Multi-redexes and multi-treks induce residual systems

least upper bounds and left-cancellation up to homotopy

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Abstract

Residual theory in rewriting goes back to Church, Rosser and Newman at the end of the 1930s. We investigate an axiomatic approach to it developed in 2002 by Melliès. He gave four axioms (SD) *self-destruction*, (F) *finiteness*, (FD) *finite developments*, and (PERM) *permutation*, showing that they entail two key properties of reductions, namely having (i) *least upper bounds* (*lubs*) and (ii) *left-cancellation*.[1] These properties are shown to hold *up to* the equivalence generated by identifying the legs of local confluence diagrams inducing the same residuation, which corresponds to Lévy's permutation equivalence. Melliès in fact presented two sets of axioms, one for *redexes* as in classical residual theory and another more general one for *treks*. We show his results factor through the theory of *residual* systems we introduced in 2000, in that any rewrite system satisfying the four axioms (for redexes or treks) can be enriched to a residual system such that (i) and (ii) follow from the theory of residual systems. We exemplify the axioms are sufficient but not necessary.

1 Residual systems

We are interested in the theory of computation based on rewriting. As this requires to have computations as first-class citizens, we use rewrite *systems* [21],[27, Def. 8.2.2] (not rewrite *relations*), whose *steps* have *sources* and *targets*. We recapitulate *residual* systems [27, Def. 8.7.2].

Definition 1. *A* residual *system (RS)* $\langle \to, 1, / \rangle$ *comprises a rewrite system* $\to$ *and a* residual *function* $/$ *having* 1 *as* unit*:* 1 *is a function from objects to steps such that* $\mathsf{tgt}(1_a) = a = \mathsf{src}(1_a)$ *and for co-initial steps* ϕ, ψ, χ*, the* residual *identities (1)–(3) in Tab. 1 must be satisfied. The* projection *order* $\lesssim$ *is defined by* $\phi \lesssim \psi$ *if* $\phi/\psi = 1$ *for co-initial steps* ϕ, ψ*.*

The projection order $\lesssim$ is a quasi-order [27, Lem. 8.7.23] inducing *projection* equivalence $\simeq := \lesssim \cap \gtrsim$. Examples of rewrite systems that can be equipped with residual structure abound.

Example 1. *For the following rewrite systems* $\to$*, residual structure is obtained from the proof of the diamond property for an appropriate rewrite system that is between* $\to$ *and its reflexive–transitive closure: i) the* $\lambda\beta$*-calculus induces a residual system by the Tait–Martin-Löf proof that* $\geq_1$ *has the diamond property [2]; ii)* β*-steps in the* linear $\lambda\beta$*-calculus have the diamond property themselves; iii)* parallel *steps* $\twoheadrightarrow\!\!\!\!\!\shortmid\;$/multisteps $\circ\!\!\to$ *in orthogonal first/higher-order term rewrite systems [14, 27, 4]; iv) positive braids with* parallel crossings *of strands [27, Sect. 8.9].*

Here we show multi-redexes and multi-treks as in Melliès' axiomatic residual theory naturally induce residual systems, entailing the results of [20] *via* the theory of residual systems [27]. We use $\phi, \psi, \chi, \ldots$ and $\gamma, \delta, \epsilon, \ldots$ to range over steps respectively reductions. We denote *finite* reductions by $\twoheadrightarrow$. They can be identified [27, Def. 8.2.10] with formal compositions $(\cdot)$ of steps

[1] Instead of the order-theoretic setting employed here, Melliès employs a category-theoretic setting and the corresponding terminology of having *pushouts* and *epis*.

(whose targets, sources match) modulo the monoid *identities*. Orienting these into the *rules* (4)–(6) of Tab. 1 gives a complete 2-rewrite system[2] so unique representatives of such reductions.

Proposition 1. *Any residual system on $\to$ extends to a residual system on $\twoheadrightarrow$, defining residuation by normalisation w.r.t. the 2-rewrite system with rules (4)–(8) of Tab. 1.*

$$
\begin{array}{rcl@{\qquad}rcl@{\quad}rcl}
\phi/1 & \overset{(1)}{=} & \phi & (\gamma\cdot\delta)\cdot\epsilon & \overset{(4)}{\Rightarrow} & \gamma\cdot(\delta\cdot\epsilon) & \gamma/(\delta\cdot\epsilon) & \overset{(7)}{\Rightarrow} & (\gamma/\delta)/\epsilon \\
\phi/\phi & \overset{(2)}{=} & 1 & \gamma\cdot 1 & \overset{(5)}{\Rightarrow} & \gamma & (\delta\cdot\epsilon)/\gamma & \overset{(8)}{\Rightarrow} & (\delta/\gamma)\cdot(\epsilon/(\gamma/\delta)) \\
(\phi/\psi)/(\chi/\psi) & \overset{(3)}{=} & (\phi/\chi)/(\psi/\chi) & 1\cdot\gamma & \overset{(6)}{\Rightarrow} & \gamma & & &
\end{array}
$$

Table 1: Residual identities, monoid rules, and residual rules for formal composition

Example 2. *The classical example of a term rewrite system is Combinatory Logic (CL) having the three rules, in applicative notation, $\iota(x) : Ix \to x$, $\kappa(x,y) : Kxy \to x$, and $\varsigma(x,y,z) : Sxyz \to xz(yz)$. We call a term over the signature extended with the so-called* rule symbols *[27, Ch. 8] ι,κ,ς (having as arities the number of variables in the respective rules) a* multistep*, as it can be assigned a source/target by mapping all such rule symbols in it to their lhs/rhs. This naturally induces a residual system on multisteps [27, Prop. 8.7.7], which by the above extends to one on reductions (of multisteps). For example, $\gamma \coloneqq \varsigma(K,y,Iz)\cdot\kappa(Iz,y(Iz))$ and $\delta \coloneqq SKI\iota(z)$ are co-initial reductions from $SKy(Iz)$ to Iz respectively $SKyz$. Both these targets are reduced to z by the respective residual reductions: $\delta/\gamma \coloneqq \iota(z)$ and $\gamma/\delta \coloneqq \varsigma(K,y,z)\cdot\kappa(z,y(z))$.*

Remark 1. *We introduced the idea of multisteps as terms over the signature extended with rule symbols in [27, Ch. 8] as a generic tool in* structured *rewrite systems, like* string *[12, p. 226],* higher-order term *[4, p. 127], and* graph *[27, Rem. 9.4.30] rewrite systems.*

Then $\twoheadrightarrow$ is a residual system *with composition* [27, Def. 8.7.38], $\simeq$ is a congruence for $/$ and $\cdot$ and quotienting $\simeq$ out yields a residual system whose projection order is a partial order [27, Lem. 8.7.41]. Projection equivalence [27] can alternatively be defined as the homotopy *generated* by the diamond property. This will allow us below to relate the former to local homotopy [20].

Definition 2. Square homotopy *equivalence $\equiv$ on reductions having the same sources/targets, is generated by closing $\phi \sqcup \psi \equiv \psi \sqcup \phi$ for local peaks ϕ,ψ under composition: if $\gamma \equiv \gamma'$ then $\delta\cdot\gamma\cdot\epsilon \equiv \delta\cdot\gamma'\cdot\epsilon$. Here $\phi \sqcup \psi \coloneqq \phi\cdot(\psi/\phi)$. Correspondingly, we define $\gamma \sqsubseteq \delta$ if $\gamma\cdot\epsilon \equiv \delta$ for some ϵ.*

Lemma 1. *$\simeq\, = \,\equiv$ and $\lesssim\, = \,\sqsubseteq$.*

Example 3. *For γ,δ in Ex. 2 we have $\gamma\cdot(\delta/\gamma) \equiv \varsigma(K,y,Iz)\cdot K(\iota(z))(y\iota(z))\cdot\kappa(z,yz) \equiv \delta\cdot(\gamma/\delta)$.*

Theorem 1. *$\twoheadrightarrow$ up to square homotopy has lubs (δ',γ' is an* upper bound *of γ,δ if $\gamma\cdot\delta' \equiv \delta\cdot\gamma'$;* least *if $\delta' \sqsubseteq \delta''$, $\gamma' \sqsubseteq \gamma''$ for all upper bounds δ'',γ'') and left-cancellation (if $\gamma\cdot\delta \equiv \gamma\cdot\epsilon$ then $\delta \equiv \epsilon$).*

[2] What we refer to as 2-rewrite systems have formal expressions of compositions (and residuations) as objects. Their rules transform such expressions into *reductions* of an ordinary (1-)rewrite system $\to$, i.e. into formal compositions in normal form with respect to the monoid rules. This set-up generalises the 2-rewrite systems as found in the literature by not giving special status to composition, not *assuming* rules to operate on reductions only but on formal expressions. Working *modulo* the monoid identities yields proper 2-rewrite systems.

Proof. By Lem. 1 it follows from the same for projection equivalence $\simeq$ instead of square homotopy $\equiv$, which holds by virtue of $\twoheadrightarrow$ being a residual system with composition [27, Ex. 8.7.52]. We do that exercise: Left-cancellation follows from (see also the proof of Prop. 1):

$$(\gamma\cdot\delta)/(\gamma\cdot\epsilon) \Rightarrow ((\gamma\cdot\delta)/\gamma)/\epsilon \Rightarrow ((\gamma/\gamma)\cdot(\delta/(\gamma/\gamma)))/\epsilon \Rrightarrow (1\cdot(\delta/1))/\epsilon \Rrightarrow (1\cdot\delta)/\epsilon \Rightarrow \delta/\epsilon$$

That $\delta/\gamma, \gamma/\delta$ is an upper bound up to $\simeq$ of γ, δ, holds by $\twoheadrightarrow$ being a residual system. To see it is least consider any δ'', γ'' such that $\gamma\cdot\delta'' \simeq \delta\cdot\gamma''$. Then $(\gamma\cdot\delta'')/(\delta\cdot\gamma'') \Rightarrow (\gamma/(\delta\cdot\gamma''))\cdot(\delta''/((\delta\cdot\gamma'')/\gamma)) = 1$. Therefore [27, Ex. 8.7.40(iii)] both components must be 1 in particular the 1st $\gamma/(\delta\cdot\gamma'') \Rightarrow (\gamma/\delta)/\gamma'' = 1$. By symmetry $(\delta/\gamma)/\delta'' = 1$ and we conclude.[3] □

2 Multi-redexes and multi-treks

In [20] rewrite systems are equipped with a notion of residuation inducing a notion of *local* homotopy on reductions, based on the four axiomatic properties (SD), (F), (FD), and (PERM). The properties guarantee that multi-redexes/treks can be *developed* into reductions, that such developments have the diamond property, that all developments are locally homotopic, and finally (the main result) that reductions have lubs and left-cancellation up to local homotopy (Thm. 2). In fact *two* sets of four axioms are given in [20], the first one for *multi-redexes* and the second more general one for *multi-treks*. We show that in both cases the main results of [20] follow by known residual theory for a naturally associated residual system (in the sense of Sect. 1) on so-called developments, in particular from Thm. 1. We first develop enough notation to formally *express* the properties required of a rewrite system $\to$ for *multi-redexes* [20, Section 2], which informally read:

(*self-destruction*, SD) no step has a residual after itself;

(*finiteness*, F) every redex has finitely many residuals after a step;

(*finite developments*, FD) developments of multi-redexes are finite; and

(*permutation*, PERM) every peak ϕ, ψ of steps can be completed by a valley of complete developments of the residuals of ψ after ϕ, respectively the residuals of ϕ after ψ, such that both legs of the resulting local confluence diagram induce the same redex-trace relation.

We then show that these properties induce a residual system (Def. 1) on developments whose square homotopy corresponds to local homotopy on reductions, i.e. that Thm. 1 entails Thm. 2:

Theorem 2 (SD,FD,PERM; [20])**.** *$\twoheadrightarrow$ up to local homotopy has lubs and left-cancellation.*

Here *local* homotopy is generated (Def. 5) from the *local* confluence diagrams given by (PERM), instead of the *square* diamonds generating *square* homotopy (Def. 2). As in the statement of this main theorem, we qualify (intermediate) results throughout with the properties used, to enable illustrating that properties are sufficient but not necessary. In [20] residuation is captured by *tracing* a *redex* along a step to its *residuals*.

Definition 3. *A* redex-*trace relation is a function* $[\![\cdot\rangle\!\rangle$ *mapping each step* $\phi : a \to b$ *to a relation* $[\![\phi\rangle\!\rangle$ *between the redexes of* a *and* b*, where (*multi-*)*redexes *are reified (sets of) steps.*

[3] That gives a *pushout* as witnessed by $\epsilon := \delta''/(\delta/\gamma)$: On the one hand, $(\delta/\gamma)\cdot\epsilon \simeq \delta'\cdot((\delta/\gamma)/\delta'') \simeq \delta''$ follows from having a residual system and $\delta/\gamma \lesssim \delta'$. On the other hand, $(\gamma/\delta)\cdot\epsilon \simeq \epsilon''$ follows by left-cancellation from $\delta\cdot(\gamma/\delta)\cdot\epsilon \simeq \gamma\cdot(\delta/\gamma)\cdot\epsilon \simeq \gamma\cdot\delta'' \simeq \delta\cdot\epsilon''$ where the 2nd equivalence holds by the above and the others by assumption.

(SD) is formalised as $(\phi\ [\![\phi\rangle\!\rangle) = \varnothing$ and (F) as $(\psi\ [\![\phi\rangle\!\rangle)$ is finite, for any step ϕ and redex ψ. Here we use *section* notation for partial application of relations. The *left* section of a binary relation for an object a is $(a\ R) \coloneqq \{b \mid a\ R\ b\}$. Similarly, the *right* section is $(R\ a) \coloneqq \{b \mid b\ R\ a\}$. This is lifted pointwise to sets by $(A\ R) \coloneqq \bigcup_{a\in A}(a\ R)$ and $(R\ A) \coloneqq \bigcup_{a\in A}(R\ a)$. Trace relations naturally extend to reductions and conversions since relations constitute an involutive (typed) monoid with respect to composition, the identity relation, and converse, so we may e.g. write $\langle\!\langle\leftarrow]\!]$ for the trace relation of $\leftarrow$. We proceed with reifying tracing, labelling objects of the rewrite system with sets of redexes, which allows to recover the notion of *development* of [20].

Definition 4. *Consider the labelled rewrite system [27, Def. 8.4.5] having for each set Φ of redexes of a the object a^{Φ}, and for each step $\phi{:}a \to b$ the step ϕ^{Φ} from a^{Φ} to $b^{(\Phi[\![\phi\rangle\!\rangle)}$. A reduction γ from an object a is a* development of Φ *if it lifts to a $[\![\to\rangle\!\rangle$-reduction γ^{Φ} from a^{Φ}, where $[\![\to\rangle\!\rangle$ is the restriction of the labelled rewrite system to steps ϕ^{Φ} such that $\phi\in\Phi$. We say γ is a* complete development *of Φ if its lifting ends in a $\varnothing$-labelled object.*

(FD) is formalised by all developments are finite,[4] and (PERM) by every local peak ϕ,ψ is completed by some valley γ,δ of complete developments of $(\psi\ [\![\phi\rangle\!\rangle), (\phi\ [\![\psi\rangle\!\rangle)$ with $[\![\phi\cdot\gamma\rangle\!\rangle = [\![\psi\cdot\delta\rangle\!\rangle$.

Remark 2. *The lifting γ^{Φ} of the reduction γ in Def. 4 is unique. Formally, this is a consequence of the labelling given being a* rewrite *labelling in the sense of [27, Def. 8.4.5].*

Lemma 2 (FD,PERM)**.** *$\langle \multimap\!\!\!\to, 1, / \rangle$ is a residual system with binary joins/diagonals, for $\multimap\!\!\!\to$ the rewrite system having as objects the objects of $\to$, and as steps a* multi-redex *$a^{\Phi} : a \multimap\!\!\!\to b$ if there is a complete development of Φ from a to b; 1_a defined as $\varnothing$; residual Φ/Ψ defined as $(\Phi\ [\![\Psi\rangle\!\rangle)$, and the binary join/diagonal given by $\Phi \cup \Psi$ (cf. [27, Def. 8.7.28]).*

Denoting a multi-redex a^{Φ} by just Φ in the lemma, is justified by that a is the source common to all steps in Φ, and that all complete developments of Φ have the same target. The join being a step from the source to the target of a residual diamond, justifies calling it a diagonal.

Remark 3. *Parallel rewriting $\multimap\!\!\!\!\!\shortparallel\!\!\!\to$ [13] does constitute a residual system for orthogonal TRSs, so does give rise to good residual theory [27], but $\multimap\!\!\!\!\!\shortparallel\!\!\!\to$ does not have joins, e.g. the join of the single/parallel steps $\iota(Ix)$ and $I\iota(x)$* should *be $\iota(\iota(x))$ but although that is a* multi*step it is not a* parallel *step as it nests ι in itself. Hence, by Lem. 2 it cannot be obtained* via *multi-redexes; a first indication that the properties in [20] are too strong.*[5]

Definition 5 ([20])**.** Local homotopy $\equiv_l$ *on reductions with the same sources/targets, is the equivalence generated by closing $\phi\cdot\gamma \equiv_l \psi\cdot\delta$ for peaks ϕ,ψ and valleys γ,δ given*[6] *by (PERM) under composition: if $\gamma \equiv_l \gamma'$ then $\delta'\cdot\gamma\cdot\epsilon' \equiv_l \delta'\cdot\gamma'\cdot\epsilon'$. We define $\gamma \sqsubseteq_l \delta$ if $\gamma\cdot\epsilon \equiv_l \delta$ for some ϵ.*

We show *local* homotopy $\equiv_l$ on finite $\to$-reductions is the same as *square* homotopy $\equiv$ on finite $\multimap\!\!\!\to$-reductions. Observe we may embed $\to\ \subseteq\ \multimap\!\!\!\to$ by mapping a step $\phi : \phi \to \psi$ to $\phi^{\{\phi\}} : a \multimap\!\!\!\to b$ assuming (SD), and vice versa $\multimap\!\!\!\to\ \subseteq\ \twoheadrightarrow$ by mapping each multi-redex a^{Φ} to an arbitrary but fixed complete development of Φ from a. Below the corresponding coercions (and their stepwise

[4] Since in [20] only *finite* reductions are defined, (FD) is (must be) circumscribed there as the absence of infinite *sequences* of steps all of whose *prefixes* are developments of the given set.

[5] Following the rewrite approach, residual systems do *not* assume that steps are closed under composition. Indeed, parallel steps are not, but *reductions* of parallel steps do have compositions and therefore also joins as follows from Proposition 1. In our example, both reductions $\iota(Ix)\cdot\iota(x)$ and $I\iota(x)\cdot\iota(x)$ along the two legs of the diamond are (equivalent) joins of $\iota(Ix)$ and $I\iota(x)$.

[6] For a peak, the choice of valley witnessing (PERM) may be non-deterministic. Essentially this follows since FD makes Newman's Lemma apply 'locally' to developments, allowing to show that independently of the choice the induced redex-trace relation is the same; see the proof of Lem. 3 and cf. [22, Prop. 2.4.16] and [24, Thm. 2].

extensions to $\to$-reductions respectively $\mathrel{-\!\!\circ\!\!\!\to}$-reductions) are denoted by overlining respectively underlining, but we omit them as much as possible. Note $\gamma = \underline{(\overline{\gamma})}$ for any γ.

Remark 4. *(FD) entails the equivalence closures of $\to, \mathrel{-\!\!\circ\!\!\!\to}$ are the same, but their reflexive–transitive closures may differ if (SD) does not hold: for steps $\phi : a \to b$ and $\phi' : b \to c$ with only $\phi \,[\![\phi\rangle\!\rangle\, \phi'$ non-empty, we have $a \to b$ and indeed also $a \mathrel{-\!\!\circ\!\!\!\to} c \mathrel{\leftarrow\!\!\!\circ\!\!-} b$, but not $a \mathrel{-\!\!\circ\!\!\!\twoheadrightarrow} b$.*

Lemma 3 (SD,FD,PERM)**.** *$\equiv\, = \,\equiv_l$ and $\sqsubseteq\, = \,\sqsubseteq_l$ (after embedding; in both directions).*

The main result on multi-redexes of [20] is now a matter of chaining the above results:

Proof of Thm. 2. Lem. 2 for $\to$ induces a residual system on $\mathrel{-\!\!\circ\!\!\!\to}$. By Prop. 1 that induces a residual system with composition on $\mathrel{-\!\!\circ\!\!\!\twoheadrightarrow}$, which by Thm. 1 has lubs and left-cancellation up to *square* homotopy. Hence $\twoheadrightarrow$ has lubs and left-cancellation up to *local* homotopy by Lem. 3. $\square$

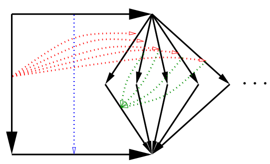

Figure 1: Rewrite system satisfying (SD), (FD) and (PERM) but not (F)

Fig. 1 illustrates the result for a system for which (F) does not hold, a second indication the properties in [20] are too strong. To recover Thms. 1 and 2 of [20] *exactly*, using (F), it suffices to observe that the above can be relativised to a collection $\mathcal{R}$ of sets of redexes such that $\varnothing, \{\phi\} \in \mathcal{R}$ for all redexes ϕ, and $\Phi \cup \Psi, (\Phi\,[\![\Psi\rangle\!\rangle) \in \mathcal{R}$ for all co-initial $\Phi, \Psi \in \mathcal{R}$, and note that the *finite* sets of co-initial steps constitute such a collection. The example in Fig. 1 is rather artificially infinite, but note that although the notion of multi-redex extends naturally (under some provisos) and are at the basis of *infinitary* confluence [27, Ch. 12], (FD) fails for them, a third indication the properties in [20] are too strong.

We generalise *redexes* to *treks* [20], employing $\mathfrak{t}, \mathfrak{s}, \ldots$ $(\mathfrak{T}, \mathfrak{S}, \ldots)$ to range over (sets of) them.

Definition 6 ([20])**.** *A* trek*-trace relation maps each step $\phi : a \to b$ to a relation $[\![\phi\rangle\!\rangle$ between the treks of a and b, where (*multi-*)*treks *of a are elements (subsets) of a set $T(a)$ quasi-ordered by $\le_a$ having the redexes of a as its minimal elements, and such that $\ge_a \cdot [\![\phi\rangle\!\rangle \subseteq [\![\phi\rangle\!\rangle \cdot \ge_b$.*

Intuitively, a trek is a representation of a reduction and $\le$ a *causal* order on the redexes contracted; the condition $\ge_a \cdot [\![\phi\rangle\!\rangle \subseteq [\![\phi\rangle\!\rangle \cdot \ge_b$ then captures that if a redex has a residual so do the redexes it causes. Accordingly, we restrict $\phi^{\mathfrak{T}}$ in $[\![\to\rangle\!\rangle$ (Def. 4) to steps ϕ in the $\le$-*downward closure* of $\mathfrak{T}$. After these changes and replacing *redex* by *trek* everywhere[7], *everything* above carries over verbatim, in particular Def. 4, Rem. 2, Lem. 2, Rem. 3, Def. 5, Rem. 4, Lem. 3, the main result Thm. 2, and their proofs, using the following remark in the proof of Lem. 2:

Remark 5. *The properties of $\le$ make $[\![\to\rangle\!\rangle$ a labelling of itself: if $\phi^{\mathfrak{T}}$ is a $[\![\to\rangle\!\rangle$-step and $\mathfrak{T} \le \mathfrak{T}'$, i.e. $\mathfrak{t} \le \mathfrak{T}'$ for all $\mathfrak{t} \in \mathfrak{T}'$, then $\phi^{\mathfrak{T}'}$ is a $[\![\to\rangle\!\rangle$-step by transitivity of $\le$ (if only $\mathfrak{T} \subseteq \mathfrak{T}'$ then transitivity of $\le$ is not needed) and $(\mathfrak{T}\,[\![\phi\rangle\!\rangle) \le (\mathfrak{T}'\,[\![\phi\rangle\!\rangle)$ by $\ge \cdot [\![\cdot\rangle\!\rangle \subseteq [\![\cdot\rangle\!\rangle \cdot \ge$ and $[\![\cdot\rangle\!\rangle$ being defined pointwise.*

[7] And also $\in$ into $\le$ when appropriate, and references to [20, Sect. 2] into corresponding ones to [20, Sect. 3].

Thus we have shown that the axiomatisation of [20] is sufficient but not necessary for obtaining a good residual theory. Although one often *may* factor residual theory through these axioms, there usually is no need to do so, and residual systems can be constructed directly and inductively [14, 27]. We conclude with two remarks on the (FD) axiom:

(FD) was not included among the axioms of residual systems [27] as we did not see a motivation for it. More generally, it is an open question whether finiteness or termination axioms have a place in analysing causality, cf. [28]. Of course, since they give rise to induction measures, they may be practically useful, and we are indeed happy to use them if and when available. For instance, in [22] we showed (FD) to be a consequence of *termination* of the so-called *substitution calculus* (SC) [25] underlying a rewrite format. But for infinitary rewrite systems termination of the SC and hence (FD) are surely too strong, despite that infinitary confluence of orthogonal systems is still based on causality/multi-redexes (up to some provisos).

(FD) may be hard to attain. The application of multi-treks to deal with Lévy's extraction theory for the $\lambda\beta$-calculus in [20, Section 6] is beautiful,[8] but in that application (FD) boils down [20, p. 46] to finiteness of family developments (FFD), cf. [23]. (FFD) is a key result in term rewriting at the basis of standardisation, (hyper-)normalisation of strategies, the theory of optimality, and more, but it also is subtle: It was formulated for the $\lambda\beta$-calculus by Lévy, forming the basis of his beautiful theory of optimality [17], but he resorted [8] to asking the Dutch, van Daalen (whose proof is employed in [17, Sect. II.1.5]) and de Vrijer [7, Stellingen], to prove it.[9] Melliès showed [19, Section 6.2.2] the result [16, Thm. 6.2.4] underlying the proof of (FFD) for Klop's combinatory reduction systems (CRSs) to be incorrect, leaving it and its consequences such as standardisation in limbo. We proved (FFD) for HRSs, hence CRSs, by adapting van Daalen's nifty proof [23], cf. [5].[10]

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[8] It seems worthwhile to adapt it to structured rewrite systems such as TRSs, HRSs, and GRSs.

[9] For first-order term rewrite systems (FFD) is due to Maranget [18]; then it is a simple consequence of RPO.

[10] In view of the subtleties it seems of interest to formalise a proof of (FFD) for HRSs in some proof assistant.

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A Proofs and remarks omitted from the main text

Proof of Prop. 1. Assume $\to$ is a residual system (on steps) for residuation / and unit 1. We show $\twoheadrightarrow$ is a residual system (on reductions) with residuation defined by $\Rightarrow$-normalisation, and unit 1.

To that end, we first show that for any pair (t, s) of co-initial formal compositions of composable $\to$-steps (denoted by *), the expressions /(t,s) and /(s,t) *can* be reduced to co-final such formal compositions t' and s' having as sources the targets of s respectively t', in the 2-TRS having rules (in term rewriting-tool format) for composites:

```
(VAR x y z)
(RULES
  /(*(x,y),z) -> *(/(x,z),/(y,/(z,x)))
  /(x,*(y,z)) -> /(/(x,y),z)
)
```

extended with rules of shape /(a,b) -> a' of the residual system on steps, one a' for each pair (a, b) of constants, i.e. pair of co-initial steps of the rewrite system $\to$. The proof is by induction on the sum of the left-sizes of t and s, where the left-sizes of an epxression f(t1,...,tn) is the sum of the sizes of the ti plus one, except that it is the left-size of t1 in case f is /.

- If neither t nor s contains a *, then they must both be constants (representing $\to$-steps), say a respectively b. Since they are co-initial by assumption, per construction there are rewrite rules /(a,b) -> a' and /(b,a) -> b' having the desired properties by the assumption that $\to$ constitutes a residual system;
- If one of t nor s contains a *, say w.l.o.g. t is *(t1,t2), then we may reduce /(*(t1,t2),s) to *(/(t1,s),/(t2,/(s,t1))) and /(s,*(t1,t2)) to /(/(s,t1),t2). By the induction hypothesis for the pair (t1, s), co-initial by construction, the expressions /(t1,s) and /(s,t1) can be reduced to co-final formal compositions t1' and s' such that their sources are the targets of s respectively t1. Correspondingly, *(/(t1,s),/(t2,/(s,t1))) reduces further to *(t1',/(t2,s')) and /(/(s,t1),t2) to /(s',t2). By inspecting the rules we see that the left-size of an expression does not increase by reduction (in fact, it stays the same for the rules for composition and

strictly decreases for the rules on constants). Therefore the induction hypothesis applies to the pair (`t2`,`s'`), co-initial since by the IH the source of `s'` is the target of `t1`, so the expressions `/(t2,s')` and `/(s',t2)` can be reduced to co-final formal compositions `t2'` and `s''` such that their sources are the targets of `s'` respectively `t2`. Correspondingly, `*(t1',/(t2,s'))` reduces further to `*(t1',t2')` and `/(s',t2)` to `s''`. From this we conclude, since `t1'`, `t2'`, and `s''` are formal compositions per construction and the constraints on the sources and targets hold per assumption and the IHs.

The above only shows that for co-initial `t` and `s` we *can* reduce `/(t,s)` and `/(s,t)` to co-final `t'` and `s'` such that their sources are the targets of `s` respectively `t`. But `t'` and `s'` are only appropriate *formal compositions*, they need not be *formal reductions*, i.e. elements of $\twoheadrightarrow$, yet. Moreover, them being defined by means of rewriting, they in principle need not be unique.

The former can be easily achieved by further reducing the formal compositions to normal form with respect to the monoid rules, turning any tree of formal compositions into a 'right-comb' without internal units:

```
(VAR x y z)
(RULES
  *(e,x) -> x
  *(x,e) -> x
  *(*(x,y),z) -> *(x,*(y,z))
)
```

The latter, uniqueness, follows by standard rewriting meta-theory, when combining the above two TRSs with the rules `/(e,x) -> e` and `/(x,e) -> x`. These two rules are admissable in the sense that for any formal composition `t` the rules of the first TRS may be used to reduce `/(e,t)` to `e` and `/(t,e)` to `t`, as one easily shows by induction on `t`. The combined TRS (having 7 rules) is terminating and confluent as current TRS tools show automatically (e.g., AProVE [9] and ACP [1]). (This is not immediate since the rules for composites have a critical pair: $(\phi{\cdot}\phi'{\cdot}\psi))/\chi \Leftarrow ((\phi{\cdot}\phi'){\cdot}\psi)/\chi \Rightarrow ((\phi{\cdot}\phi')/\chi){\cdot}(\psi/(\chi/(\phi{\cdot}\phi')))$. However it can be completed by $\Rightarrow$-reductions to $(\phi/\chi){\cdot}((\phi'/(\chi/\phi)){\cdot}(\psi/((\chi/\phi)/\phi')))$.) Our final 2-rewrite system $\Rightarrow$ is obtained by adjoining the rules `/(a,b) -> a'` for pairs of co-initial $\rightarrow$-steps. This is unproblematic, in the sense that confluence and termination are preserved. Termination is not affected as can be seen by replacing each constant with `e`, and confluence is not affected as these rules do not give rise to new critical peaks (stated differently, because of having adjoined the two admissable rules we need *not* adjoin rules for `/(e,a)` or `/(a,e)`.)

Finally, to show that reductions, i.e.$\twoheadrightarrow$-steps, constitute a residual system with composition it suffices (by completeness of $\Rightarrow$) to show that the left- and right-hand sides of all residual laws are $\Rightarrow$-convertible. Most laws are trivial, e.g. the composition laws are immediate since they are *instances* of $\Rightarrow$-rules. Only the cube law is not immediate, but it holds by induction on the sum of the left-sizes of the three expression involved, as was shown in the proof of [27, Lemma 8.7.47]. $\square$

Proof of Lem. 1. Since projection equivalence is a congruence for composition, to show that $\equiv \subseteq \simeq$ it suffices to show that the lhs and rhs of its generating relation are projection equivalent, which holds for any residual system with composition per [27, Ex. 8.7.40(iii)]. From this $\sqsubseteq \subseteq \lesssim$ follows since if $\gamma \sqsubseteq \delta$ then $\gamma \cdot \epsilon \equiv \delta$ for some ϵ, hence $\gamma \cdot \epsilon \simeq \delta$ so $\gamma/\delta = 1$ by [27, Ex. 8.7.40(vii)].

Vice versa, assuming $\gamma \simeq \delta$, we have $\gamma = \gamma \cdot (\delta/\gamma) \equiv \delta \cdot (\gamma/\delta) = \delta$ as desired, where the identities hold since the assumption entails both $\gamma/\delta = 1$ and $\delta/\gamma = 1$, and square homotopy follows from the claim that $\gamma \cdot (\delta/\gamma) \equiv \delta \cdot (\gamma/\delta)$ for all co-initial reductions γ, δ. Assuming $\gamma \lesssim \delta$, then $\gamma \cdot (\delta/\gamma) \equiv \delta \cdot (\gamma/\delta) = \delta$, hence $\gamma \sqsubseteq \delta$ as witnessed by δ/γ.

The proof of the claim is by induction on the sum of the lengths of γ and δ:

– If neither γ nor δ has length greater than 1, then we conclude by assumption/generation (in case both have length 1) or by reflexivity (otherwise, i.e. if one of them is empty);

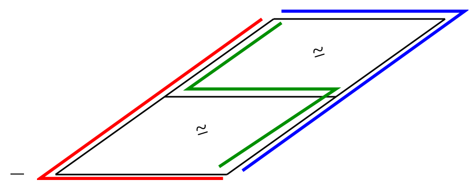

Figure 2: Square homotopy for compositions

If at least one of γ, δ is composite, say w.l.o.g. $\gamma = \gamma' \cdot \gamma''$, then $\gamma \cdot (\delta/\gamma) = (\gamma' \cdot \gamma'') \cdot (\delta/(\gamma' \cdot \gamma'')) = \gamma' \cdot (\gamma'' \cdot ((\delta/\gamma')/\gamma'')) \equiv \gamma' \cdot ((\delta/\gamma') \cdot (\gamma''/(\delta/\gamma'))) \equiv \delta \cdot ((\gamma'/\delta) \cdot (\gamma''/(\delta/\gamma'))) = \delta \cdot ((\gamma' \cdot \gamma'')/\delta) = \delta \cdot (\gamma/\delta)$,[11] where the two square homotopies hold by the IH, see Fig. 2, and the equalities follow from the identities holding in any residual system $\twoheadrightarrow$ as constructed per Prop. 1. $\square$

Remarks on residuation and tracing (Defs. 3 and 6). The idea to interpet each step as a *residual* relation, i.e. as a relation between the steps from its source and its target, is a special case of the idea of so-called *trace* or *descendant* relations, where a step is interpreted as a *transformation* between the *spaces* of its source and its target, see Fig.3. From this perspective, a *redex* is a step *from* an object reified into an element *in* the space of the object. Indeed, often

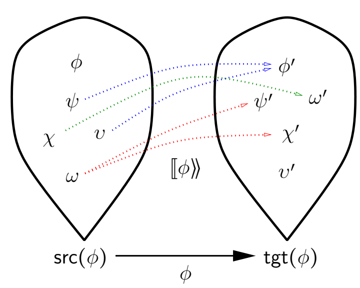

Figure 3: Tracing

in the literature residual relations are defined *via* tracing of symbols. E.g. Huet in [15] defines a residual relation for β-redexes by tracing their *application* symbol and contrasts that with literature where that is done by tracing their *abstraction* symbol, more generally in [22] it is shown that in orthogonal first- and higher-order term rewriting systems the residual relation can be defined by tracing *head* symbols of redexes. In a slogan (sometimes definition): residuals are descendants of redexes that are redexes again.

This need not be restricted to orthogonal rewrite systems, though then one typically needs to trace notions other than mere steps, e.g. when the rewrite system *refines* an orthogonal system.

[11] This proof augments the usual proof that join is commutative up to projection equivalence, i.e. it augments the usual proof of confluence by commutation with a property stating that the two legs of a the constructed confluence diagram are homotopic, cf. [24, Thm. 2].

For instance, in λ-calculi with explicit substitutions the residual of a Beta-redex is typically *not* a redex after distributing an explicit substitution just over its application-node; it only *becomes* one again after distributing the explicit substitution further also over the abstraction-node. In such cases we would like to say that the final Beta-redex is a residual of the one before the two explicit substitution steps, via some *substrate* in the intermediate term. To us this indicates that an approach based exclusively on tracing steps or even reductions will not do, for lack of a suitable (Beta-)redex at the intermediate stage. Tracing other objects or properties may work however; in this case, tracing the application- and abstraction-symbols separately. For a basic example illustrating the idea of tracing *via* symbols, consider the associativity rule. Since it is self-overlapping it is non-orthogonal. Still, as displayed in Fig. 4, the *parallel* associativity rule

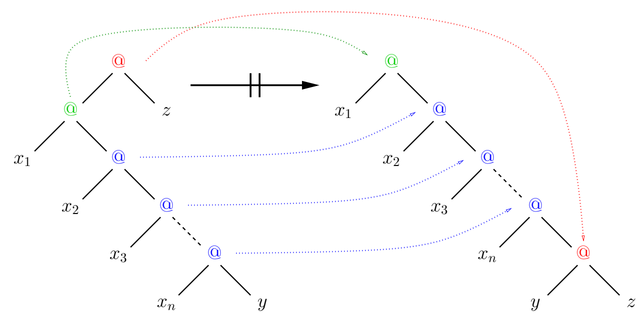

Figure 4: Parallel associativity step with tracing

$$x_1(x_2(\dots(x_n y)\dots))z \twoheadrightarrow\!\!\!\!\!\!\shortmid\;\; x_1(x_2(\dots(x_n(yz))\dots))$$

has the diamond property and to show this parallel associativity steps may by traced via pairs $(p, p12^n)$ of (positions of) their *head* and *tail* symbols as indicated, as one easily checks.

Tracing has been employed throughout the rewriting literature to trace properties along reductions. For instance, they are used by Klop as a fundamental tool throughout his work [16, 3]. Melliès initially insisted on only allowing to trace *steps* [19], but then switched to tracing *reductions* in [20]. For an algebraic approach to defining trace relations from proof terms (a term representation of steps) for term rewriting, see [27, Chapter 8]. $\square$

Proof of claim in Rem. 2. Let $\mathbb{B}$ relate every object a to (a, Φ), and every step $\phi : a \to b$ to the step (a, Φ) from (a, Φ) to $(b, (\Phi\, [\![\phi\rangle\!\rangle))$, for each subset Φ of the redexes in a. $\mathbb{B}$ is a *bisimulation*:

- We first verify the (*back*) and (*forth*) conditions [27, Def. 8.4.5], i.e. that $\mathbb{B} \cdot [\![\to\rangle\!\rangle \subseteq \to \cdot \mathbb{B}$ and $\leftarrow \cdot \mathbb{B} \subseteq \mathbb{B} \cdot \langle\!\langle\leftarrow]\!]$ hold. To show (back) note that per construction of $[\![\to\rangle\!\rangle$ if $a\ \mathbb{B}\ a'$ then a' has shape a^{Φ}, and each step from a^{Φ} has shape ϕ^{Φ} with target $b^{(\Phi[\![\phi\rangle\!\rangle)}$, for some $\phi : a \to b$. By definition of $\mathbb{B}$, then $\phi\ \mathbb{B}\ (\phi, \Phi)$ and $b\ \mathbb{B}\ b^{(\Phi[\![\phi\rangle\!\rangle)}$. The (forth) condition holds by analogous reasoning.
- To show (*relator*), i.e. bisimulation commutes with taking sources/targets of steps, note per construction if $\phi\ \mathbb{B}\ (\phi, \Phi)$, then $\mathsf{src}(\phi)\ \mathbb{B}\ (\mathsf{src}(\phi), \Phi)$ and $\mathsf{tgt}(\phi)\ \mathbb{B}\ (\mathsf{tgt}(\phi), (\Phi\, [\![\phi\rangle\!\rangle))$.

That $\mathbb{B}$ not only is a bisimulation but gives rise to a rewrite labelling is checked by:

- Per definition there is a *unique* object $\mathbb{B}$-related to any (a, Φ), namely a, and also per construction given $\phi : a \to b$ and $a\ \mathbb{B}\ (a, \Phi)$ there is a *unique* step ϕ' such that $\phi\ \mathbb{B}\ \phi'$, namely (ϕ, Φ) from (a, Φ) to $(b, (\Phi\, [\![\phi\rangle\!\rangle))$.

– We may take as *initial* labelling of an object a the pair $(a, \varnothing)$. $\square$

Remark on multi-redexes in Lem. 2. The intuition for multi-redexes is that they consist of sets of co-initial steps that could be performed simultaneously, but are performed sequentially. Trace relations serve to mediate between both by allowing to express that co-initial steps leave a trace after each other. For a multi-redex, this give rise to its set of developments [6] corresponding intuitively to all its sequentialisations. Vice versa, multi-redexes can be thought of as developments up to permutation. In light of the failure of the cube property for developments, due to the latter being too concrete (cf. [27, Fig. 8.53]), we have mostly abandoned working with developments[12] in favour of their more intrinsic (inductive) representation as parallel steps $\mathrel{+\!\!\!\!\twoheadrightarrow}$ or multi-steps $\mathrel{\circ\!\!\!\!\twoheadrightarrow}$ (cf. [27, Ch. 8]). $\square$

Proof of Lem. 2. We first show that the structure $\langle \mathrel{\circ\!\!\!\!\twoheadrightarrow}, 1, / \rangle$ is well-defined. To that end, we show that every multi-redex a^Φ of $\mathrel{\circ\!\!\!\!\twoheadrightarrow}$ is parametric complete in $[\![\to\rangle\!\rangle$, that is, it is both *complete*, i.e. *terminating* and *confluent*, and *parametric*, i.e. all reductions to normal form induce the same residual relation (cf. [22, Def. 2.4.9 and Prop. 2.4.16]). This yields well-definedness in that completeness yields that the target of the $\mathrel{\circ\!\!\!\!\twoheadrightarrow}$-step a^Φ exists uniquely, and parametricity that $(\Psi\ [\![\Phi\rangle\!\rangle)$ is independent of the development of Φ chosen, for any Ψ. That justifies identifying a multi-redex a^Φ with Φ as the latter uniquely determines both the source a, as the source of all steps in Φ,[13] and the target.

That $[\![\to\rangle\!\rangle$ is terminating follows immediately from (FD) since we have (Rem. 2) a bisimulation between a^Φ-reductions in $[\![\to\rangle\!\rangle$ and developments of Φ from a in $\to$ [27, Exercise 8.4.11]. By Newman's Lem. and by parametricity being a closed property in the sense of [24, Thm. 2], i.e. holding for trivial diagrams and being preserved by diagram composition, to show parametric completeness it suffices to show parametric *local* confluence of $[\![\to\rangle\!\rangle$. Consider a local peak from a^X comprising $[\![\to\rangle\!\rangle$-steps ϕ' and ψ'. Per construction of $[\![\to\rangle\!\rangle$, then $\phi' = \phi^X$ with target $b^{(X [\![\phi\rangle\!\rangle)}$ and $\psi' = \psi^X$ with target $c^{(X [\![\psi\rangle\!\rangle)}$ for some $\phi : a \to b$ and $\psi : a \to c$ with $\phi, \psi \in X$. By (PERM) for the peak ϕ, ψ there exists a valley γ, δ of complete developments of $(\psi\ [\![\phi\rangle\!\rangle), (\phi\ [\![\psi\rangle\!\rangle)$ such that $[\![\phi \cdot \gamma\rangle\!\rangle = [\![\psi \cdot \delta\rangle\!\rangle$. Per definition of complete developments, the valley lifts to a pair of $[\![\to\rangle\!\rangle$-reductions $\gamma^{(\psi [\![\phi\rangle\!\rangle)}, \delta^{(\phi [\![\psi\rangle\!\rangle)}$ that end in some $d^\varnothing$ (for treks this also uses Rem. 5). By residuation being defined pointwise and since $\phi, \psi \in X$, the valley lifts also to a pair of $[\![\to\rangle\!\rangle$-reductions $\gamma^{(X [\![\phi\rangle\!\rangle)}, \delta^{(X [\![\psi\rangle\!\rangle)}$, which constitute a $[\![\to\rangle\!\rangle$-valley as they must have the same target $d^{((X [\![\phi\rangle\!\rangle) [\![\gamma\rangle\!\rangle)} = d^{((X [\![\psi\rangle\!\rangle) [\![\delta\rangle\!\rangle)}$ by parametricity $[\![\phi \cdot \gamma\rangle\!\rangle = [\![\psi \cdot \delta\rangle\!\rangle$ and $[\![\cdot\rangle\!\rangle$ being homomorphic.

Apart from entailing well-definedness of $\langle \mathrel{\circ\!\!\!\!\twoheadrightarrow}, 1, / \rangle$, also that that structure is a residual system (Def. 1) is a consequence of parametric completeness.

– We first show that $\mathrel{\circ\!\!\!\!\twoheadrightarrow}$ has the diamond property, with the join/diagonal of two multi-redexes given by taking their union, for residual function $/$. To that end, consider a peak Φ, Ψ of multi-redexes from a. We have $\Phi/\Psi := (\Phi\ [\![\Psi\rangle\!\rangle)$ and $\Psi/\Phi := (\Psi\ [\![\Phi\rangle\!\rangle)$. Both are seen to be complete developments by noting that they are equal (as sets) to $((\Phi \cup \Psi)\ [\![\Psi\rangle\!\rangle)$ respectively $((\Phi \cup \Psi)\ [\![\Phi\rangle\!\rangle)$, so by completeness of $[\![\to\rangle\!\rangle$ have some normal form d^X as common reduct, namely the normal form of $\Phi \cup \Psi$. To see $X = \varnothing$, suppose $\phi\ [\![\Phi \cup \Psi\rangle\!\rangle\ \chi$ for some $\phi \in \Phi \cup \Psi$, say $\phi \in \Phi$, then $\phi\ [\![\Phi\rangle\!\rangle\ \psi\ [\![(\Psi\ [\![\Phi\rangle\!\rangle)\rangle\!\rangle\ \chi$ for some ψ contradicting that $\Phi\ [\![\Phi\rangle\!\rangle$ is empty, and symmetrically for $\phi \in \Psi$. The above shows already that $(\Phi\ [\![\Psi\rangle\!\rangle)$, $(\Psi\ [\![\Phi\rangle\!\rangle)$, and the join/diagonal $\Phi \cup \Psi$ have the same target.

  The source of $(\Phi\ [\![\Psi\rangle\!\rangle)$ and the target of Ψ both are a target of a complete development of Ψ from a, which exists uniquely by completeness, so they must coincide. $\Phi \cup \Psi$ is a multi-redex from a and its target is the target of a complete development of it.

[12] Only reinforced by our qualms about (FD).

[13] Except in case Φ is empty when we should be explicit but will be sloppy.

- the first law (1) of residual system holds by $\Phi/\varnothing = (\Phi\ [\![\varnothing\rangle\!\rangle) = \Phi$;
- law (2) holds by $\Phi/\Phi = (\Phi\ [\![\Phi\rangle\!\rangle) = \varnothing$;
- the proof the cube law (3) holds, follows the argumentation for the diamond property:

$$(\Phi/\Psi)/(X/\Psi) = ((\Phi\ [\![\Psi\rangle\!\rangle)\ [\![X/\Psi\rangle\!\rangle) = (\Phi\ [\![\Psi\cdot(X/\Psi)\rangle\!\rangle) = (\Phi\ [\![\Psi\cup X\rangle\!\rangle) =$$

$$(\Phi\ [\![X\cup\Psi\rangle\!\rangle) = (\Phi\ [\![X\cdot(\Psi/X)\rangle\!\rangle) = ((\Phi\ [\![X\rangle\!\rangle)\ [\![\Psi/X\rangle\!\rangle) = (\Phi/X)/(\Psi/X)$$ □

Remark on parametricity. The assumption that complete developments should be parametric, i.e. induce the same trace relation, originates in the setting of abstract rewriting with Axiom 0 of [11]. Its reformulation as parametric completeness for the associated labelled rewrite system as in Lem. 2 originates with [22, Def. 2.4.9].

Being parametric does not entail that *every* confluence diagram is parametric. For instance, in the $\lambda\beta$-calculus the local peak $Ix \leftarrow I(Ix) \to Ix$ is *accidentally* a confluence diagram; it is not parametric as its two legs trace different redexes in its source to the one in the target. However, the completion of the peak by the valley $Ix \to x \leftarrow Ix$ *is* parametric. □

A partial inverse holds to the construction in the proof of Lem. 2 as shown in the proposition below: if $[\![\to\rangle\!\rangle$ is parametric complete, then properties (FD) and (PERM) hold, but (PERM) only under the assumption that also (SD) holds. To see that is needed, consider steps $\phi,\psi{:}a\to b$ and $\phi',\psi'{:}b\to c$ with only non-empty residuals $\phi\ [\![\phi\rangle\!\rangle\ \phi'$ and $\psi\ [\![\psi\rangle\!\rangle\ \psi'$ (cf. Rem. 4).[14] Then $[\![\to\rangle\!\rangle$ is parametric complete and properties (SD), (F) and (FD) hold but (PERM) fails: the peak ϕ,ψ is completed by the empty valley developing $\varnothing,\varnothing$ but $(\phi\ [\![\phi\rangle\!\rangle) = \phi' \neq \varnothing = (\phi\ [\![\psi\rangle\!\rangle)$. For this example, $\to$ and $\multimap\!\!\to$ do not present the same quasi-order as $a \to b$ but not $a \multimap\!\!\to b$, although we do have $a \multimap\!\!\to c \leftarrow\!\!\multimap b$.

Proposition 2. *If $[\![\to\rangle\!\rangle$ is parametric complete, then properties (FD) and (PERM) hold, assuming for the latter that property (SD) holds.*

Proof. As was already noted (FD) corresponds to termination of $[\![\to\rangle\!\rangle$.

To show (PERM) under the assumption $[\![\to\rangle\!\rangle$ is parametric complete, consider a local $\to$-peak $\phi : a \to b$ and $\psi : a \to c$. The peak lifts to a $[\![\to\rangle\!\rangle$-peak $\phi^{\{\phi,\psi\}},\psi^{\{\phi,\psi\}}$ having targets $b^{(\{\phi,\psi\}[\![\phi\rangle\!\rangle)} = b^{(\psi[\![\phi\rangle\!\rangle)}$ and $c^{(\{\phi,\psi\}[\![\psi\rangle\!\rangle)} = c^{(\phi[\![\psi\rangle\!\rangle)}$ by (SD) and definition of the residual relation. By completeness of $[\![\to\rangle\!\rangle$ this peak can be completed by a $[\![\to\rangle\!\rangle$-valley $\gamma^{(\psi[\![\phi\rangle\!\rangle)},\delta^{(\phi[\![\psi\rangle\!\rangle)}$ comprising reductions to normal form. Since $[\![\to\rangle\!\rangle$-normal forms have the empty set as label, γ,δ is a valley of complete developments of $(\psi\ [\![\phi\rangle\!\rangle),(\phi\ [\![\psi\rangle\!\rangle)$ and $[\![\phi\cdot\gamma\rangle\!\rangle = [\![\psi\cdot\delta\rangle\!\rangle$ by parametricity. □

Proof of same equivalence closure in Rem. 4. We show that $\to$ and $\multimap\!\!\to$ induce the same equivalence relation assuming (FD). Suppose $\phi : a \to b$ and consider a reduction $\gamma^{\{\phi\}}$ from $a^{\{\phi\}}$ to normal form starting with the step $\phi^{\{\phi\}}$ to $b^{(\phi[\![\phi\rangle\!\rangle)}$. It exists by (FD), and its tail $\delta^{(\phi[\![\phi\rangle\!\rangle)}$ is a reduction to the same normal form. Thus γ,δ is a local valley of $\multimap\!\!\to$-steps having a,b as source, i.e. a two-step $\multimap\!\!\to$-conversion between a and b. □

Remark on not assuming (SD). The absence of (SD) entails further consequences:

Even if a $\to$-reduction arises from completely developing the multi-redexes in an $\multimap\!\!\to$-reduction, local homotopy might not preserve that. To see this consider a peak ϕ,ψ such that the valley ψ',ϕ' witnesses (PERM) for it, and let $\chi\ [\![\chi\rangle\!\rangle\ \phi$, then $\chi\cdot\phi\cdot\psi'$ is a complete development, but $\chi\cdot\psi\cdot\phi'$ is not despite being locally homotopic.

[14] Note the system does not satisfy Axiom 4 of [26] and the acyclicity axiom of [10].

To check whether a given $\to$-reduction is a complete development one may proceed greedily: If γ is empty, then answer yes. Otherwise, search for a maximal set Φ of steps from the source such that Φ has a complete development that is a (non-empty) prefix of γ. If that exists, repeat the procedure on the suffix. If no such exists, then answer no. That this procedure is correct, follows from that the tail of any complete development is a complete development again. $\square$

Proof of Lem. 3. Both directions, namely that if $\gamma \equiv_l \delta$ for finite $\to$-reductions γ, δ, then $\overline{\gamma} \equiv \overline{\delta}$, and that if $\gamma \equiv \delta$ for finite $\mathrel{-\!\!\circ\!\!\!\rightarrow}$-reductions γ, δ, then $\overline{\gamma} \equiv \overline{\delta}$, are shown by induction on the generation of $\equiv_l$ respectively $\equiv$. Both will be based on the claim that all complete developments of a multi-redex $X : a \mathrel{-\!\!\circ\!\!\!\rightarrow} b$ are $\equiv_l$-equivalent, and that their embeddings are $\equiv$-equivalent to X. We first prove this claim, by well-founded induction on X ordered by $\llbracket\to\rangle\!\rangle$, which is well-founded by (FD).

If X is empty, the claim is trivial since there is only one complete development of it, namely the empty reduction. Otherwise, a complete development of X has shape $\phi \cdot \gamma$ for some $\phi \in X$, $\phi : a \to a'$ completely developing $\{\phi\}$, and complete development $\gamma : a' \twoheadrightarrow b$ of $X/\{\phi\}$.

- The second part of the claim then follows by $X = X \cdot \varnothing = X \cdot (\{\phi\}/X) \equiv \{\phi\} \cdot (X/\{\phi\}) \equiv \{\phi\} \cdot \overline{\gamma} = \overline{\phi \cdot \gamma}$ using that $X/\{\phi\} \equiv \overline{\gamma}$ by the (second part of the) IH for the multi-redex $X/\{\phi\}$ and its complete development γ.

- For the first part of the claim, suppose to have another complete development of X having shape $\psi \cdot \delta$ for some $\psi \in X$, $\psi : a \to a''$ completely developing $\{\psi\}$, and complete development $\delta : a'' \twoheadrightarrow b$ of $X/\{\psi\}$. By (PERM), the peak ϕ, ψ can be completed by a valley γ', δ' of complete developments of $(\psi \llbracket \phi \rangle\!\rangle), (\phi \llbracket \psi \rangle\!\rangle)$ such that $\llbracket \phi \cdot \gamma' \rangle\!\rangle = \llbracket \psi \cdot \delta' \rangle\!\rangle$. Letting ϵ be a complete development of $(X \llbracket \{\phi, \psi\} \rangle\!\rangle)$, we conclude by $\phi \cdot \gamma \equiv_l \phi \cdot \gamma' \cdot \epsilon \equiv_l \psi \cdot \delta' \cdot \epsilon \equiv_l \psi \cdot \delta$ where the first and third equivalences hold by the IH for $X/\{\phi\}$ and $X/\{\psi\}$ respectively, and the second equivalence by definition of local homotopy.

The base case for $\gamma \equiv_l \delta$ is (PERM), i.e. meaning that both γ and δ are complete developments of some multi-redex $\{\phi, \psi\}$. Then we conclude to $\overline{\gamma} \equiv \{\phi, \psi\} \equiv \overline{\delta}$ by the claim. In the generation case, concluding from $\gamma \equiv_l \gamma'$ that $\delta \cdot \gamma \cdot \epsilon \equiv_l \delta \cdot \gamma' \cdot \epsilon$, we have $\overline{\gamma} \equiv \overline{\delta}$ by the IH hence $\overline{\delta \cdot \gamma \cdot \epsilon} = \overline{\delta} \cdot \overline{\gamma} \cdot \overline{\epsilon} \equiv \overline{\delta} \cdot \overline{\gamma'} \cdot \overline{\epsilon} = \overline{\delta \cdot \gamma' \cdot \epsilon}$ by the embedding being stepwise and by generation for $\equiv$ using that by (SD) single $\to$-steps embed as (singleton) multi-redexes of $\mathrel{-\!\!\circ\!\!\!\rightarrow}$.

The base case for $\gamma \equiv \delta$ is that $\gamma = \Phi \cdot (\Psi/\Phi)$ and $\delta = \Psi \cdot (\Phi/\Psi)$ for some multi-redexes Φ, Ψ. By the claim $\underline{\gamma} = \underline{\Phi \cdot (\Psi/\Phi)} = \underline{\Phi} \cdot \underline{(\Psi/\Phi)} \equiv_l \underline{\Phi \cup \Psi} \equiv_l \underline{\Psi} \cdot \underline{(\Phi/\Psi)} = \underline{\Psi \cdot (\Phi/\Psi)} = \underline{\delta}$ since each of $\underline{\gamma}$, $\underline{\delta}$, and $\underline{\Phi \cup \Psi}$ is a complete development of $\Phi \cup \Psi$. In the generation case, concluding from $\gamma \equiv \gamma'$ that $\delta \cdot \gamma \cdot \epsilon \equiv \delta \cdot \gamma' \cdot \epsilon$, we have $\underline{\gamma} \equiv_l \underline{\delta}$ by the IH hence $\underline{\delta \cdot \gamma \cdot \epsilon} = \underline{\delta} \cdot \underline{\gamma} \cdot \underline{\epsilon} \equiv_l \underline{\delta} \cdot \underline{\gamma'} \cdot \underline{\epsilon} = \underline{\delta \cdot \gamma' \cdot \epsilon}$ by the embedding being multi-redex-wise and by generation for $\equiv_l$ using the assumption that each multi-redex is embedded as an arbitrary but fixed $\to$-reduction.

That $\sqsubseteq = \sqsubseteq_l$ in both directions as well, follows from the above. For instance, if $\gamma \sqsubseteq_l \delta$ then by definition $\gamma \cdot \epsilon \equiv_l \delta$ for some ϵ, hence by the above and embedding acting stepwise $\overline{\gamma} \cdot \overline{\epsilon} = \overline{\gamma \cdot \epsilon} \equiv \overline{\delta}$ showing $\overline{\gamma} \sqsubseteq \overline{\delta}$. The other direction is analogous. $\square$

Remarks on the orders on treks. In [20, Def. 6] it is stipulated that $\leq_a$ *partially orders the set of treks, whose minimal elements are the redexes in* a. The stipulation leaves it unclear whether or not there is a minimal element below *every* element. Although we have phrased our definitions and results (in particular on parametric completeness) so as not to depend on that, note that if it needs not be the case, then not every development of a set can be completed into a complete development. For instance, take a single object a with treks the integers ordered by less-than-or-equal. This rewrite system satisfies all properties, but the only development of $\{0\}$ from a

is the empty reduction, which is not complete as the label is non-empty; the conditions on the trek-trace relation are not violated simply because there is *no* minimal integer.

Since we did not use anti-symmetry we require the $\leq_a$ only to be quasi-orders, not necessarily partial orders. $\square$