Confluence via Critical Valleys

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A recent result due to Hirokawa and Middeldorp expresses that a left-linear first-order term rewriting systems is confluent, if its critical pairs are joinable and its *critical pair* system, comprising the steps of the critical peaks as rules, is relatively terminating with respect to the original term rewriting system. That result captures both confluence of orthogonal first-order term rewriting systems and of terminating left-linear first-order term rewritings having joinable critical pairs. Here we extend it in three ways:

- we generalise the result from first- to higher-order rewriting;
- we show that instead of the critical pair system, it suffices to consider only a *critical valley* system, comprising as rules reductions from the source of a critical peak to the targets of the first multisteps (if these exist) of the valley completing the peak; and
- we show that *development closed* critical pairs, where the target of the inner step of a critical peak reduces in a multistep to the target of the outer step of the peak, need not be considered when constructing the critical valley system.

1 Confluence via Critical Valleys

Let a *critical valley* system for a locally confluent term rewriting system $\mathscr{R}$ be a system $\mathscr{S}$ over the same signature comprising for *each* critical peak $\underline{s}_0 \leftarrow_{\mathscr{R},\mathrm{root}} \underline{t} \rightarrow_{\mathscr{R}} \underline{r}_0$ such that not $\underline{s}_0 \mathrel{\leftarrow\!\!\!\circ\!\!-}_{\mathscr{R}} \underline{r}_0$, and *some* valley $\underline{s}_0 \mathrel{-\!\!\circ\!\!\!\to}^n_{\mathscr{R}} \underline{s}_n = \underline{r}_m \mathrel{\leftarrow\!\!\!\circ\!\!-}^m_{\mathscr{R}} \underline{r}_0$ completing it, rules $\underline{t} \rightarrow \underline{s}_1$ if $n \geq 1$ and $\underline{t} \rightarrow \underline{r}_1$ if $m \geq 1$. Referring the reader to [2] for no(ta)tions and results used, we generalize [1, Thm. 16 and p. 497]:

Theorem (Critical Valley)**.** *A left-linear locally confluent first- or higher-order term rewriting system $\mathscr{R}$ is confluent if $\mathscr{S}/\mathscr{R}$ is terminating for some critical valley system $\mathscr{S}$ for $\mathscr{R}$.*

Proof. Since $\rightarrow_{\mathscr{R}} \subseteq \mathrel{-\!\!\circ\!\!\!\to}_{\mathscr{R}} \subseteq \twoheadrightarrow_{\mathscr{R}}$ holds for all term rewriting systems, it suffices [4, Proposition 1.1.11 and Lemma 11.6.24] to show confluence of $\mathrel{-\!\!\circ\!\!\!\to}_{\mathscr{R}}$, for which in turn it suffices [3, Theorem 3] to show that its labelling defined by $t \blacktriangleright_{\hat{t}} s$ if $\hat{t} \twoheadrightarrow_{\mathscr{R}} t \mathrel{-\!\!\circ\!\!\!\to}_{\mathscr{R}} s$, is decreasing with respect to the order $(\mathscr{S}/\mathscr{R})^+$. In particular, we show that for given $\hat{t}_i$, a peak $t_0 \blacktriangleleft_{\hat{t}_0} t \blacktriangleright_{\hat{t}_1} t_1$ contracting the multi-redexes U_0,U_1, can be completed into a decreasing diagram by a conversion of shape $\blacktriangleright_{\hat{t}_1} \cdot \blacktriangleleft\!\!\blacktriangleright^* \cdot \blacktriangleleft_{\hat{t}_0}$, where all steps in the conversion $\blacktriangleleft\!\!\blacktriangleright^*$ have labels $\mathscr{S}/\mathscr{R}$-smaller than a $\hat{t}_i$, by induction on the amount of overlap between the patterns of redexes in U_0,U_1:

(0) Then $U_0 \cup U_1$ is a set of non-overlapping redexes and contracting them in t yields a common $\mathrel{-\!\!\circ\!\!\!\to}$-reduct t' of the t_i by the Triangle Theorem 10 of [2], so $t_0 \blacktriangleright_{\hat{t}_1} t' \blacktriangleleft_{\hat{t}_0} t_1$, since $\hat{t}_i \twoheadrightarrow_{\mathscr{R}} t \twoheadrightarrow_{\mathscr{R}} t_{1-i}$.

(>0) Let $u_i \in U_i$ with $s_0 \leftarrow_{u_0} t \rightarrow_{u_1} r_0$ be induced by a critical peak $\underline{s}_0 \leftarrow_{\mathscr{R}} \underline{t} \rightarrow_{\mathscr{R}} \underline{r}_0$ with, w.l.o.g., u_1 innermost, and distinguish cases on whether $\underline{s}_0 \mathrel{\leftarrow\!\!\!\circ\!\!-}_{\mathscr{R}} \underline{r}_0$ or not:

($\top$) By Claim 23 of [2] there exists a peak $t_0 \blacktriangleleft_{\hat{t}_0} r_0 \blacktriangleright_{\hat{t}_1} t_1$ contracting multi-redexes U'_0,U'_1 having a smaller amount of overlap than U_0,U_1 had, and we conclude by the induction hypothesis.

($\perp$) There is a valley $\underline{s}_0 \mathrel{-\!\!\circ\!\!\!\to}^n_{\mathscr{R}} \underline{s}_n = \underline{r}_m \mathrel{\leftarrow\!\!\!\circ\!\!-}^m_{\mathscr{R}} \underline{r}_0$ such that $\underline{t} \rightarrow \underline{s}_1$ if $n \geq 1$ and $\underline{t} \rightarrow \underline{r}_1$ if $m \geq 1$.

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If $n \geq 1$, the induction hypothesis can be applied to $t_0 \blacktriangleleft_{\hat{t}_0} s_0 \blacktriangleright_{\hat{t}_1} s_1$ as $\underline{t}$ and $U_0 - \{u_0\}$ do not overlap in t by innermostness of u_1 and the tree-structure of terms so neither do their descendants in s_1 after u_0, yielding a decreasing diagram $t_0 \blacktriangleright_{\hat{t}_1} \cdot \blacktriangleleft\!\blacktriangleright^* \cdot \blacktriangleleft_{\hat{t}_0} s_1$ hence, relabeling its last step, also $t_0 \blacktriangleright_{\hat{t}_1} \cdot \blacktriangleleft\!\blacktriangleright^* \cdot \blacktriangleleft_{s_1} s_1$, where all steps except the first have labels $\mathscr{S}/\mathscr{R}$-smaller than a $\hat{t}_i$.

If $m \geq 1$ the induction hypothesis can be applied to $r_1 \blacktriangleleft_{\hat{t}_0} r_0 \blacktriangleright_{\hat{t}_1} t_1$ as $\underline{t}$ and V_0 overlap more in t than $\underline{r}_0$ and the residuals of V_0 after v_0 do in r_0 by innermostness of t and the tree-structure of terms, yielding a decreasing diagram $r_1 \blacktriangleright_{\hat{t}_1} \cdot \blacktriangleleft\!\blacktriangleright^* \cdot \blacktriangleleft_{\hat{t}_0} t_1$ hence, relabeling its first step, also $r_1 \blacktriangleright_{r_1} \cdot \blacktriangleleft\!\blacktriangleright^* \cdot \blacktriangleleft_{\hat{t}_0} t_1$ where all steps except the last have labels $\mathscr{S}/\mathscr{R}$-smaller than a $\hat{t}_i$.

- If $n,m \geq 1$ then we may join the above conversions by the following labelling induced by a suffix of the local confluence valley $s_1 \blacktriangleright_{s_1}^{n-1} s_n = r_m \blacktriangleleft_{r_1}^{m-1} r_1$;
- If $n = 0$ and $m \geq 1$, then we may join $t_0 \blacktriangleleft\!\blacktriangleleft_{r_1} r_1$ and the second conversion above;
- If $n \geq 1$ and $m = 0$, then we may join the first conversion above and $s_1 \blacktriangleright\!\blacktriangleright_{s_1} t_0$; and
- The case that $n = 0 = m$ cannot occur as then $\underline{s}_0 \leftarrow\!\circ\!\!-_{\mathscr{R}} \underline{r}_0$. □

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