# Properties of Needed Strategies

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## Neededness Intuition

## Four formalizations of neededness

- Permutation
- Standardisation
- Labelling
- Tracing

# Motivation for Neededness

Do not contract redexes not needed to reach result

# Typical Result

### Theorem

*The needed strategy is normalising for combinatory logic.*

$$
\begin{aligned}
(K \cdot x) \cdot y &\longrightarrow x \\
((S \cdot x) \cdot y) \cdot z &\longrightarrow (x \cdot z) \cdot (y \cdot z)
\end{aligned}
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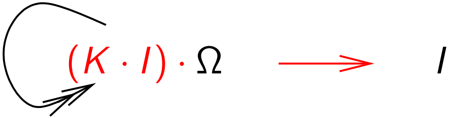
### Example




$$
\begin{array}{rcl}
I & = & (S \cdot K) \cdot K \\
\Omega & = & ((S \cdot I) \cdot I) \cdot ((S \cdot I) \cdot I)
\end{array}
$$

# Needed redexes need not exist

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$$\begin{aligned} a &\rightarrow b \\ f(b,x) &\rightarrow c \\ f(x,b) &\rightarrow c \end{aligned}$$

$f(a, a)$ not needed: $f(a, a) \rightarrow f(a, b) \rightarrow c$

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f(x, b) & \to & c
\end{array}
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$f(a, a)$ no needed redex!

# Needed reduction may be counterproductive

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$$\begin{aligned} a &\rightarrow b \\ f(x) &\rightarrow g(x, x) \\ g(a, b) &\rightarrow c \\ g(b, b) &\rightarrow f(a) \end{aligned}$$

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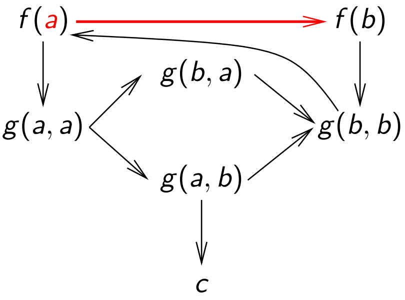

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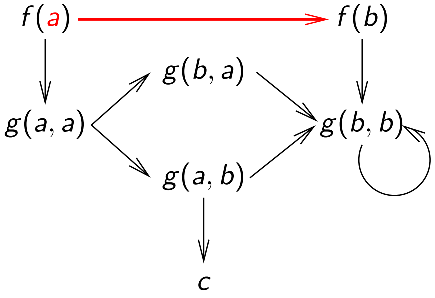

# When is needed reduction interesting?

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for orthogonal term rewriting systems

- combinatory logic

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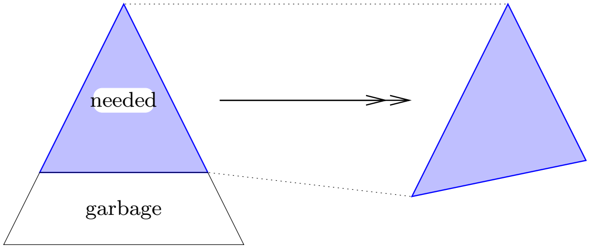
Intuition: reductions to normal form contract same redexes
(only order of contraction differs)

# Neededness intuition



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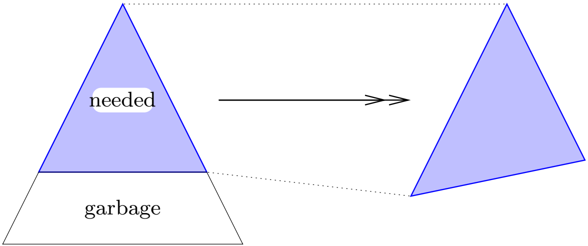


Reduction to normal form splits term into
needed top and garbage bottom

# Formalizing intuition

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1. Permutation

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permutation equivalent to

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- $\vartheta(x, y) : (K \cdot x) \cdot y \to x$
- $\phi : \Omega \to^* \Omega$

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- $\vartheta(x, y) : (K \cdot x) \cdot y \to x$
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$((K \cdot I) \cdot \phi) \circ \vartheta(I, \Omega)$ permutation equivalent to $\vartheta(I, \Omega)$

# Permutation equivalence on proof terms

$$
\begin{array}{rcl}
1 \circ \phi & \approx & \phi \\
\phi \circ 1 & \approx & \phi \\
(\phi \circ \psi) \circ \chi & \approx & \phi \circ (\psi \circ \chi) \\
f(\phi_1, \ldots, \phi_n) \circ f(\psi_1, \ldots, \psi_n) & \approx & f(\phi_1 \circ \psi_1, \ldots, \phi_n \circ \psi_n) \\
\varrho(\phi_1, \ldots, \phi_n) & \approx & l(\phi_1, \ldots, \phi_n) \circ \varrho(t_1, \ldots, t_n) \\
\varrho(\phi_1, \ldots, \phi_n) & \approx & \varrho(s_1, \ldots, s_n) \circ r(\phi_1, \ldots, \phi_n)
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where $\varrho : l \to r$ and $\phi_i : s_i \to^* t_i$

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Example

$((K \cdot I) \cdot \phi) \circ \vartheta(I, \Omega) \approx \vartheta(I, \phi) \approx \vartheta(I, \Omega) \circ I \approx \vartheta(I, \Omega)$

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**Example**

$((K \cdot I) \cdot \phi) \circ \vartheta(I, \Omega) \Rightarrow \vartheta(I, \Omega) \circ I \approx \vartheta(I, \Omega)$

**Theorem**

*Removing anti-standard pairs terminates*

**Proof.**

Like sorting by inversions but more difficult (duplication) □

# Neededness via labelling

**Definition**

Redex not needed if labelling it yields no labels in target

**Example**

labelling $\Omega$ by underlining in:

$$(K \cdot I) \cdot \Omega \rightarrow^* (K \cdot I) \cdot \Omega \rightarrow I$$

does not give rise to labels in the target:

$$(K \cdot I) \cdot \underline{\Omega} \rightarrow^* (K \cdot I) \cdot \underline{\Omega} \rightarrow I$$

# Labelling ARSs

Attach information; behaviour should be the same

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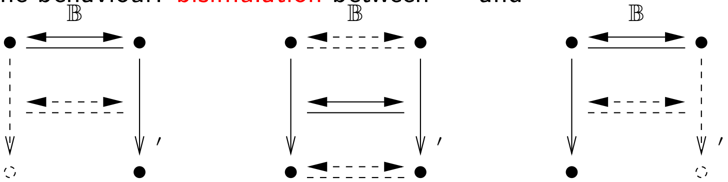
Same behaviour: bisimulation between $\rightarrow$ and $\rightarrow'$



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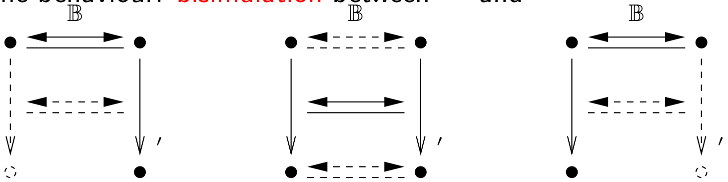
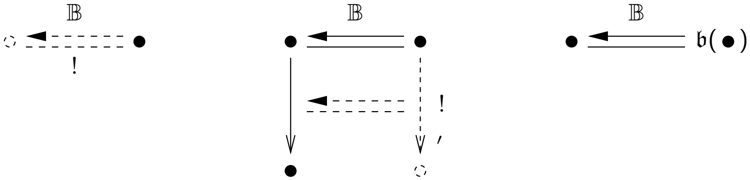



Attach information: labelling of $\to$ to $\to'$



# Labelled ARSs

Example

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- *termination preserved and reflected by bisimulation*

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- *termination preserved and reflected by bisimulation*
- *normalisation preserved and reflected by bisimulation*

# Labelled ARSs

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- Semantic labelling of TRSs
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## Lemma

- *termination preserved and reflected by bisimulation*
- *normalisation preserved and reflected by bisimulation*
- *confluence reflected by labelling*

# Labelling TRSs

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Labelling of signature, lifting this to rules, steps

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Example

Semantic labelling

# Hyland–Wadsworth labelling

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Label symbol with its creation level

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$\varrho : \ a \rightarrow f(a)$

$\varrho_{a^i} : \ a^i \rightarrow f^{i+1}(a^{i+1})$, for all $i \in \mathbb{N}$

# Hyland–Wadsworth labelling

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$$\mathcal{R} : \ a \rightarrow f(a) \rightarrow f(f(a)) \rightarrow f(f(f(a))) \rightarrow \cdots$$

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$$\mathcal{R} : \ a \to f(a) \to f(f(a)) \to f(f(f(a))) \to \cdots$$

$$\mathfrak{HW}(\mathcal{R}) : \ a^0 \to f^1(a^1) \to f^1(f^2(a^2)) \to f^1(f^2(f^3(a^3))) \to \cdots$$

Theorem

*Upper bounding creation level implies termination*

Proof.

use recursive path orders □

# Hyland–Wadsworth vs. Arts–Giesl

**Lemma**

*If non-terminating, there's an infinite creation chain*

**Proof.**

zoom-in □

**Corollary**

*If non-terminating, then no upper bound on creation level*

**Corollary**

*If non-terminating, then infinite dependency chain*

# Neededness via Tracing

**Definition**

Redex not needed if its positions do not trace to target

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$\Omega$ does not trace to target in

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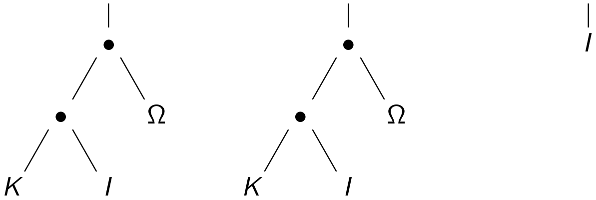

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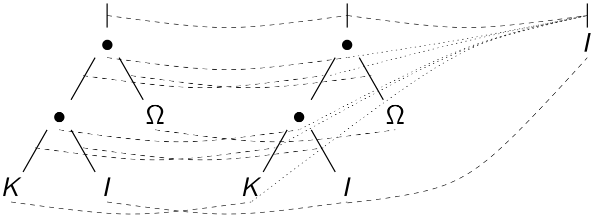
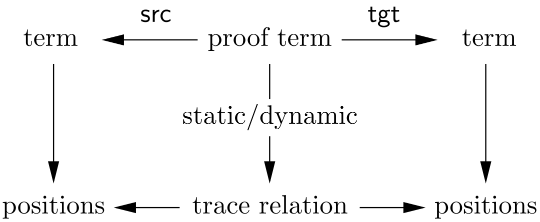


# Tracing as proof term algebra



# Equivalence of notions of neededness

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**Theorem**

*Permutation non-needed* $\Leftarrow$ *Standardisation non-needed*

**Proof.**

Standardisation is permutation □

# Equivalence of notions of neededness

**Theorem**

*Standardisation non-needed* $\Leftarrow$ *Labelling non-needed*

**Proof.**

Permutation does not change labelling of target

Contracting any labelled redex would label target □

# Equivalence of notions of neededness

**Theorem**

*Labelling non-needed* $\Leftarrow$ *Tracing non-needed*

**Proof.**

Labels trace (internal vs. external) □

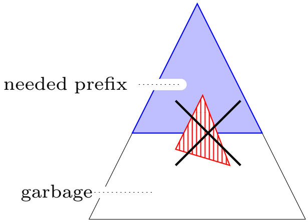
# Equivalence of notions of neededness

### Theorem

*Tracing non-needed* $\Leftarrow$ *Permutation non-needed*

### Proof.

Tracing invariant wrt. permutation (redex-pattern closed)




□

# Can needed redexes be computed?

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No

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How to evaluate $g(s_1, s_2, s_3)$ in CL + Gustave's TRS?

$$
\begin{aligned}
g(a, b, x) &\rightarrow c \\
g(x, a, b) &\rightarrow c \\
g(b, x, a) &\rightarrow c
\end{aligned}
$$

(recall: word problem for CL is undecidable)

# Neededness via externality

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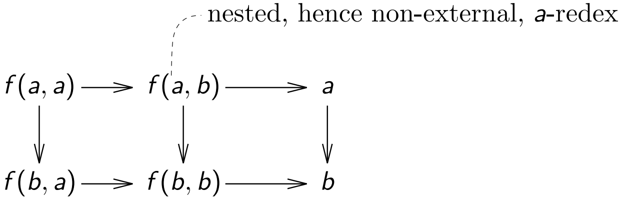
Redex is external if outermost until contracted

TRS: $a \to b$, $f(x, b) \to x$



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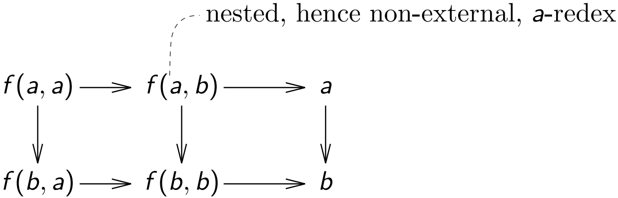


## Lemma

*For orthogonal fully-extended HRSs*
*every reducible term contains an external redex*

## Proof.

Idea: cannot mutually erase each other □

# Computable cases of neededness

**Lemma**

*External redexes are needed (needed strategy is strategy!)*

**Proof.**

Trivial □

- combinatory logic: leftmost-outermost redex external

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- combinatory logic: leftmost-outermost redex external
- $\lambda$-calculus: idem
- left-normal orthogonal fully-extended HRSs: idem

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- sequential TRSs (but sequentiality not decidable)

# Computable cases of neededness

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Proof.

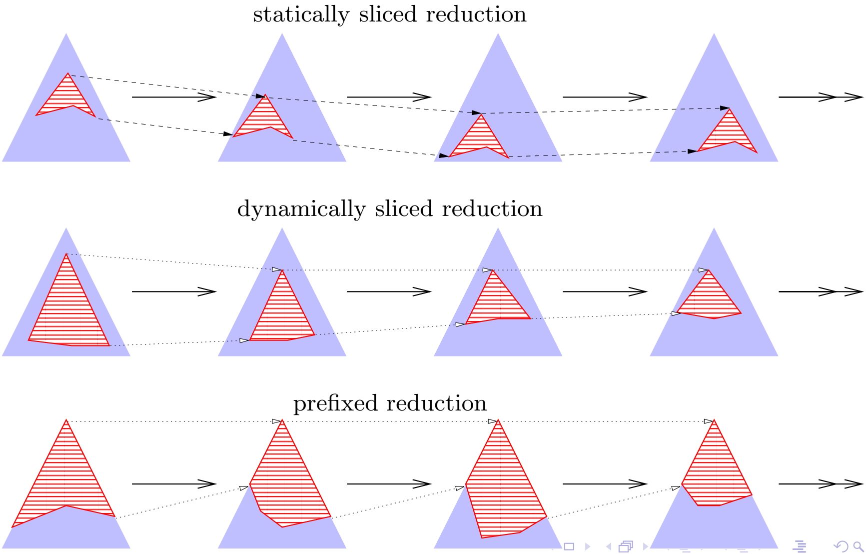
Trivial ☐

- combinatory logic: leftmost-outermost redex external
- $\lambda$-calculus: idem
- left-normal orthogonal fully-extended HRSs: idem
- sequential TRSs (but sequentiality not decidable)
- strongly sequential TRSs (both decidable).

# Prefix property

## Theorem

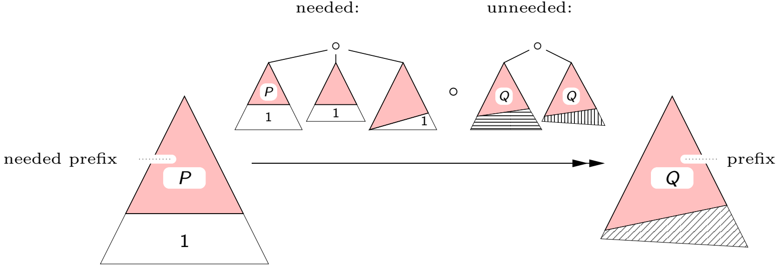
*Ancestor of prefix is prefix*



# Factorisation (semi-standardisation)

## Theorem

*Reductions factorise into needed/garbage for given prefix of target*



# Neededness summary

Formalized four neededness intuitions

- Permutation (inverting steps)
- Standardisation (sorting steps outside-in)
- Labelling (attach info, preserving behaviour)
- Tracing (see what happens)

Formalizations are equivalent