Properties of Needed Strategies

Vincent van Oostrom

Theoretical Philosophy
Utrecht University
The Netherlands

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Literature

Term Rewriting Systems
Terese
Cambridge University Press, 2003

Abstract Rewriting Strategies

Term Rewriting Strategies

Structured Rewriting Strategies

Motivation for Strategies

Controlling non-determinism

Typical Result

Theorem

The needed strategy is normalising for combinatory logic.

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\begin{aligned}
(K \cdot x) \cdot y &\longrightarrow x \\
((S \cdot x) \cdot y) \cdot z &\longrightarrow (x \cdot z) \cdot (y \cdot z)
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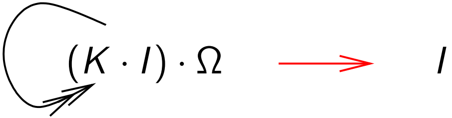
Example



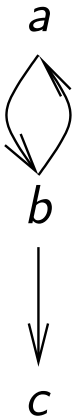
$$
\begin{aligned}
I &= (S \cdot K) \cdot K \\
\Omega &= ((S \cdot I) \cdot I) \cdot ((S \cdot I) \cdot I)
\end{aligned}
$$

ARS strategy

Definition

Strategy is sub-ARS having same objects and normal forms.

sub-ARS example



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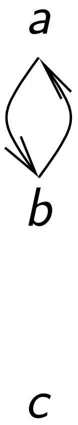

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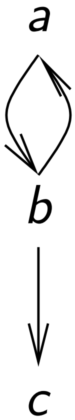

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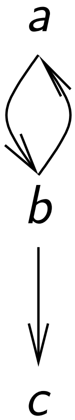

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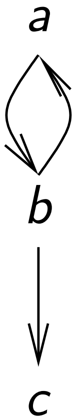
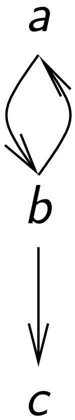


sub-ARS example

a

c

sub-ARS example



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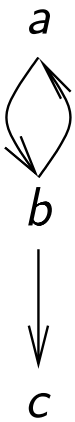

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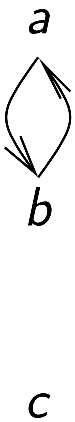

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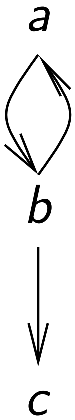

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Examples of strategies

- Innermost (‘call by value’)

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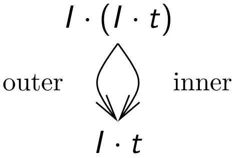

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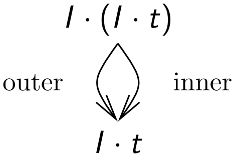

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Want steps

Abstract Rewriting Systems

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ARS is a binary relation on a set.

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extensional (existence of steps)

Abstract Rewriting Systems redefined

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ARS is $\langle A, \Phi, \mathrm{src}, \mathrm{tgt}\rangle$

- A set of objects
- Φ set of steps
- src, tgt : $\Phi \longrightarrow A$
  source and target functions

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$\phi : a \to b$ denotes

ϕ is step with source a and target b

Abstract Rewriting Systems redefined?

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Equivalently

- Directed graph
- Category without composition (no monoid laws)

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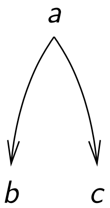

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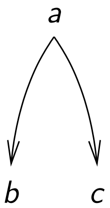

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Lemma

a deterministic strategy always exists
(simply choose one step from each source)

Facts on strategies

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- normalising reflected, not preserved
- confluence neither preserved nor reflected

Reduction sequences

Many-step ARS $\to^+$:

- Objects: objects of $\to$
- Steps: non-empty reduction sequences of $\to$
- source of sequence is source of first step
  target of sequence is target of last step

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reduction sequences can be composed (associative)

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Many-step strategy for $\rightarrow$ is strategy for $\rightarrow^+$

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No, not in general: ARSs lack structure to express

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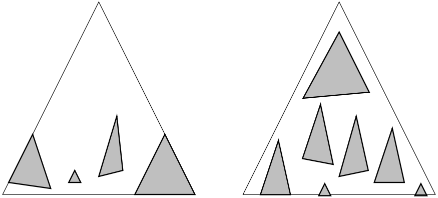

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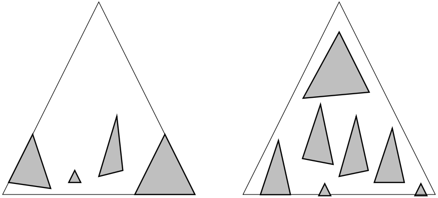
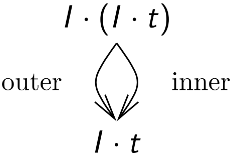


Need structured objects
terms, graphs, . . .

ARS underlying a TRS



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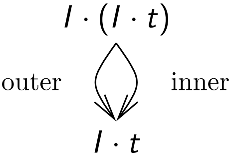

Want to prevent syntactic accidents systematically
(instead of ad hoc)

Equational logic inference system

$$\frac{s \rightarrow t}{s = t}\,(\text{rule}) \qquad \frac{s = t}{s^\sigma = t^\sigma}\,(\text{substitution})$$

$$\frac{s_1 = t_1 \quad \ldots \quad s_n = t_n}{f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)}\,(\text{congruence})$$

$$\frac{}{s = s}\,(\text{reflexive}) \quad \frac{s = t}{t = s}\,(\text{symmetric}) \quad \frac{s = t \quad t = u}{s = u}\,(\text{transitive})$$

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Theorem

$t \approx s \iff t \leftrightarrow^* s \iff t = s$

Equational logic inference example

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Distinct proofs!

Idea: Proofs as steps

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Symmetry never needed in rewriting

Rewriting logic inference system

Equational logic inference system without (symmetric)

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How to represent proofs?

Idea: Proofs as terms

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What is represented by $\varrho(I \cdot t) \circ \varrho(t)$, and by $\varrho(t \circ t)$?

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- Parallel step $\twoheadrightarrow\!\!\!\!\!+$: no transitivity, no nested rules
- Multi-step $\multimap$: no transitivity
- Many-step $\to^+$: transitivity only at root

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- Multi-step: full-substitution (Gross–Knuth)

Higher-order rewriting strategies

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3. Strategies as restriction of higher-order proof terms.

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Other structures: graphs, ...

Strategies summary

- Abstract rewrite relations vs. systems
  (extensional vs. intensional)
- Strategy as sub-ARS
  (same objects, normal forms)
- Term rewrite strategies as ARS strategies
  (via proof terms for rewrite logic)