Some symmetries of commutation diamonds

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Abstract

We study commutation of rewrite systems under swapping, reversing.

1 Introduction

Commutation may be used as a building block in various settings ranging from the abstract to the concrete. For instance, causality may be analysed as the independence of events which may be modeled as their commutation, and correctness of compiler optimisations may be modeled as commutation between evaluation and optimisation. Commutation generalises confluence, which itself may be used for establishing consistency of and deciding equational theories.

In this short paper we investigate commutation under *swapping* and *reversing* of the rewrite systems. We show that at the level of abstract rewriting this perspective, although bringing nothing new *per se*, may serve to bring unity into disparate results in the literature. At the level of term rewriting some fun is to be had, albeit simple, because in the commutation-by-critical-pair-analysis (due to Knuth and Bendix and Huet for first-order term rewriting, and extended to higher-order term rewriting by Nipkow), rules are commonly assumed to conform to certain restrictions, restrictions that may not be preserved when reversing them.

Example 1. *Reversing the term rewrite rules:*

$$f(x) \to x \qquad f(x) \to a \qquad f(x) \to g(x,x) \qquad (\lambda x.F(x))G \to F(G)$$

violates the respective requirements that: the lhs should not be a variable (x is),[1] *variables in the rhs should occur in the lhs (x does not occur in a), the lhs should be linear (x occurs twice in $g(x,x)$),*[2] *the lhs should be a pattern (in the sense of Miller; $F(G)$ is not a pattern).*

Instead of a conclusion we have **positions** after each subtopic.[3]

2 Diamond symmetries in abstract rewriting

We assume knowlege of abstract and term rewriting [7]. We employ arrow-like notations $\blacktriangleright$, $\triangleright$, ... for abstract rewrite systems or relations on a (the same) set of objects, reversing these notations $\blacktriangleleft$, $\triangleleft$, ... to denote the reverse systems,and their repetitions $\blacktriangleright\!\!\blacktriangleright$, $\triangleright\!\!\triangleright$, ... to denote the reflexive–transitive closures. We define $\blacktriangleleft\!\triangleright := \blacktriangleleft \cup \triangleright$ and $\to := \blacktriangleright \cup \triangleright$. As in [4], we extend terminology and notation for a single rewrite system $\blacktriangleright$ to the *diagonal* $\blacktriangleright,\blacktriangleright$, i.e. such that $P(\blacktriangleright)$ iff $P(\blacktriangleright,\blacktriangleright)$ for P a property of (pairs of) rewrite systems.

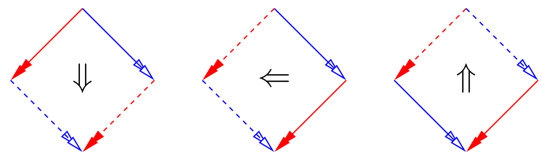
Definition 1. *The pair $\blacktriangleright,\triangleright$ has the* diamond *property if $\blacktriangleleft \cdot \triangleright \subseteq \triangleright \cdot \blacktriangleleft$, it* commutes *if $\blacktriangleright\!\!\blacktriangleright,\triangleright\!\!\triangleright$ has the diamond property, and has the* Church–Rosser *property if $(\blacktriangleleft \cup \triangleright)^* \subseteq \triangleright\!\!\triangleright \cdot \blacktriangleleft\!\!\blacktriangleleft$.*

[1] Variable-lhss are allowed by Huet in his Critical Pair Lemma, but not in Nipkow's higher-order version.

[2] Although non-left-linear rules are usually allowed in term rewriting, they do not go well with commutation: Hirokawa and Shintani show in IWC 2015, for them commutation is not even preserved by signature extension.

[3] For lack of space we often only provide hyperlinks for informal references to the literature.

We may refer to the diagonal of P as *self-*P. For instance, self-commutation is conventionally known as *confluence*. The name of the diamond property derives from that it can be visualised like that. The 8 symmetries of such a diamond (the dihedral group D_4) give rise to 8 instances of the diamond property, obtained by *swapping* and *reversing* the rewrite systems $\blacktriangleright, \triangleright$, and hence also of commutation. We have visualised[4] three of these, together with their respective (conventional) names:[5]



commutation factorisation upward commutation

Formally, letting P range over the properties in Definition 1, left–right *reflecting* the diamond corresponds to *swapping* the pair $\blacktriangleright, \triangleright$ into $\triangleright, \blacktriangleright$ and is a symmetry: $P(\blacktriangleright, \triangleright)$ iff $P(\triangleright, \blacktriangleright)$.[6] Clockwise *rotating* the diamond corresponds to first *swapping* the pair $\blacktriangleright, \triangleright$ and then *reversing* the first element yielding $\triangleleft, \blacktriangleright$. The first displayed rotation turns commutation into *factorisation*, $\triangleright\!\triangleright \cdot \blacktriangleright\!\blacktriangleright \subseteq \blacktriangleright\!\blacktriangleright \cdot \triangleright\!\triangleright$, which is equivalent to the Church–Rosser property of $\triangleleft, \blacktriangleright$, i.e. that a reduction $a \twoheadrightarrow b$ *factors* as $a \blacktriangleright\!\blacktriangleright \cdot \triangleright\!\triangleright\, b$. Symmetries preserve results. E.g. that the diamond property implies the Church–Rosser property is the same result as that $\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleright$ implies factorisation.

Position 1. *Results for diamond and commutation are to be taken up to symmetries.*

How to show the Church–Rosser property of $\blacktriangleright, \triangleright$? Based on a note from 1978 by De Bruijn, we introduced the *decreasing diagrams* (DD) technique in my PhD thesis, requiring for each *local*[7] *peak* $a \blacktriangleleft \cdot \triangleright b$ existence of a suitably constrained *valley* $a \triangleright\!\triangleright \cdot \blacktriangleleft\!\blacktriangleleft b$, a result we extended in 2008 to allow *conversions* instead of valleys:

Definition 2. $\blacktriangleright, \triangleright$ *is* decreasing*, if* $\blacktriangleright := \bigcup_{i \in I} \blacktriangleright_i$, $\triangleright := \bigcup_{j \in J} \triangleright_j$ *for families* $(\blacktriangleright_i)_{i \in I}, (\triangleright_j)_{j \in J}$ *and some well-founded strict order* $<$ *on* $I \cup J$*, such that for all* $i \in I, j \in J$ ${}_i\blacktriangleleft \cdot \triangleright_j \subseteq \blacktriangleleft\!\triangleright^*_{\curlyvee i} \cdot \triangleright^=_j \cdot \blacktriangleleft\!\triangleright^*_{\curlyvee\{i,j\}} \cdot {}_i\blacktriangleleft^= \cdot \blacktriangleleft\!\triangleright^*_{\curlyvee j}$*, where* $\curlyvee K := \{k \in I \cup J \mid \exists \ell \in K\ \ell > k\}$ *and* $\curlyvee k := \curlyvee\{k\}$*.*

Theorem 1 (Decreasing Diagrams, DD [5])**.** $\blacktriangleright, \triangleright$ *commute if decreasing.*

Since then, DD has found wide application in the literature in the study of confluence and commutation (see below), but somewhat surprisingly (given their symmetry), as far as we know, not yet within the study of factorisation. Here and in Section 3 we give examples[8] illustrating the power of DD also for establishing factorisation results, and at the same time how DD allows one to focus on extracting appropriate families and orders on them, easing applicability.

Example 2. $\blacktriangleright := \bigcup_i \blacktriangleright_{i \in I}$ *and* $\triangleright := \bigcup_{j \in I} \triangleright_j$ *commute if for all* i, j*,* $\triangleright_j$ *is terminating and*

$$ {}_i\blacktriangleleft \cdot \triangleright_j \subseteq \triangleright_j \cdot ({}_i\blacktriangleleft \cup \triangleright_i)^* \tag{1}$$

To see this, first note it suffices to show $\blacktriangleright_i, \triangleright$ *commute for all* i*. Fixing* i *allows to turn (1) into a DD by decomposing* $\triangleright$ *into* $\triangleright_i$ *and* $\triangleright_{\neg i} := \bigcup_{k \neq i} \triangleright_k$*, ordering* $\triangleright_{\neg i}$*-steps above others, and ordering* $\blacktriangleright_i$*- and* $\triangleright_i$*-steps via their targets by* $(\blacktriangleright\!\blacktriangleright_i \cdot {}_i\triangleleft \cdot \blacktriangleright\!\blacktriangleright_i)^+$*, well-founded by (1) [7, Exc. 1.3.19].*

[4] As usual, ordinary/dashed arrows represent universally/existentially quantified steps and reductions.

[5] Other names of factorisation are *postponement* (from a $\triangleright$-perspective) or *preponement* ($\blacktriangleright$-perspective).

[6] This is apparent via left–right reflection of our formal notations, as these mirror those of the diamond.

[7] Following Newman's 1942 in *localising* properties P, we use *local* P to refer to P with its assumption restricted to single steps ($\blacktriangleright, \triangleright$) instead of general reductions ($\blacktriangleright\!\blacktriangleright, \triangleright\!\triangleright$). E.g., *local* commutation is $\blacktriangleleft \cdot \triangleright \subseteq \triangleright\!\triangleright \cdot \blacktriangleleft\!\blacktriangleleft$.

[8] Found by us over the past 20 years, but only privately communicated and circulated.

That families $\to_\bullet := (\rightsquigarrow_\bullet, \rightarrowtail_\bullet)$ *and* $\to_\circ := (\rightsquigarrow_\circ, \rightarrowtail_\circ)$ *satisfy the conditions of a* square factorisation system *[1], directly entails* $\rightsquigarrow_\circ, \rightarrowtail_\circ$ *are terminating and (1) holds for*[9] ${}_\bullet\!\leftarrow, \to_\circ$, *yielding the main abstract factorisation result of that paper [1, Thm. 5.2].*

Note that using DD not only allowed our statement and proof to be (much) more compact,[10] but also our result to be (much) more general, not just because of allowing arbitrary size families; already for families of size 2 it is more general: e.g. where square factorisation systems require $\rightsquigarrow_\bullet \cdot \rightsquigarrow_\circ$ to be contained in $\rightsquigarrow_\circ^+ \cdot \rightsquigarrow_\bullet^+$, (1) only requires it to be contained in $\rightsquigarrow_\circ \cdot (\rightsquigarrow_\bullet \cup \rightsquigarrow_\circ)^+$ and similarly for $\rightarrowtail$. In [5] we showed that in fact *all* 'local commutation $\Rightarrow$ commutation' results we knew of then, could be obtained as instances of DD. We did so by introducing various basic techniques for finding families and orders such as *self*- and *rule*-labelling. For instance, the Lemma of Hindley–Rosen was obtained by taking for $>$ the *empty* order on families [5, Example 13], and the commutation version (due to Backhouse and Doornbos) of Newman's Lemma was obtained by *self*-labelling a step $a \to b$ by its *source* and ordering labels by $\leftarrow^+$ [5, Example 12].[11] DD has been formalised in Isabelle and is part of the AFP, due to Zankl for Theorem 1 and to Felgenhauer for a *proof order* version. DD has been automated in tools such as ACP and CSI, and many commutation results have been *factored through* DD and often generalised by it, e.g. Toyama's famous modularity of confluence result; see the introduction of [2] for more examples. Still we think that (much) more leverage could be gotten out of DD.

Remark 1. *A main feature of the TRS confluence tool CoLL [6] is that it is not built on* confluence *but on* commutation *criteria. The idea is to (rule-based) decompose a TRS* $\mathcal{R}$ *into a family* $(\mathcal{R}_i)_i$ *of TRSs. Confluence of* $\mathcal{R}$ *follows by the Lemma of Hindley–Rosen from commutation of all pairs (including the diagonal)* $\mathcal{R}_i, \mathcal{R}_j$ *of family members. We suggest also there employing DD, instead of the lemma of Hindley–Rosen, could be beneficial. In fact, an example of such a decomposition based on DD was already given as [5, Thm. 5]. Whether/how such decompositions could be found automatically remains to be investigated. Moreover, it seems interesting to consider decompositions other than rule-based ones, e.g. ones obtained by instantiation or by strategies.*

Position 2. *DD is the swiss-army-knife for commutation, its symmetries (factorisation), and its instances (confluence). Trying to extract appropriate families and orders on them for establishing these properties via DD, results often in (more) powerful yet (more) compact results.*

How powerful is the DD technique exactly, for showing confluence and commutation?

Theorem 2.[12] *The DD technique is* complete *for confluence of rewrite systems. More precisely, every countable*[13] *confluent rewrite system* $\to$, *is decreasing for some family and order.*

As the name suggests, more than being a 1-dimensional confluence result, DD establishes a 2-*dimensional* diagrammatic confluence result: Every peak $b \twoheadleftarrow a \twoheadrightarrow c$ *can* be completed by some valley $b \twoheadrightarrow d \twoheadleftarrow c$ into a confluence *diagram*, by means of repeatedly adjoining *locally* decreasing *diagrams*; even stronger, repeatedly adjoining such locally decreasing diagrams *must* terminate after finitely many steps into a decreasing confluence diagram. We now show that countable

[9] Note the reversal for the relations of the first family, turning commutation into factorisation.

[10] 2 lines vs. 8 lines respectively 3 lines vs. 3 pages.

[11] Pous showed in 2005 that termination of $(\blacktriangleleft^+ \cdot \triangleleft^+)^+$ instead of of $\leftarrow^+$ suffices; again DD applies, but using *step* labelling instead of source-labelling [5, Example 17].

[12] This was established independently by Ken Mano and the author (see Remark 2.3.29 in my PhD thesis). The proofs employed a decomposition into a natural–number-indexed family. Recently Klop asked the question whether smaller families suffice, upon which Endrullis, Klop and Overbeek showed that in fact doubletons do.

[13] The case of uncountable systems, conjectured to be false in my PhD thesis, remains open.

confluence suffices even to obtain a 3-*dimensional* (decreasing) diagrammatic confluence result, by a process we dub *cutting*[14] *faces*, which will be discussed more extensively below.[15]

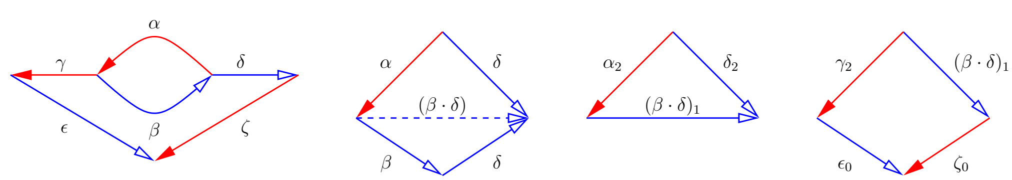
Lemma 1.[16] *Every countable confluent rewrite system* $\to$ *admits a residual system [7, Sect. 8.7]* $(\mathrel{\circ\!\!\!\twoheadrightarrow}, 1, /)$ *with* $\to \subseteq \mathrel{\circ\!\!\!\twoheadrightarrow} \subseteq \twoheadrightarrow$.

Proof. By countable confluence $\to$ has a *spanning forest* F [2, Lem. 1]. Let the relation $\mathrel{\circ\!\!\!\twoheadrightarrow}$ comprise the *steps* of $\to$ not in F, and the *reductions* of F. Viewing steps of an F-reduction as its *faces* its transformation into a single $\mathrel{\circ\!\!\!\twoheadrightarrow}$-step can be viewed as *cutting* faces, in this case resulting in a *diamond*: Define the *residual* ϱ/ϕ of $\varrho : a \mathrel{\circ\!\!\!\twoheadrightarrow} b$ *after* $\phi : a \mathrel{\circ\!\!\!\twoheadrightarrow} c$ by cases on their sources and targets to be ϱ if $c = a$, 1 if $a = b$, and $c \mathrel{\circ\!\!\!\twoheadrightarrow} d$ otherwise with d the *least common descendant* of b, c in F. One checks that the (designated) diamond property, i.e. if $\varrho : a \mathrel{\circ\!\!\!\twoheadrightarrow} b$, $\phi : a \mathrel{\circ\!\!\!\twoheadrightarrow} c$ then $\varrho/\phi : c \mathrel{\circ\!\!\!\twoheadrightarrow} d$, $\phi/\varrho : b \mathrel{\circ\!\!\!\twoheadrightarrow} d$ for some d, and the (3-dimensional) *cube* identity $(\varrho/\phi)/(\psi/\phi) = (\varrho/\psi)/(\phi/\psi)$ hold by uniqueness of least common descendants in F. $\square$

The process of *tiling* by 2-dimensional local diagrams, introduced in [3] and later resumed by Melliès, for the purpose of establishing confluence and standardisation, need *not* entail the 3-dimensional cube property; the edges of the 6 plane surfaces obtained by 2-dimensional tiling need *not* match up. It fails both for terms [7, Fig. 8.53] and (positive) braids [7, Sect. 8.9].

Position 3. *It is of interest to study residual systems for commutation (* coloured *cubes).*

Is DD also complete for *commutation*? We raised this question in [5] and at ISR 2008. It was summarily confuted, then and there, by two participants, Endrullis and Grabmayer:



The rewrite system on the left is not (and cannot be made) decreasing. Even stronger, although the system commutes, that cannot be shown by *tiling* with local commutation diagrams, as it embeds (omit ϵ, ζ) the standard counterexample against 'local commutation $\Rightarrow$ commutation'. As in the previous section, cutting faces comes to the rescue: Looking at the commutation diagram (2nd from left) for the local peak $_{\alpha}\!\blacktriangleleft \cdot \triangleright_{\delta}$ we see it is not yet a diamond since it needs the 2-step *reduction* comprising β and δ. However, *cutting* the reduction into a fresh *face* (step) we name $(\beta \cdot \delta)$, the local diagram (3rd from left) cuts a better figure/diamond.[17] Adjoining $(\beta \cdot \delta)$ and, symmetrically, $(\alpha \cdot \gamma)$ gives rise to new peaks, but we see these can also be completed into diamonds (right diagram). The rewrite system obtained is decreasing for the labels in the natural numbers ordered by $\leq$, as displayed in the rightmost two diagrams. Cutting is a common process, e.g., *parallel* steps $\mathrel{+\!\!\!\!\twoheadrightarrow}$ for TRSs and *multi-steps* $\mathrel{\circ\!\!\!\twoheadrightarrow}$ for HRSs and braids [7] can be seen as being obtained by repeatedly cutting diagrams into diamonds.[18]

Position 4. Minimally[19] *cutting faces is useful to get diamonds, decreasing diagrams and cubes.*

[14] Our (tentative) naming is based on that used for processing raw diamonds. However, adjoining *transitive* inferences (cutting corners), is at the basis of both our *cuts* and those in proof theory, where, e.g., $\Gamma \vdash \Delta$ may be obtained by *cutting* the occurrence of A between $\Gamma \vdash A$ and $A \vdash \Delta$.

[15] We learned the technique of cutting faces in 1995 from Hans Zantema, for braids.

[16] This result was obtained in 2008, in our collaboration with Patrick Dehornoy on the Z-property.

[17] The diamond is still not square, but could be made so by adjoining *empty* reductions reified into steps.

[18] Somewhat miracuously, in these 3 cases even cubes are obtained.

[19] Cutting reductions into steps *by need* only; if $\blacktriangleright, \triangleright$ commutes, simply taking $\blacktriangleright\!\blacktriangleright, \triangleright\!\triangleright$ as *steps* yields a diamond.

3 Diamond symmetries in term rewriting

Commutation between *term* rewrite systems is standardly reduced to an analysis of their *critical peaks*, cf. [6]. Symmetry suggests the same applies to factorisation, but then for peaks with respect to the *reverse* of the second system. Despite that standard term rewriting theory is not well-adapted to reverse rules, cf. Example 1, it can often be easily adapted, as we illustrate:

Example 3. *Factorisation holds in the untyped λ-calculus for $\blacktriangleright := \to_\beta$, $\triangleright := \to_\eta$. Note the reverse $P \to \lambda y.Py$ of the η-rule is not a higher-order pattern rule in the sense of Nipkow, as its left-hand side P is a* ***variable****. Still, we do have a (single) critical peak* à la Huet *with the β-rule: $Q := (\lambda y.(\lambda x.M)y)N \triangleright (\lambda x.M)N \blacktriangleright M[x{:=}N] =: R$; in rewriting terminology, the $\triangleright$-step is said to* create *the $\blacktriangleright$-step. Toward factorisation, note the critical peak can be completed as $Q \blacktriangleright (\lambda x.M)N \blacktriangleright R$. Observing the first $\blacktriangleright$-step is* affine *(non-duplicating) we decompose $\blacktriangleright$ into affine $\blacktriangleright_1$ and non-affine steps $\blacktriangleright_2$. We claim that then all local peaks are decreasing for the order $\blacktriangleright_1 < \triangleleft < \blacktriangleright_2$. For the above critical diagram this holds per construction of our order. A non-critical, i.e. non creating, local peak $\triangleright \cdot \blacktriangleright_i$ can be completed by $\blacktriangleright_i \cdot \triangleright\!\!\triangleright$ by standard residual theory, using that the η-rule is* ***linear****, which again yields a decreasing diagram.*

Position 5. *Extending Nipkow's higher-order critical pair lemma to allow for variable-lhss (à la Huet) is useful. Lifting results should be unproblematic (exception: development-closedness).*

Example 4. *To show* head,internal-*factorisation for untyped $\lambda\beta$-calculus, first note that although critical peaks seem intricate as the reverse of the β-rule is not a* ***pattern****-rule (Example 1), here there are in fact none as a step creating a head-step outside it must be head itself, so not internal. Next, note it suffices to show $\blacktriangleright,\triangleright$-factorisation for $\blacktriangleright := \to_h$ and $\triangleright := \multimap_i$ since $\to_i \subseteq \multimap_i \subseteq \twoheadrightarrow_i$. Finally, we conclude by DD ordering $\blacktriangleright < \triangleleft$ since $\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \multimap \subseteq \blacktriangleright \cdot \blacktriangleright\!\!\blacktriangleright \cdot \triangleright^=$ where the 1st inclusion holds by Church–Rosser as the $\blacktriangleright$-step must be a* unique *residual as $\to_i$-steps can neither create (noted) nor* ***replicate*** *$\blacktriangleright$-steps, and the 2nd[20] by exhaustively (it stops by Finite Developments) selecting $\blacktriangleright$-steps from $\multimap$ until the residual is a $\triangleright$-step or empty.[21]*

Position 6. *Factorisation of term rewrite system(strategie)s is best[22] analysed by a critical peak analysis between rules and reverse rules.*

References

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[20] Thanks to the IWC reviewers for pointing out that by mistake I had oversimplified this (part of the) proof.

[21] A similar analysis applies to strategies other than head, e.g. spine or left.

[22] As observed by Geuvers in (the 'Stellingen' going with) his PhD thesis, the above critical peak between the reverse of η and β, was missed by Barendregt in the proof of Corollary 15.1.5 in his book The Lambda Calculus.