \tau$ by Lemma 5, or

(sub) , $M \twoheadrightarrow_{\mathcal{P}^\omega} Q^\upsilon$, and $\upsilon^\sigma \twoheadrightarrow_{\mathcal{P}^\omega} \sigma' \rightarrow_{\mathcal{P}^\omega} \tau$ and we are done. □

Remark. Using the decomposition of PRS steps into partial PRS steps and β-steps [Oos96], the base cases of the lemma are seen to rely on FFD for β-steps.

The proof for showing finiteness of family developments for PRSs extends the one for Λ [Daa80], and roughly runs as follows. Consider a minimal term M such that M^σ is not terminating for some terminating substitution σ. Due to minimality eventually there must be some rewrite step taking place at the head.

1. The substitution did contribute to this head step.
   (a) If the head of M is a function symbol, then the label of the right-hand side of the head step is smaller than that of M, contradicting minimality of M.
   (b) If the head of M is a variable, then the rôles of term and substitution are interchanged. More precisely, if $M =^{\mathrm{def}} x(\mathbf{M})$ and $x^\sigma =^{\mathrm{def}} \mathbf{y}.b(\mathbf{N})$, then we consider termination of $b(\mathbf{N})$ with substitution $[\mathbf{y} \mapsto \mathbf{M}^\sigma]$.
2. The substitution didn't contribute to the head step. Then the 'same' head step could have been obtained from M alone contradicting minimality of M.

Theorem 16 (FFD). *$\mathcal{P}^\omega$ is terminating for every PRS $\mathcal{P}$.*

Proof. By Lemma 8 it suffices to prove *$M^\sigma_{\underset{m}{\bullet}}$ is terminating for every term M, label $m \in \omega$, and terminating substitution* σ, by induction on (m,M) ordered by the lexicographic product $< \times_{lex} \lhd$. A minimal counterexample term has an empty binder and all free variables occur exactly once in it by Lemma 7. Consider such a minimal $M =^{\mathrm{def}} a(\mathbf{M})$ for which $M^\sigma_{\underset{m}{\bullet}} =^{\mathrm{def}} m(a^\sigma(\mathbf{M}^\sigma_{\underset{m}{\bullet}}))$ is not terminating. By Lemma 14, this is the same as non-termination of $a^\sigma(\mathbf{M}^\sigma_{\underset{m}{\bullet}})$. and by induction hypothesis $\mathbf{M}^\sigma_{\underset{m}{\bullet}}$ is terminating. The proof is by cases on a.

(fun) If a is a function symbol f, then an infinite rewrite sequence looks like

$$f(\mathbf{M}^\sigma_{\underset{m}{\bullet}}) \twoheadrightarrow_{\mathcal{P}^\omega} f(\mathbf{M'}) = l^\tau_{\underset{n+1}{\circ}} \rightarrow_{\mathcal{P}^\omega} r^\tau_{\underset{n}{\bullet}} \rightarrow_{\mathcal{P}^\omega} \dots$$

for some rule $l \rightarrow r \in \mathcal{R}$ and substitution τ for the variables in the common binder of l, r. By the induction hypothesis $r^\tau_{\underset{n}{\bullet}}$ is terminating, since
1. $\mathbf{M'}$ is a vector of terminating terms by the induction hypothesis, hence the substitution τ is terminating by Lemma 7, and
2. $n < m$, by the assumption that every left-hand side contains at least two function symbols and the labels of the head symbols of the arguments of f are initially m and cannot increase by rewriting.

(var) If a is a variable not in the domain of σ, then termination follows as in the previous item. If a is in the domain of σ, then we prove termination by induction on the order $\ll_{\mathcal{P}^\omega}$ starting from a^σ, which is well-founded by the termination assumption on σ and Lemma 7.[5] If $a^\sigma =^{\mathrm{def}} \mathbf{y}.b(\mathbf{N})$, then

[5] In the case of TRSs we are done with the proof now. In the case of Λ we have arrived at the *substitution theorem* [NGV94, p. 511].

$a^\sigma(\mathbf{M}^\sigma_{\underset{m}{\bullet}}) =^{\mathrm{def}} b(\mathbf{N})^{\sigma'}$ with $\sigma' =^{\mathrm{def}} [\mathbf{y} \mapsto \mathbf{M}^\sigma_{\underset{m}{\bullet}}]$. By the (inner) induction hypothesis, we have that $\mathbf{N}^{\sigma'}$ is terminating. The proof is by cases on b.

(lab) If b is a label, then termination follows from Lemma 14.

(fun) If b is a function symbol $f \in \mathcal{F}$, then an infinite sequence looks like

$$f(\mathbf{N}^{\sigma'}) \twoheadrightarrow_{\mathcal{P}^\omega} f(\mathbf{N'}) = l^\tau_{\underset{n+1}{\circ}} \rightarrow_{\mathcal{P}^\omega} r^\tau_{\underset{n}{\bullet}} \rightarrow_{\mathcal{P}^\omega} \dots$$

for some rule $\mathbf{z}.l \rightarrow \mathbf{z}.r \in \mathcal{R}$ where $l =^{\mathrm{def}} f(\mathbf{P})$, and substitution τ having domain $\mathbf{z}$. By Lemma 15, either the substitution σ' having m as head label did contribute to one among $\mathbf{P'} =^{\mathrm{def}} \mathbf{P}_{\underset{n+1}{\circ}}$ or it didn't.

(int) If $\bigwedge \mathbf{P'} = n{+}1 \leq m$, then $n < m$, and by Lemma 7 the outer induction hypothesis applies to $r_{\underset{n}{\bullet}}$ and τ.

(ext) If $\mathbf{N} \twoheadrightarrow_{\mathcal{P}^\omega} \mathbf{P'}^\upsilon$ and $\upsilon^{\sigma'} \twoheadrightarrow_{\mathcal{P}^\omega} \tau$ for some substitution υ, then

$$f(\mathbf{N}) \twoheadrightarrow_{\mathcal{P}^\omega} f(\mathbf{P'}^\upsilon) = l^\upsilon_{\underset{n+1}{\circ}} \rightarrow_{\mathcal{P}^\omega} r^\upsilon_{\underset{n}{\bullet}} \text{ and } r^{\upsilon^{\sigma'}}_{\underset{n}{\bullet}} \twoheadrightarrow_{\mathcal{P}^\omega} r^\tau_{\underset{n}{\bullet}}$$

by Lemma 5. Since the former sequence contains at least one step (the final one), the inner induction hypothesis applies to $r^\upsilon_{\underset{n}{\bullet}}$ and σ', yielding termination of $r^\tau_{\underset{n}{\bullet}}$ by the latter sequence.

(var) If b is a variable not among $\mathbf{y}$, then termination follows as in the previous item. If $b = y_i$, and $M_i =^{\mathrm{def}} \mathbf{z}.c(\mathbf{O})$, then $b(\mathbf{N})^{\sigma'} =^{\mathrm{def}} c(\mathbf{O})^{\sigma\,[\mathbf{z} \mapsto \mathbf{N}^{\sigma'}]}_{\underset{m}{\bullet}}$ which is terminating by the outer induction hypothesis, since $c(\mathbf{O})$ is a proper subterm of M, σ is terminating by assumption, and $\mathbf{N}^{\sigma'}$ is terminating by the inner induction hypothesis. □

There should be a simpler proof of this in the style of [Vri87, Stelling 1].

Remark. It was shown in [Ter96] that allowing the head of the right-hand side to carry the label of the left-hand side (instead of a strictly smaller one) does not affect validity of FFD in the case of Λ. By inspection of our proofs it will be clear that the same holds for PRSs. (Formally, define $(\mathbf{x}.a(\mathbf{M}))_{\underset{m}{\odot}} =^{\mathrm{def}} \mathbf{x}.m{+}1(a(\mathbf{M}_{\underset{m}{\bullet}}))$ and replace everywhere in the text $\bullet$ by $\odot$.)

Applications. The applications of FFD to Λ in [Bar84, Section 14.2] of FFD scale up to PRSs immediately.

1. A first consequence of FFD is the finiteness of developments theorem for PRSs [Oos94], which will be discussed in the next section.
2. Another application is the standardisation theorem in [Bar84, Theorem 14.2.10], roughly expressing that rewrite sequence can be sorted in an outside-in manner. The proofmethod presented there is essentially abstract and can be lifted to arbitrary (left-linear) PRSs.
3. Mapping the labels of Lévy's labelled lambda calculus [Lév78] to their 'nesting depth', yields Hyland-Wadsworth's labelled lambda calculus (Λ^ω). The same holds for PRSs for a suitable extension of Lévy labelling [Oos96].

Many more applications should be around, for example we expect that FFD implies finiteness of superdevelopments [Raa96], since intuitively a superdevelopment from some term should be bounded by the labelling such that each symbol is labelled by its distance to the root in the term.

Related work on FFD. Since TRSs, Λ, CRSs, ISs, and ERSs are all embeddable in PRSs (see above), they enjoy FFD by our main result. For TRSs and Λ this was known [Mar92, Daa80]. For the other systems we are aware of only two partial results. In [Klo80, Remark II.6.2.7.16(i)] it is stated that FFD can be shown to hold for orthogonal CRSs by adapting the methods presented there. However, the method is highly technical and full of pitfalls (cf. [Mel96, Section 6.2.2] where one of the proofsteps of [Klo80] was shown to be erroneous) rendering it not very flexible (although we claim it can be adapted to PRSs). Moreover, the method is confined to orthogonal CRSs in an essential way. To remedy all this, an axiomatic approach to FFD was introduced in [Mel96] and it is conjectured there that the axioms are verified by CRSs.

Remark. We conjecture that the axioms for FFD in [Mel96] are not verified by PRSs. More precisely, we think his notion of *contexte λ-clos* is a second order notion and expect that the axioms for these will not be verified by say third order PRSs (cf. the remark on gripping below).

3 Finiteness of Developments

In this section we present two proofs of the finiteness of developments theorem (FD) for PRSs [Oos94]. In the first paragraph, we show that FD is a consequence of FFD. In the second paragraph we present a direct proof of FD via reducibility [Tai67] and its specialisation to TRSs and Λ, yielding particularly simple proofs of FD for those.

Consider a term and outline some left-hand sides of rules occurring in it (such that outlines do not cross and do not contain one another). Then FD expresses that every rewrite sequence contracting outlined left-hand sides only will always terminate.

Example 4. Consider the TRS having rules $\mathtt{a} \to \mathtt{a}$, $\mathtt{a} \to \mathtt{b}$, and $f(X) \to g(X, X)$. Examples of outline rewrite sequences starting from the outline term $\boxed{f}(\boxed{\mathtt{a}})$ are:[6]

$$\boxed{f}(\boxed{\mathtt{a}}) \to \boxed{f}(\mathtt{b}) \to g(\mathtt{b}, \mathtt{b}), \text{ and}$$

$$\boxed{f}(\boxed{\mathtt{a}}) \to g(\boxed{\mathtt{a}}, \boxed{\mathtt{a}}) \to g(\mathtt{a}, \boxed{\mathtt{a}}) \to g(\mathtt{a}, \mathtt{b})$$

Note that the final term of the latter sequence is in outline normal form, but not in normal form and that in this sequence different rules were applied to 'residuals' of the same outlined left-hand side $\boxed{\mathtt{a}}$.

[6] In examples, only heads of left-hand sides are outlined.

Definition 17. The outline version $\boxed{\mathcal{P}} =^{\text{def}} (\boxed{\mathcal{F}},\boxed{\mathcal{R}})$ of a PRS $\mathcal{P} =^{\text{def}} (\mathcal{F},\mathcal{R})$ is:

1. $\boxed{\mathcal{F}}$ is $\mathcal{F}$ extended with a symbol $\boxed{l}$ of type θ for every $l \to r \in \mathcal{R}$ of type θ.
2. For every rule $l \to r \in \mathcal{R}$, there's a rule $\boxed{l} \to r \in \boxed{\mathcal{R}}$.

Analogous to unlabelling there's an obvious notion of inlining having natural properties, allowing for the usual definition of developments.

Definition 18. The *inlining* function $\dagger : \mathcal{T}(\boxed{\mathcal{F}}, \mathcal{X}) \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ is defined as the homomorphic extension of the mapping $\boxed{l} \mapsto l$, for every $l \to r \in \mathcal{R}$.

Lemma 19. *If* $M \to_{\boxed{\mathcal{P}}} N$*, then* $\dagger(M) \to_{\mathcal{P}} \dagger(N)$*.* □

Definition 20. A *development* is the projection $\dagger(M_1) \to_{\mathcal{P}} \dagger(M_2) \to_{\mathcal{P}} \dagger(M_3) \to_{\mathcal{P}} \dots$ of some outline rewrite sequence $M_1 \to_{\boxed{\mathcal{P}}} M_2 \to_{\boxed{\mathcal{P}}} M_3 \to_{\boxed{\mathcal{P}}} \cdots$.

Below we will present two proofs of the fact that developments in PRSs are always terminating, or stated differently that outline PRSs are terminating.

Theorem 21 (FD for PRSs). *$\mathcal{P}$-developments are terminating for any PRS $\mathcal{P}$.*

FD via FFD. Finiteness of developments should follow from finiteness of family developments, since in a development never newly created redexes are contracted hence it suffices to use only labels {0,1}.

Definition 22. Define the mapping $\boxed{\omega} : \mathcal{T}(\boxed{\mathcal{F}}, \mathcal{X}) \to \mathcal{T}(\mathcal{F}^{\omega}, \mathcal{X})$ as the homomorphic extension of the map $\boxed{l} \mapsto l_{\underset{1}{\circ}}$.

Lemma 23. *If* $M \to_{\boxed{\mathcal{P}}} N$*, then* $\boxed{\omega}(M_{\underset{0}{\bullet}}) \to^{+}_{\mathcal{P}^{\omega}} \boxed{\omega}(N_{\underset{0}{\bullet}})$*. (So, first fully label all subterms by* 0*, then inline all left-hand sides and internally label them by* 1*.)*

Proof. By induction on the generation of $\to_{\boxed{\mathcal{P}}}$. □

Remark. Since outlines were not allowed to cross or contain one another, there are in general fewer developments starting from some term M, than there are family developments obtained by projecting a labelled rewrite sequence starting with $\boxed{\omega}(M)$. For orthogonal PRSs there's no difference.

Proof (of Theorem 21). From Definition 20, Theorem 16, and Lemma 23. □

FD à la Tait. In order to give a direct proof of Theorem 21, the following strengthening of termination will be employed.

Definition 24. A term $\mathbf{x}.a(\mathbf{M})$ is *reducible*, if $a(\mathbf{M})^{\sigma}$ is terminating for reducible σ (on a subset of $\mathbf{x}$).

This is well-defined since the types of the terms in the substitution are smaller than the type of the term. Note that reducibility implies termination (by taking a substitution with empty domain) and is closed under rewriting.

Proof (of Theorem 21). We prove *every outlined term* $\mathbf{x}.a(\mathbf{M})$ *is reducible* by induction on the pair (number of outlined symbols,number of head symbols) in a term ordered by $< \times_{lex} <$. We need to show termination of $a(\mathbf{M})^\sigma$ for reducible σ. Without loss of generality we may assume that $\mathbf{x}$ is the domain of σ since variables are reducible. By induction hypothesis we know that $\mathbf{x}.M_i$ is reducible for every M_i among $\mathbf{M}$. The proof is by cases on a.

(var) If a is a variable x_i among $\mathbf{x}$ and $\sigma(x_i) =^{\text{def}} \mathbf{y}.b(\mathbf{N})$, then $a(\mathbf{M})^\sigma = \sigma(x_i)(\mathbf{M}^\sigma) = b(\mathbf{N})^{\sigma[\mathbf{y}\mapsto\mathbf{M}^\sigma]}$, which is reducible by induction hypothesis.

(fun) If a is a function symbol f, then termination follows from the induction hypothesis on the arguments.

(rule) In a is an outline symbol $\boxed{l}$ for some rule $l \to r$ of $\mathcal{P}$, then since the arguments are terminating by the induction hypothesis, an infinite rewrite sequence must look like

$$\boxed{l}(\mathbf{M}^\sigma) \twoheadrightarrow \boxed{l}(\mathbf{M'}) \to r(\mathbf{M'}) \to \dots$$

But since r contains no outlined symbols, and $\mathbf{M'}$ is reducible by the induction hypothesis and closure of reducibility under rewriting, if $r =^{\text{def}} \mathbf{y}.b(\mathbf{N})$, then $r(\mathbf{M'}) = b(\mathbf{N})^{[\mathbf{y}\mapsto\mathbf{M'}]}$ which is reducible by the induction hypothesis, hence terminating. □

Specialising to TRSs and Λ, the following two short proofs of FD for those systems are obtained.

Theorem 25 (FD for TRSs). $\boxed{\mathcal{R}}$ *is terminating for every TRS* $\mathcal{R}$.

Proof. By Lemma 8 it suffices to prove termination of terminating instances of right-hand sides of $\boxed{\mathcal{R}}$, which follows by a trivial induction on outline terms. □

Theorem 26 (FD for Λ). $\boxed{\Lambda}$ *is terminating.*

Proof. One shows *for all outline lambda terms* M *and terminating substitutions* σ*,* M^σ *is terminating* by induction on M, from which the theorem follows by taking the identity for σ. The only non-trivial case is:

($\boxed{\lambda}$) Consider a $\boxed{\lambda}$-redex of the form $(\boxed{\lambda}x.M)N$. Since M^σ and N^σ are terminating by induction hypothesis, an infinite rewrite sequence looks like

$$(\boxed{\lambda}x.M^\sigma)N^\sigma \twoheadrightarrow (\boxed{\lambda}x.M')N' \to M'^{[x\mapsto N']} \to \dots$$

But $M'^{[x\mapsto N']} \twoheadleftarrow M^{\sigma[x\mapsto N^\sigma]}$ which is terminating by the induction hypothesis. □

Applications. Starting with [CR36], FD has been the cornerstone of confluence proofs for orthogonal rewrite systems and our results can be used for that purpose. Moreover, the FD proof given here is flexible and can be easily adapted to the enhanced versions of FD occurring in literature (e.g. FD! expressing that all maximal developments end in the same term [Bar84]). It is also possible to give explicit realizers for our proofs, yielding proofs of FD$\sharp$, i.e. (sharp) upperbounds on lengths of developments [Vri87].

Related work on FD. Since the literature on FD is quite extensive, we only discuss related work on higher order rewriting. In [Klo80, pp. 141–163] a first proof of FD is presented for the class of CRSs, which is akin to the proof of FD for ERSs in [Kha92]. Both proofs are based on the 'memory method' [Klo80, Section II.4], which has its roots in the conservation theorem [Bar84, Theorem 11.3.4]. We remark that the obvious adaptation of this theorem to PRSs fails (cf. the next example). [Oos94] introduces a modular approach to FD and presents some conditions sufficient for FD, which are verified to hold for PRSs. The method is based on explicitly manipulating residuals, i.e. a kind of realizers for the proof à la Tait above, which is tedious. Finally, [Mel96] introduces an axiomatic approach to FD, and the axioms are verified to hold for CRSs. Although the method is appealing it does not apply directly to (third order) PRSs.

Example 5. Consider the outline version of the following (orthogonal) PRS

$$\begin{aligned} g(y.X(x.y(x))) &\to X(z.X(x.z)) \\ f(x.Y(x)) &\to \dots \end{aligned}$$

where the variable X is of (third order) type $o \to o \to o$, and the outline rewrite step

$$\boxed{g}(y.(\boxed{f}(x.y(x)))) \to \boxed{f}(x.(\boxed{f}(y.x))$$

Observe that the innermost $\boxed{f}$-redex in the resulting term is *gripped* [Mel96] by the outermost one, i.e. contains a variable bound by the outer one. This implies that axioms fd-3 and either acyclicity of gripping or axiom Z-1 in [Mel96, Ch. 3] cannot hold at the same time.

It is not clear to us whether FD is a consequence of termination of rewrite systems satisfying the so-called *general schema* [JO]. Although left-hand sides of outline PRSs are algebraic, right-hand sides are not which makes the method not directly applicable.

4 Conclusion

We've presented a modular proof of finiteness of family developments (FFD) for the class of higher order pattern rewrite systems (PRS). The proof is modular in the sense that higher order rewriting as studied here is rewriting modulo simply typed λ-calculus and the proof can be viewed upon as reducing FFD for PRSs to a similar question for the simply typed λ-calculus. A question is whether FFD can be shown in an essentially different way, for example by an appeal to a higher-order generalisation of Kruskal's Tree Theorem (cf. [Mar92, Prop. 3.3.11] for the TRS case). Another question is whether systems such as proofnets or sharing graph rewrite systems (or people) enjoy FFD.

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