Abstract Rewriting

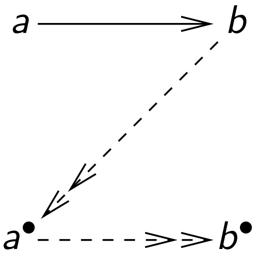
ISR 2008, Obergurgl, Austria

Vincent van Oostrom

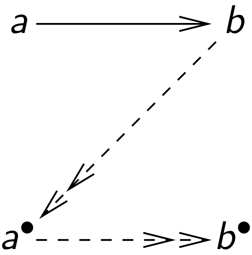
Theoretical Philosophy
Utrecht University
Netherlands

16:00 – 17:30, Mon/Wednesday July 21, ISR 2008

Z

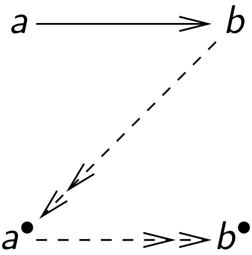


Z



A rewrite relation $\rightarrow$ has the Z-property

Z



A rewrite relation $\rightarrow$ has the Z-property
if there is a map • from objects to objects

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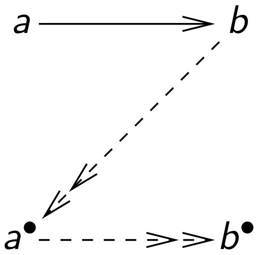
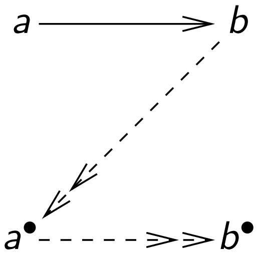


A rewrite relation $\to$ has the Z-property
if there is a map $\bullet$ from objects to objects
such that for any step $a \to b$ from a to b

Z



A rewrite relation $\to$ has the Z-property
if there is a map $\bullet$ from objects to objects
such that for any step $a \to b$ from a to b
there exists a many-step reduction $b \twoheadrightarrow a^\bullet$ from b to $a^\bullet$

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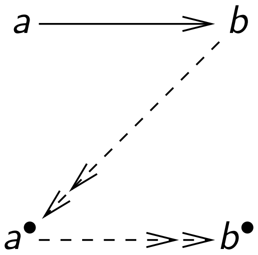
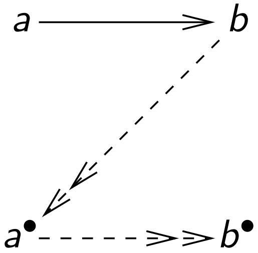


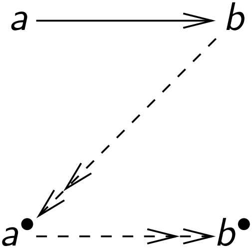
A rewrite relation $\to$ has the Z-property
if there is a map $\bullet$ from objects to objects
such that for any step $a \to b$ from a to b
there exists a many-step reduction $b \twoheadrightarrow a^\bullet$ from b to $a^\bullet$
and there exists a many-step reduction $a^\bullet \twoheadrightarrow b^\bullet$ from $a^\bullet$ to $b^\bullet$

Z



$$\exists \bullet : A \to A, \forall a, b \in A : a \to b \;\Rightarrow\; b \twoheadrightarrow a^\bullet, a^\bullet \twoheadrightarrow b^\bullet$$

Z intuitions



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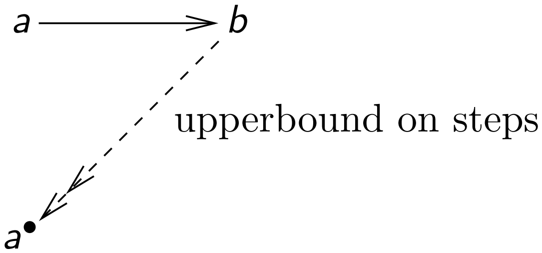

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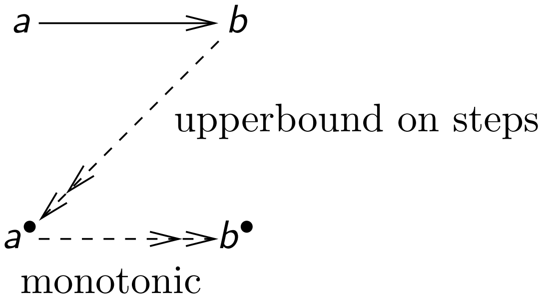

Z $\Rightarrow$ confluence

Theorem

If rewrite relation has the Z-property, then it is confluent

Proof.

□

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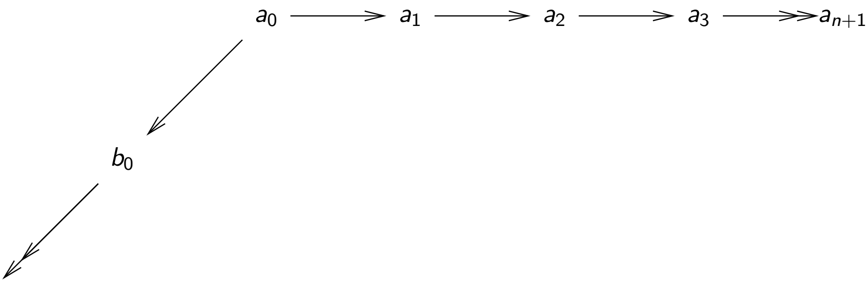

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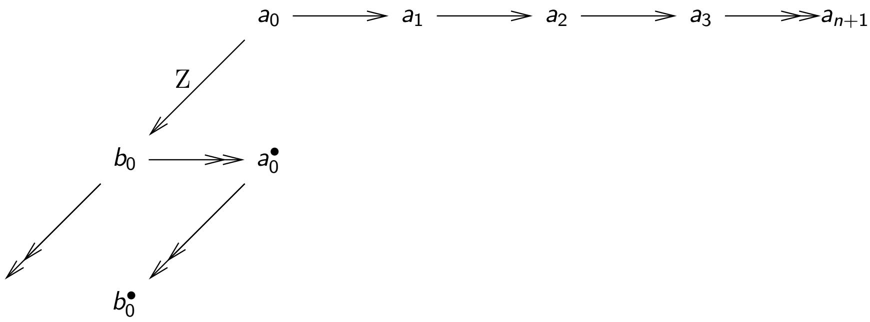

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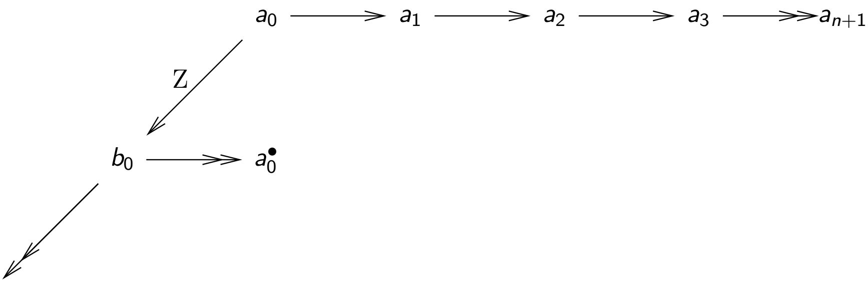

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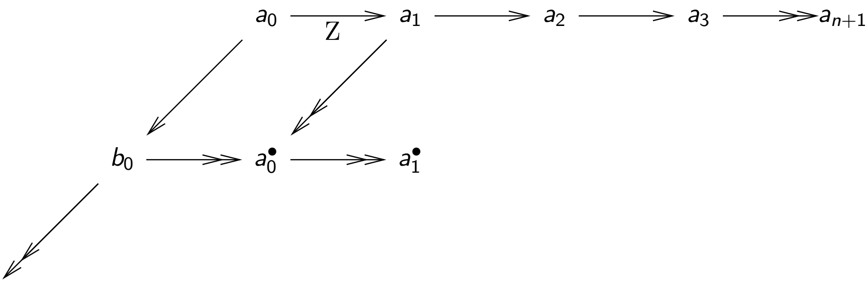

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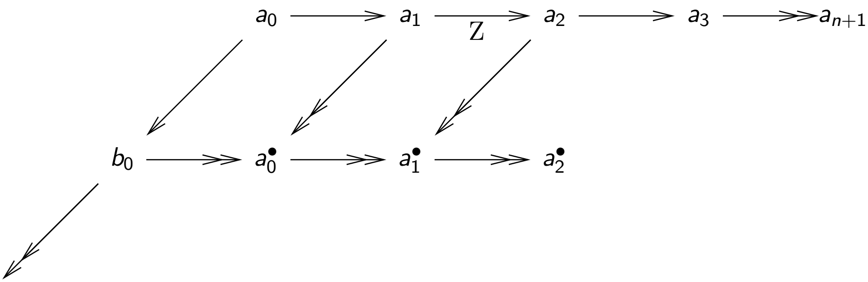

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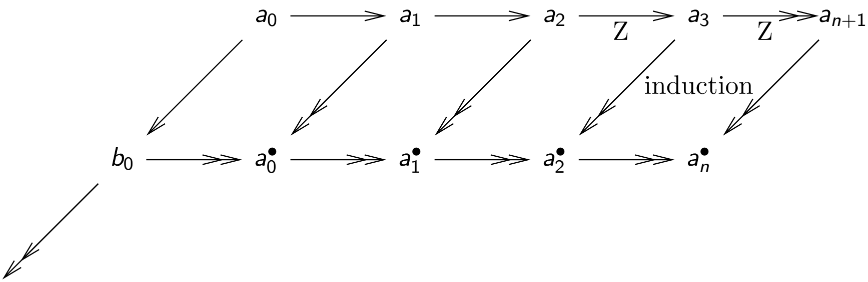

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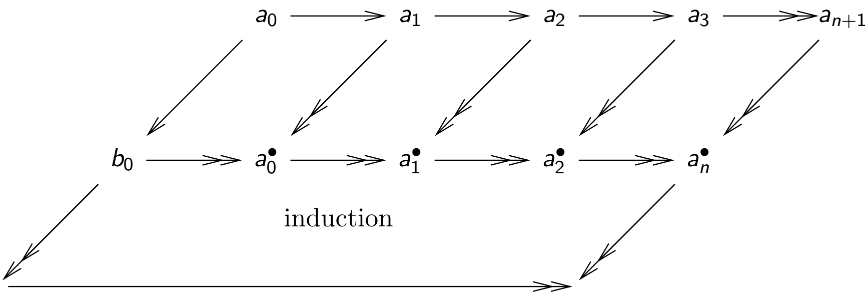


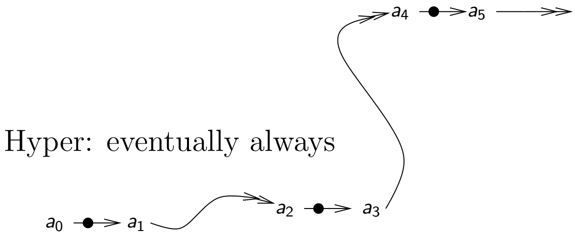
$\square$

Z $\Rightarrow$ $\xrightarrow{\bullet}$ strategy is hyper-cofinal

Definition (•-strategy)

$a \xrightarrow{\bullet} b$ if a is not a normal form and $b = a^\bullet$

Z $\Rightarrow$ $\multimap\!\!\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal



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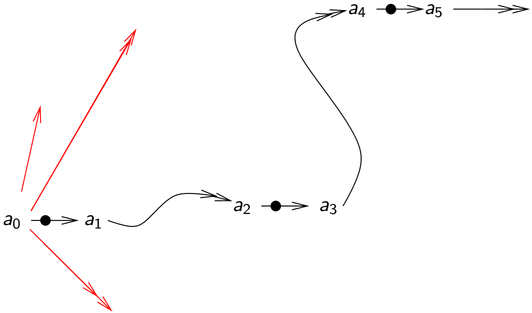

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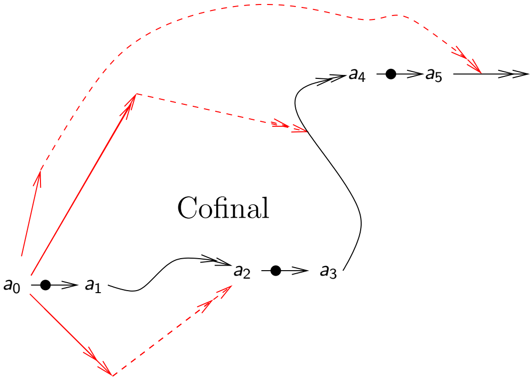

Z $\Rightarrow$ $\multimap\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal

Definition

$\multimap\!\!\bullet\!\!\rightarrow$ hyper-cofinal, if for any reduction which eventually always contains a $\multimap\!\!\bullet\!\!\rightarrow$-step, any co-initial reduction can be extended to reach the first

Z $\Rightarrow$ $\multimap\!\!\!\bullet\!\!\rightarrow$ strategy is hyper-cofinal

hyper-cofinal $\Rightarrow$

- confluent
- (hyper-)normalising
- bullet-fast ...

Z $\Rightarrow$ $\multimap\!\!\!\bullet\!\!\!\rightarrow$ strategy is hyper-cofinal

Theorem

$\multimap\!\!\!\bullet\!\!\!\rightarrow$ *is hyper-cofinal*

Proof.

$\bullet$

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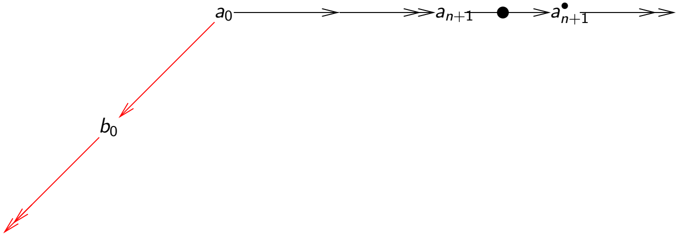

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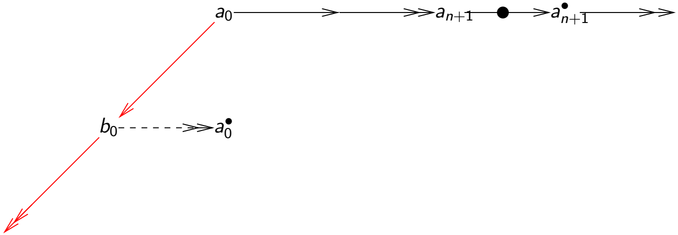

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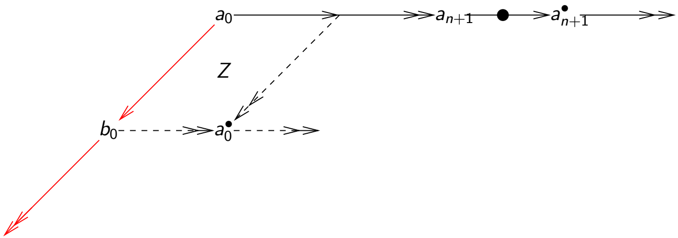

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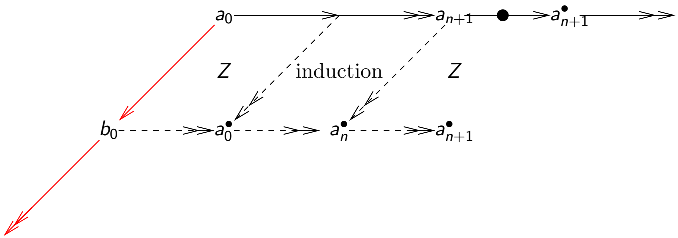

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Z $\Rightarrow$ $\multimap\!\!\!\bullet\!\!\!\to$ strategy is hyper-cofinal

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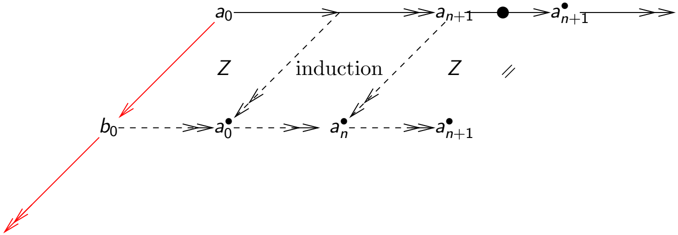

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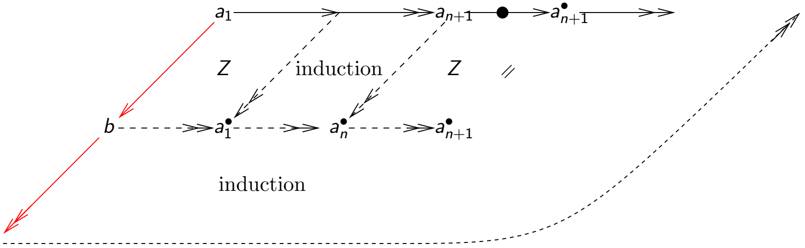

□

Examples

Example: braids

Definition

Braid rewriting: cross adjacent strands, right over left.

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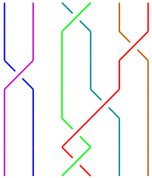

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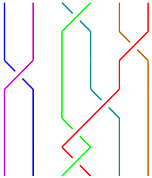
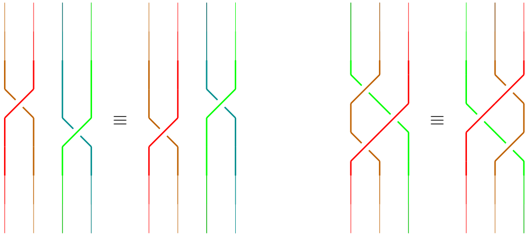


Up to topological equivalence:



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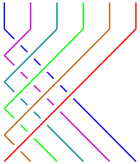

Example: braids

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Example: braids

Theorem

Braid rewriting has the Z-property, for • *full crossing*

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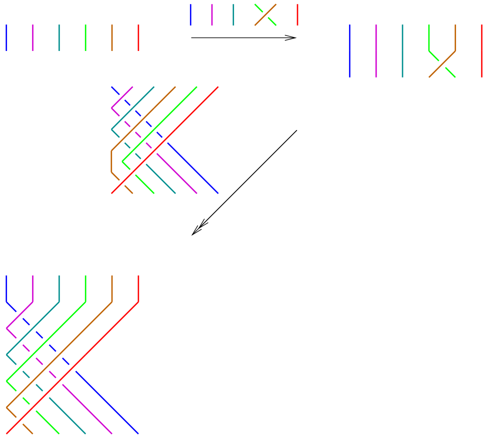

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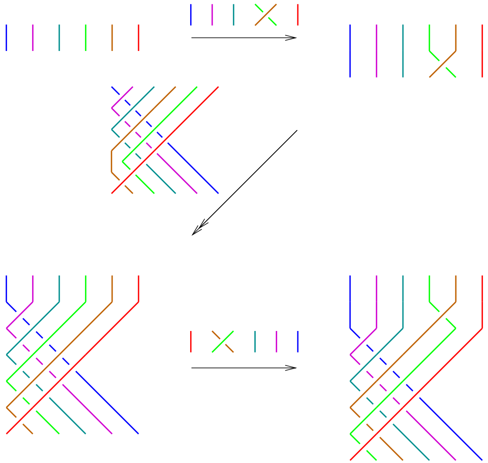

Example: self-distributivity

Definition

Self-distributivity, rewrite relation generated by $xyz \longrightarrow xz(yz)$

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Some models:

- ACI operations
- take middle of points in space
- substitution

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Some models:

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In depth: Braids and Self-distributivity (Dehornoy 2000)

Example: self-distributivity

Theorem

Self-distributivity has the Z-property, for $\bullet$ *full distribution:*

$$x^\bullet = x \qquad (ts)^\bullet = t^\bullet[s^\bullet]$$

with $t[s]$ *uniform distribution of* s *over* t*:*

$$t[x_1{:=}x_1 s, x_2{:=}x_2 s, \ldots]$$

Example: self-distributivity

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Example

- $(xy)^\bullet = x[y] = x[x{:=}xy] = xy$;
- $(xyz)^\bullet = (xy)[x{:=}xz, y{:=}yz] = xz(yz)$.

Example: self-distributivity

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Proof.

By induction on t:

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Proof.

By induction on t:

- (Sequentialisation) $\quad ts \twoheadrightarrow t[s]$;

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Proof.

By induction on t:

- (Sequentialisation) $ts \twoheadrightarrow t[s]$;
- (Substitution) $t[s][r] \twoheadrightarrow t[r][s[r]]$

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Self-distributivity has the Z-property, for $\bullet$ full distribution:

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with $t[s]$ uniform distribution of s over t:

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Proof.

By induction on t:

- (Sequentialisation) $ts \twoheadrightarrow t[s]$;
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- (Self) $t \twoheadrightarrow t^\bullet$;

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Self-distributivity has the Z-property, for • full distribution:

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with $t[s]$ uniform distribution of s over t:

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Proof.

By induction on t:

- (Sequentialisation) $ts \twoheadrightarrow t[s]$;
- (Substitution) $t[s][r] \twoheadrightarrow t[r][s[r]]$
- (Self) $t \twoheadrightarrow t^\bullet$;
- (Z) $s \twoheadrightarrow t^\bullet \twoheadrightarrow s^\bullet$, if $t \to s$.

□

Example: normalising and confluent relations

Theorem

Normalising and confluent relations have the Z-property, for $\bullet$ the full reduction map (map to normal form).

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Proof.

If $a \to b$, then $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$ since b reduces to its normal form $b^\bullet$ (normalisation) which is the same as the normal form $a^\bullet$ of a (confluence). □

Example: normalising and confluent relations

Theorem

Normalising and confluent relations have the Z-property, for $\bullet$ the full reduction map (map to normal form).

Proof.

If $a \to b$, then $b \twoheadrightarrow a^{\bullet} \twoheadrightarrow b^{\bullet}$ since b reduces to its normal form $b^{\bullet}$ (normalisation) which is the same as the normal form $a^{\bullet}$ of a (confluence). □

Corollary

Z-property for typed λ-calculi (by confluence and termination)

Example: λ-calculus

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ *has the Z-property, for* $\bullet$ *full development contracting all redexes present:*

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \text{otherwise}
\end{array}
$$

Example: λ-calculus

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$$

Example

- $I^\bullet = I$; $(I = \lambda x.x)$
- $(I(II))^\bullet = I$, $(III)^\bullet = II$;
- $((\lambda xy.x)zw)^\bullet = (\lambda y.z)w$;
- $((\lambda xy.Iyx)zI)^\bullet = (\lambda y.yz)I$;

Example: λ-calculus

Theorem

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Proof.

By induction on M:

- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;

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Example: λ-calculus

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- (Self) $M \twoheadrightarrow M^\bullet$;

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Example: λ-calculus

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Proof.

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- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and

□

Example: λ-calculus

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- (Substitution) $M[y{:=}P][x{:=}N] = M[x{:=}N][y{:=}P[x{:=}N]]$;
- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and
- (Z) $M \to N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$.

□

Example: λ-calculus

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$(\lambda x.M)N \to M[x{:=}N]$ has the Z-property, for $\bullet$ full development contracting all redexes present:

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- (Self) $M \twoheadrightarrow M^\bullet$;
- (Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and
- (Z) $M \to N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$.

□

Same method works for all orthogonal first/higher-order TRSs

Example: weakly orthogonal term rewriting systems

Definition

Rewrite system is weakly orthogonal, if only trivial critical pairs.

Example: weakly orthogonal term rewriting systems

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Example

- λ-calculus with β and $\eta : \lambda x.Mx \to M$, if $x \notin M$;
- predecessor/successor $\quad S(P(x))) \to x \quad P(S(x)) \to x$;
- parallel-or.

Example: weakly orthogonal term rewriting systems

Theorem

Weakly orthogonal first/higher-order term rewrite systems have the Z-property, for $\bullet$ full inside-out development

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Proof.

$$c(x) \to x$$

$$f(f(x)) \to f(x)$$

$$g(f(f(f(x)))) \to g(f(f(x)))$$

Then $g(f(f(c(f(f(x)))))) \to g(f(f(f(f(x)))))$ gives Z:
$g(f(f(c(f(f(x))))))^\bullet = g(f(f(x))) = g(f(f(f(f(x)))))^\bullet$ □

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Outside-in not monotonic: not $g(f(f(x))) \twoheadrightarrow g(f(f(f(x))))$!

Non-examples

Some properties of $\bullet$s

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$;

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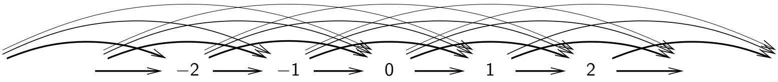
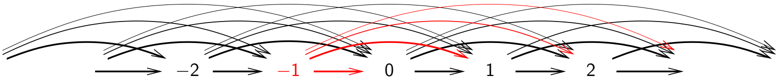
Used to get ideas about (confluent) systems which do not have Z

$\mathbb{Z}$ does not have Z



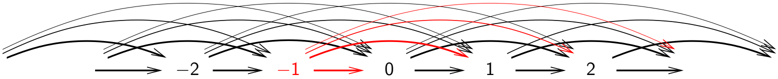
ℤ does not have Z



for given integer, no upperbound on steps from it

$\mathbb{Z}$ does not have Z

not finitely branching, no finite TRS



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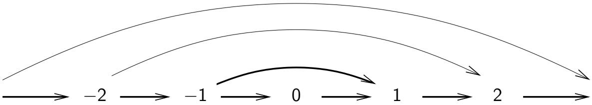

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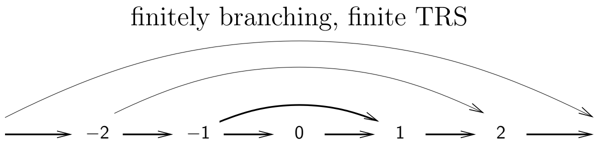

$n(x) \to p(x)$ $n(1) \to 0$ $0 \to p(1)$

$n(s(x)) \to n(x)$

$p(x) \to p(s(x))$

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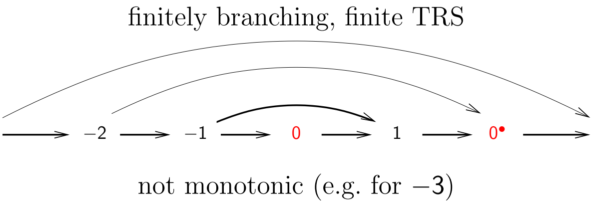

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$$\longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow$$

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finitely branching, finite TRS, no transitivity

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Z trivial ($i^\bullet = i + 1$)

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finitely branching, finite TRS, no transitivity

$$\longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow$$

Z trivial ($i^\bullet = i + 1$)

Examples show:

- confluent $\not\Rightarrow$ Z
- transitivity might be harmful

Exercises on Z

Exercise

(favourite Z?)
Does your favourite confluent rewrite system have the Z-property?

Exercises on Z

Exercise

(orthogonal systems)

- *Verify that for the λ-calculus the full-development bullet map indeed has the Z-property, by verifying (Substition), (Self), (Rhs), and (Z)*
- *Inductively define a full-development function on terms, for orthogonal TRSs, and verify that it does have the Z-property for these TRSs.*

Exercises on Z

Exercise

(superdevelopments)
Adapt the proof of the first item of the previous exercise to show:

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ *has the Z-property, for* $\bullet$ *full super-development contracting all redexes present or upward created:*

$$
\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is a term, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \text{otherwise}
\end{array}
$$

Example

- $I^\bullet = I$; $(I = \lambda x.x)$
- $(I(II))^\bullet = I$, $(III)^\bullet = I$;
- $((\lambda xy.x)zw)^\bullet = z$;
- $((\lambda xy.Iyx)zI)^\bullet = Iz$

Exercises on Z

Exercise

*(Z vs. decreasing diagrams)**

Can you prove confuence of braids or λ-calculus or orthogonal TRSs using decreasing diagrams (other than via the completeness result)?

Exercises on Z

Exercise

*($\beta\overline{\eta}$)**

Does the λ-calculus with β-reduction and restricted η-expansion, i.e. the inverse of η-reduction restricted so that it never creates a β-redex (generates a new β-redex), have the Z-property?

Exercises on Z

Exercise

(properties of $\bullet$s)

Prove the properties of $\bullet$ as given on page 75 of these slides.

Summary of first lecture

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- Decreasing diagrams (complete) for confluence

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- Decreasing diagrams (complete) for confluence
- Z-property for confluence and cofinality