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Contents

1. Goal

Extend optimality theory and practice to HRS

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Subgoals:

- TRS optimality
  theory done (Terese book)
- λ-calculus optimality
  here

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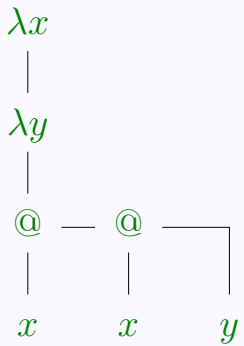
Subgoals:

- TRS optimality
  theory done (Terese book)
- λ-calculus optimality
  here

Literature: Lévy, Lamping, Gonthier, Abadi, Danos, Mackie, Regnier, Asperti, Laneve, Guerrini, Mairson, Lawall, ...
Here: analysis of De Bruijn's calculational approach

1.1. Named

Church numeral $\underline{2} = \lambda x.\lambda y.x(xy)$



1.1. Named

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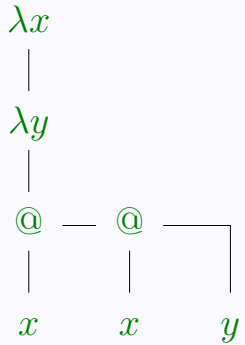
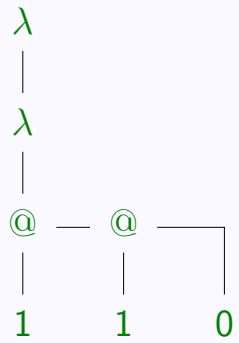


Application is exponentiation

$$\begin{aligned}
\underline{2}\,\underline{2} &\to_\beta \lambda y.\underline{2}(\underline{2}y) \\
&\to_\beta \lambda y.\underline{2}\lambda x.y(yx) \\
&\to_\beta \lambda y.\lambda z.(\lambda x.y(yx))((\lambda x.y(yx))z) \\
&\to_\beta \lambda y.\lambda z.(\lambda x.y(yx))(y(yz)) \\
&\to_\beta \lambda y.\lambda z.y(y(y(yz))) = \underline{4}
\end{aligned}$$

1.2. De Bruijn indexed

De Bruijn indexed Church numeral $\underline{2} = \lambda\lambda 1(10)$



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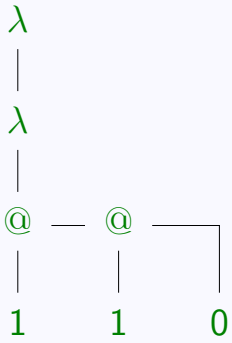
De Bruijn indexed Church numeral $\underline{2} = \lambda\lambda 1(10)$

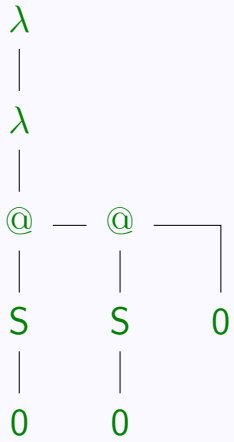


De Bruijn indexed exponentiation

$$
\begin{aligned}
\underline{2}\,\underline{2} &\rightarrow_\beta \lambda\underline{2}(\underline{2}0) \\
&\rightarrow_\beta \lambda\underline{2}\lambda 1(10) \\
&\rightarrow_\beta \lambda\lambda(\lambda 2(20))((\lambda 2(20))0) \\
&\rightarrow_\beta \lambda\lambda(\lambda 2(20))(1(10)) \\
&\rightarrow_\beta \lambda\lambda 1(1(1(10)))
\end{aligned}
$$

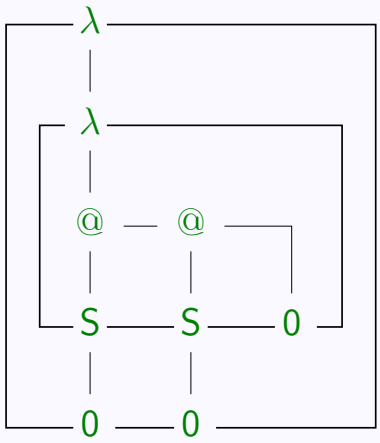
1.3. Unary indexed

Unary indexed $\underline{2} = \lambda\lambda(\mathsf{S}0)((\mathsf{S}0)0)$



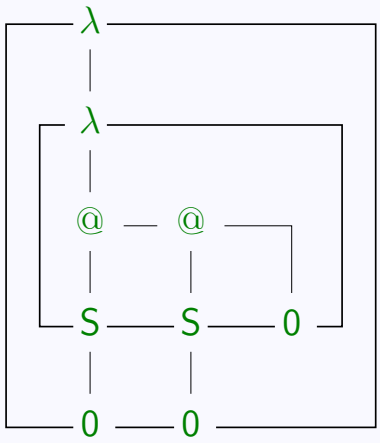
1.4. Scoped

Scoped $\underline{2} = \lambda\lambda(\mathsf{S}0)((\mathsf{S}0)0)$



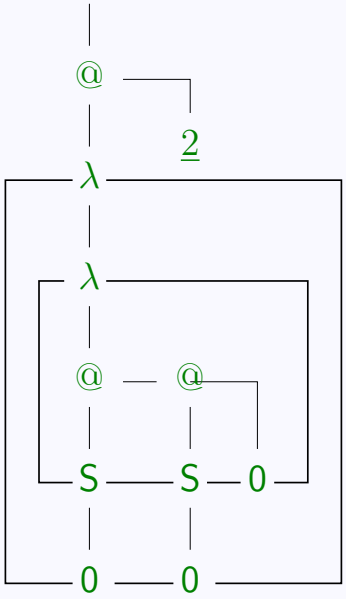
1.4. Scoped

Scoped $\underline{2} = \lambda\lambda(\mathsf{S}0)((\mathsf{S}0)0)$



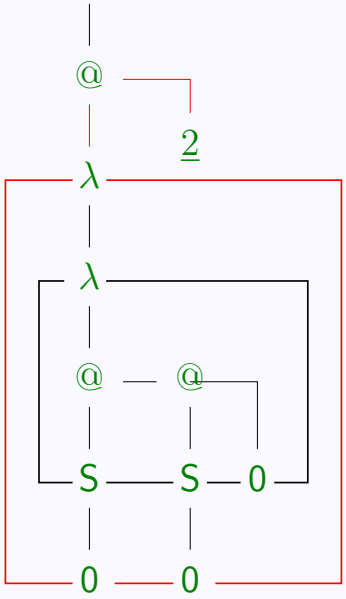
scoping is matching
binding is match with 0
end-of-scope is match with S
λ-terms are context-free trees

1.5. β-reduction



how to reduce?

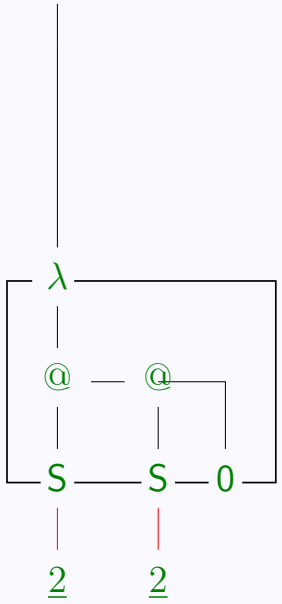
1.6. β-reduction: replication



put argument if matching 0

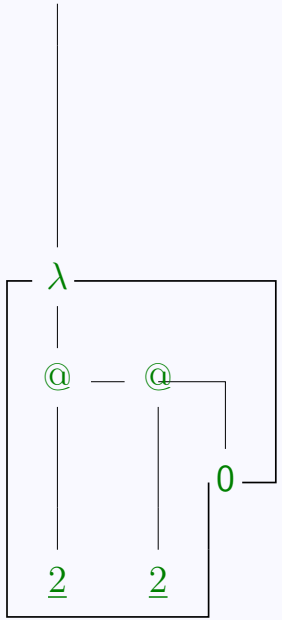
erase argument if matching S

1.7. β-reduction: yields generalised term



successors of subterms!

1.8. β-reduction: extrusion



extrude scopes to yield unary indexed terms

1.9. Generalised terms inductively

$$t \in G\Lambda \ ::= \ 0 \mid \mathsf{S}t \mid \lambda t \mid tt$$

De Bruijn indexed Λ-terms: repeated successors of 0

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$$t \in G\Lambda \ ::= \ 0 \mid \mathsf{S}t \mid \lambda t \mid tt$$

De Bruijn indexed Λ-terms: repeated successors of 0

$$\frac{\mathsf{S}i \vdash 0}{}\,0 \quad \frac{\mathsf{S}i \vdash \mathsf{S}t}{i \vdash t}\,\mathsf{S} \quad \frac{i \vdash \lambda t}{\mathsf{S}i \vdash t}\,\lambda \quad \frac{i \vdash t_1 t_2}{i \vdash t_1 \quad i \vdash t_2}\,@$$

t well-formed under stack i

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$$t \in G\Lambda \ ::= \ 0 \mid \mathsf{S}t \mid \lambda t \mid tt$$

De Bruijn indexed Λ-terms: repeated successors of 0

$$\frac{}{\mathsf{S}i \vdash 0}\,0 \quad \frac{\mathsf{S}i \vdash \mathsf{S}t}{i \vdash t}\,\mathsf{S} \quad \frac{i \vdash \lambda t}{\mathsf{S}i \vdash t}\,\lambda \quad \frac{i \vdash t_1 t_2}{i \vdash t_1 \quad i \vdash t_2}\,@$$

t well-formed under stack i

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \vdash \lambda\lambda(\mathsf{S}0)((\mathsf{S}0)0)}\,\lambda}{\mathsf{S}0 \vdash \lambda(\mathsf{S}0)((\mathsf{S}0)0)}\,\lambda}{\mathsf{SS}0 \vdash (\mathsf{S}0)((\mathsf{S}0)0)}\,@}{\dfrac{\mathsf{SS}0 \vdash \mathsf{S}0}{\mathsf{S}0 \vdash 0}\,0}\,\mathsf{S} \quad \dfrac{\mathsf{SS}0 \vdash (\mathsf{S}0)0}{\dfrac{\mathsf{SS}0 \vdash \mathsf{S}0}{\mathsf{S}0 \vdash 0}\,0}\,\mathsf{S} \quad \dfrac{\mathsf{SS}0 \vdash 0}{}\,0}\,@$$

1.10. Replication recursively

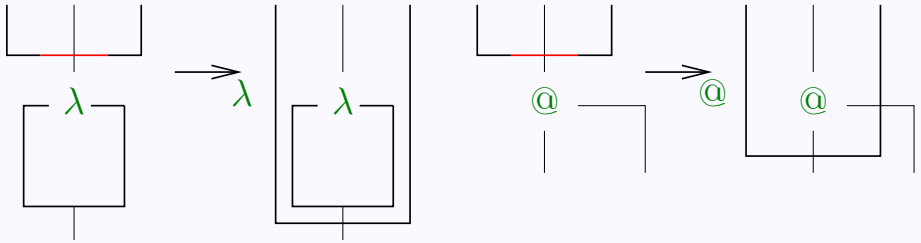
$$(\lambda t)s \to t[s]^0$$

replication $t[s]^i$ of argument s in t at depth i

$$\begin{aligned}
0[s]^0 &= s \quad \text{put argument} \\
0[s]^{Si} &= 0 \\
(St)[s]^0 &= t \quad \text{erase argument} \\
(St)[s]^{Si} &= St[s]^i \\
(\lambda t)[s]^i &= \lambda t[s]^{Si} \\
(t_1 t_2)[s]^i &= t_1[s]^i t_2[s]^i \quad \text{duplicate argument}
\end{aligned}$$

$$\underline{2}\,\underline{2} = (\lambda\lambda(S0)((S0)0))\underline{2} \to \lambda(S\underline{2})((S\underline{2})0)$$

1.11. Extrusion iteratively/recursively



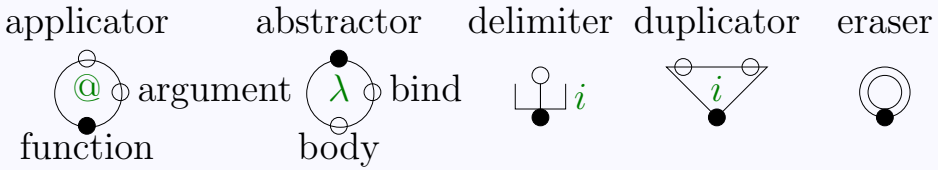
$\mathsf{S}\lambda t \to_\lambda \lambda t^{\mathsf{S}0}$ $\qquad$ $\mathsf{S}(t_1 t_2) \to_@ \mathsf{S}t_1 \mathsf{S}t_2$

minimal lifting t^i

$$
\begin{aligned}
t^0 &= \mathsf{S}t \\
0^{\mathsf{S}i} &= 0 \\
(\mathsf{S}t)^{\mathsf{S}i} &= \mathsf{S}t^i \\
(\lambda t)^{\mathsf{S}i} &= \lambda t^{\mathsf{SS}i} \\
(t_1 t_2)^{\mathsf{S}i} &= t_1^{\mathsf{S}i} t_2^{\mathsf{S}i}
\end{aligned}
$$

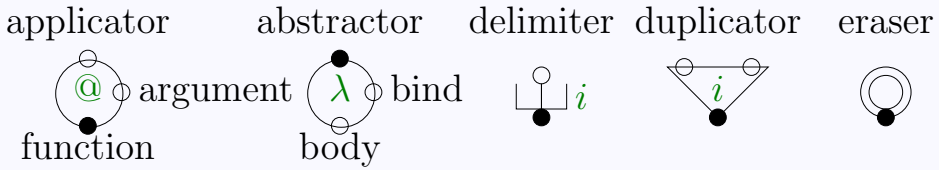
$$\lambda(\mathsf{S}\underline{2})((\mathsf{S}\underline{2})0) \to \lambda\underline{2}(\underline{2}0)$$

2. Interaction net implementation



- ○s are ports
- ●s are principal ports
- i index (default 0)

2. Interaction net implementation



○s are ports

●s are principal ports

i index (default 0)

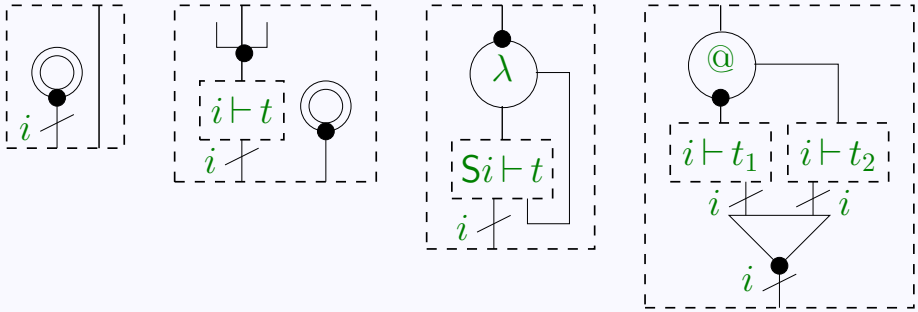
@ applicator, λ abstractor for local β

∇_i duplicator, ◎ eraser for explicit local replication

$\sqcup_i$ scope delimiter for explicit local extrusion

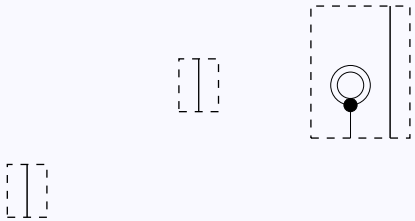
2.1. From terms to nets

$\square : \Lambda \longrightarrow \mathsf{IN}$ recursively maps closed terms to 'closed' nets

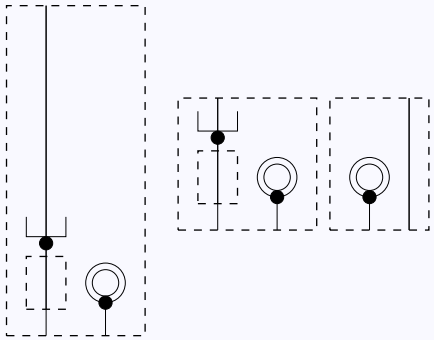


$i \vdash t$ mapped to net with $i + 1$ free ports

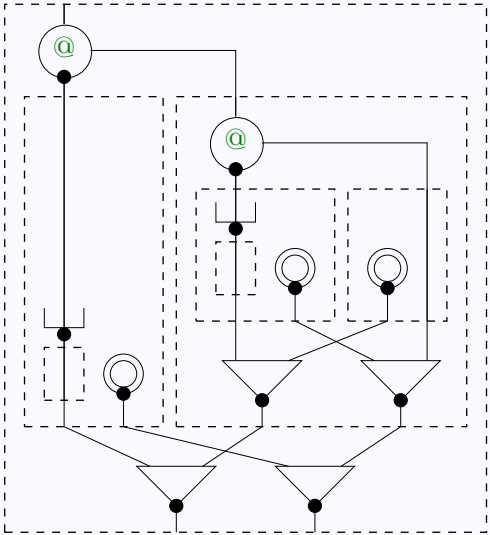
2.2. From terms to nets: translating zero



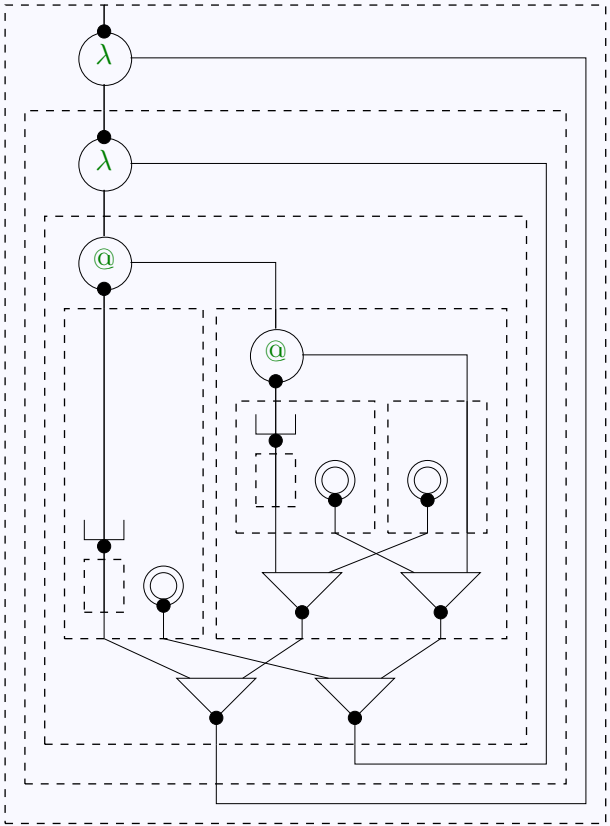
2.3. From terms to nets: translating successor



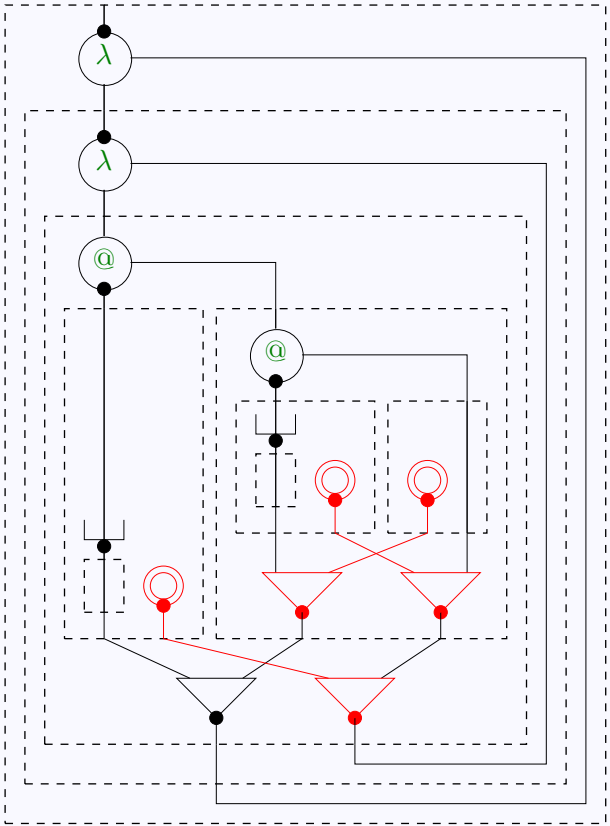
2.4. From terms to nets: translating application



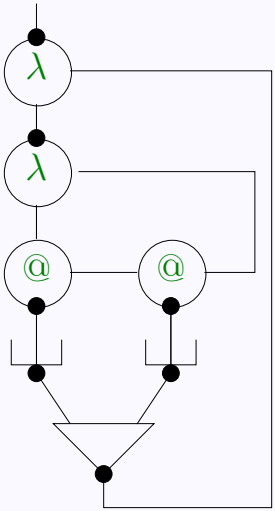
2.5. From terms to nets: translating abstraction



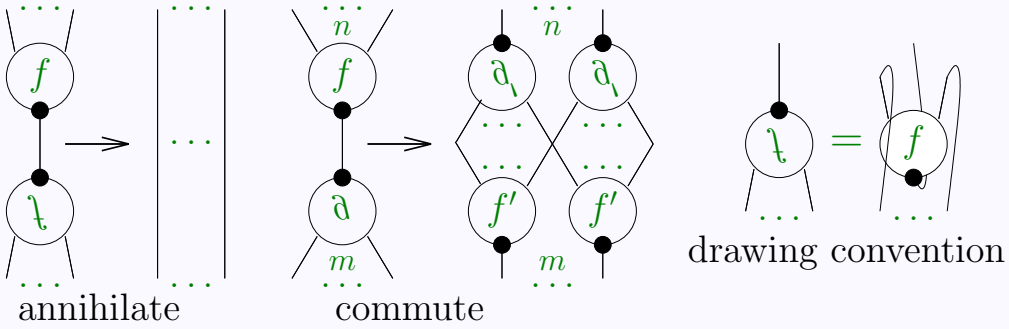
2.6. From terms to nets: useless parts



2.7. From terms to nets: useful part



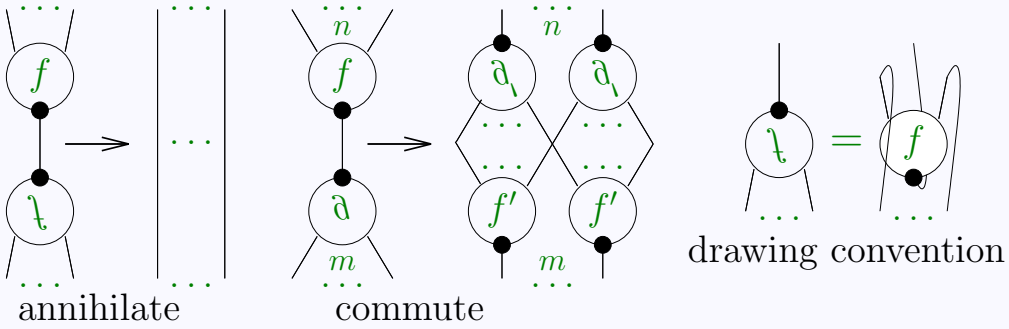
2.8. Net reduction: x-rules



f',g' identical to or updates of f,g (distinct)

Update: increment of i iff other λ or $\sqcup_j$ $(i \geq j)$

2.8. Net reduction: x-rules



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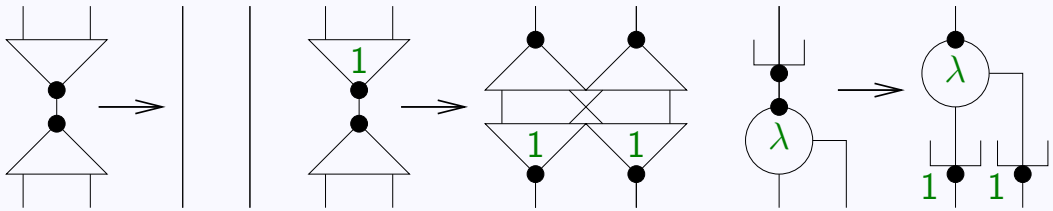
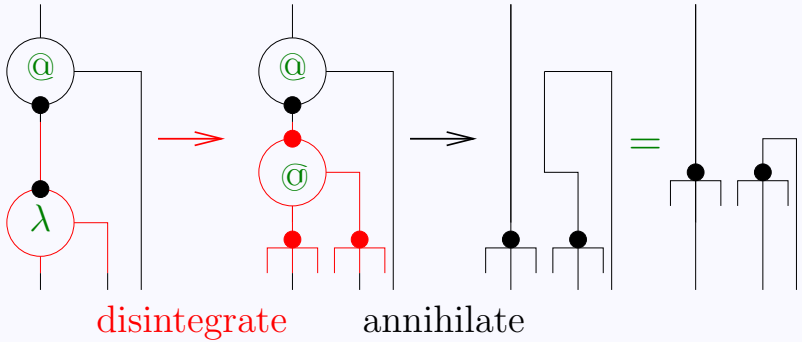
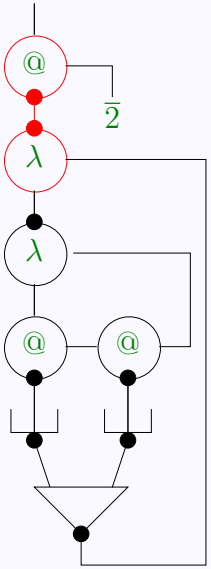


2.9. Net reduction: Beta

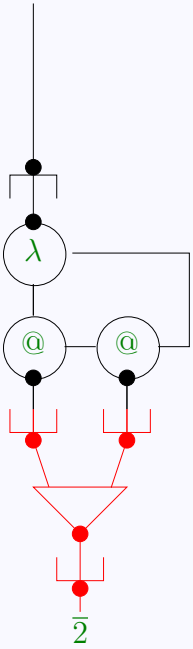


$\mathsf{B} = \mathsf{Beta} + \mathbf{x}$

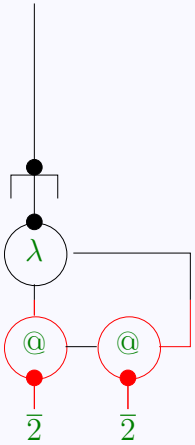
2.10. Net reduction: example Beta-step



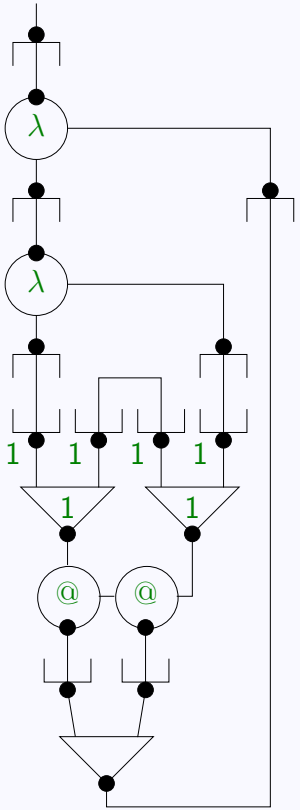
2.11. Net reduction: example x-normalisation



2.12. Net reduction: example B-normalisation



2.13. Net reduction: example normal form



2.14. From nets to terms

Why does this net represent $\underline{4}$?

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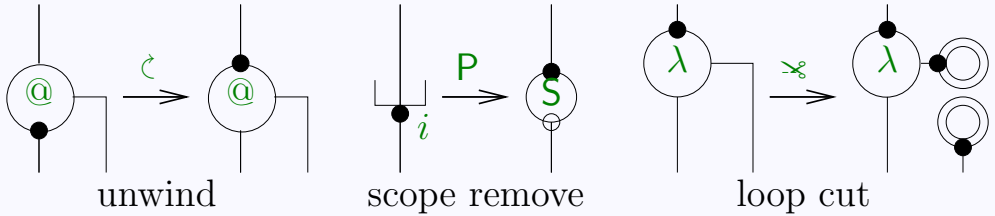
Why does this net represent $\underline{4}$?

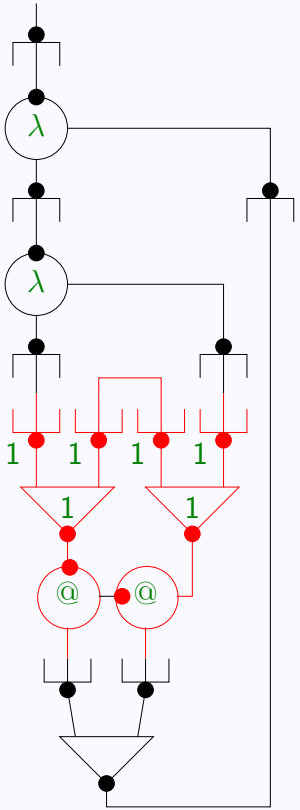
read-back map $\triangle : \mathsf{IN} \rightarrow \Lambda$ in three phases:



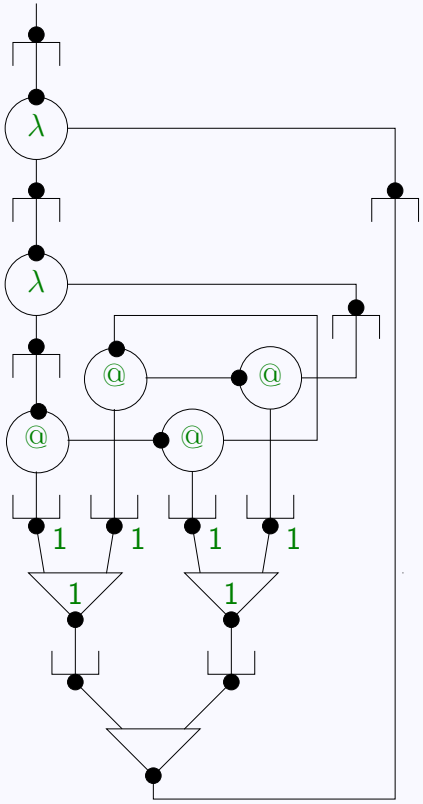
each followed by $\mathbf{x}$-normalisation

1. unwind: replicate,extrude applications
2. scope remove: remove redundant scopes
3. cut loop: elide duplicators yielding tree

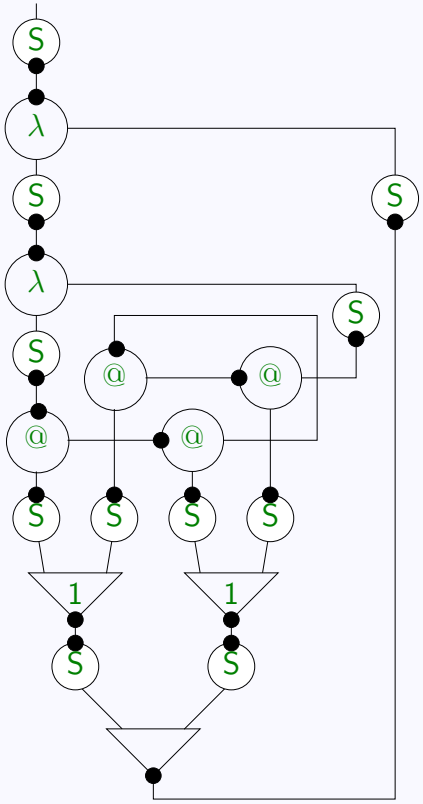
2.15. Read-back: rotating port of application



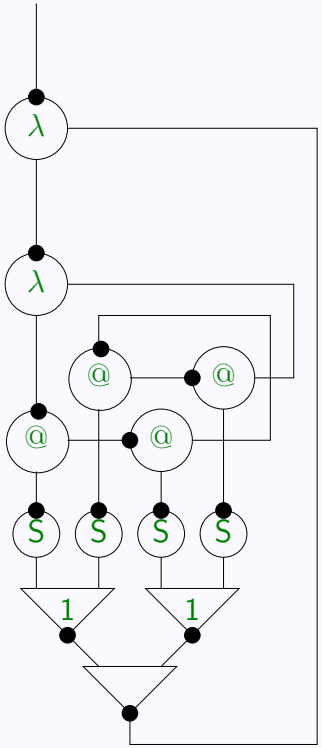
2.16. Read-back: replicate, extrude applications



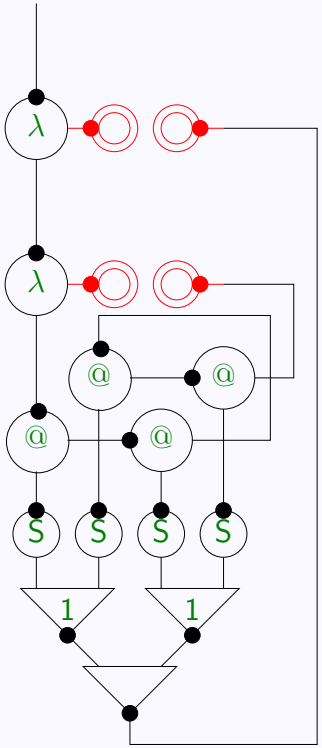
2.17. Read-back: rotating port of delimiter



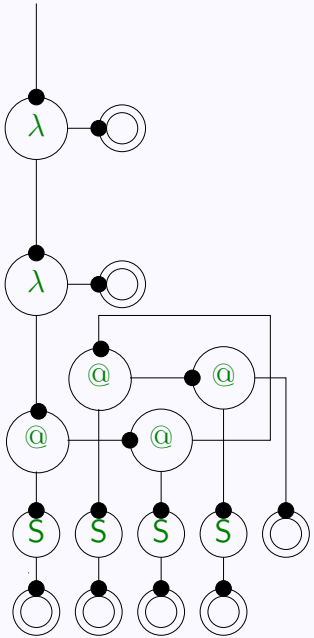
2.18. Read-back: remove redundant scopes



2.19. Read-back: cut loops



2.20. Read-back: elide duplicators



3. Correctness

No syntactic proof yet . . .

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No syntactic proof yet ...

Semantic proof via stack-based read-back

- Invariant under reduction
- Coincides with normal form (tree)

3.1. Push down automaton

Transitions between (ports on) edges of net

$$b \in B \ ::= \ i\vec{\delta} \quad \text{blocks with replication info}$$
$$\ell \in L \ ::= \ b\vec{\ell} \quad \text{levels with scoping info}$$
$$\sigma \in S \ ::= \ i\vec{\ell} \quad \text{stack with read-back info}$$

δ ranges over directors $\{L, R\}$

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Transitions between (ports on) edges of net

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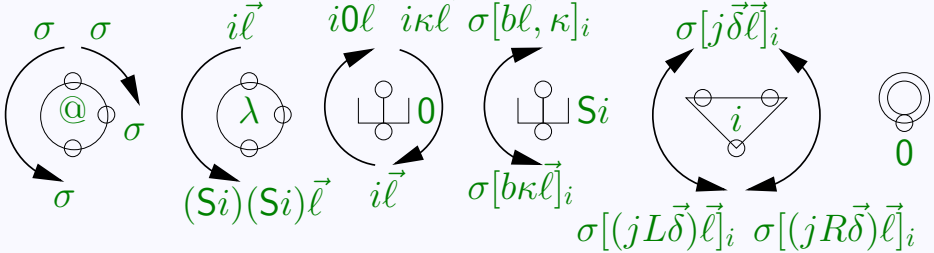
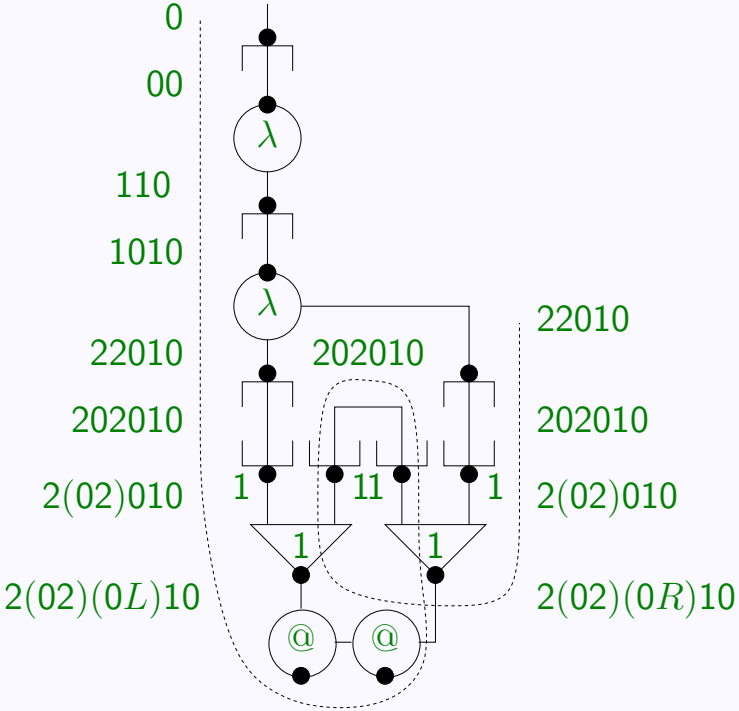
Constraints on PDA transitions:



3.2. Example path read-back



So term of shape $\lambda\lambda?(?(?(?0)))$ (in fact $?$s are 1s)

4. Optimality

Implementation is optimal in sense of Lévy

- No needless work
  via outermost strategy
- No double work
  because of local replication

Why?

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Implementation is optimal in sense of Lévy

- No needless work
  via outermost strategy
- No double work
  because of local replication

Why?

Same abstract algorithm as extant implementations

5. Efficiency

- Prototype implementation lambdascope
- Trivial: 1 day of programming (in Java)
- As efficient as (optimised version of) BOHM (outperforms functional languages on same examples)
- Solves oracle problems (brackets/croissants)
- Many possibilities for optimisation
  closedness, types etc.

6. Future work

- Fast implementaton
- Ideal explicit substitution calculus
  (orthogonal, preservation of termination etc.)
- Type system for graphs
- Extension to higher-order rewriting

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