Z

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Theoretical Philosophy
Universiteit Utrecht
The Netherlands
this month at LIX

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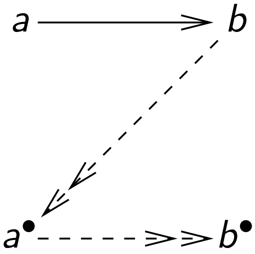
Z

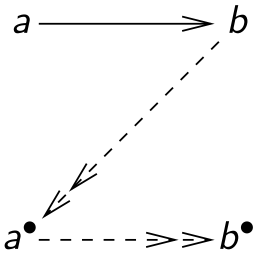
Z for λ-calculi

Z or not

Z

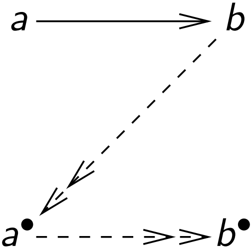


Z



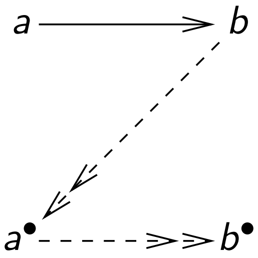
A rewrite relation $\rightarrow$ has the Z-property

Z



A rewrite relation → has the Z-property
if there is a map • from objects to objects

Z



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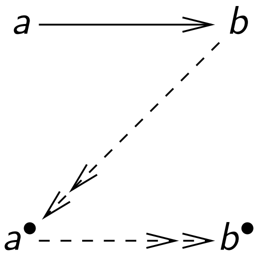
Z



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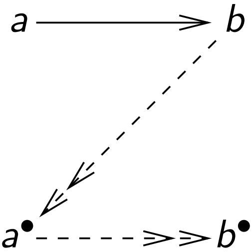
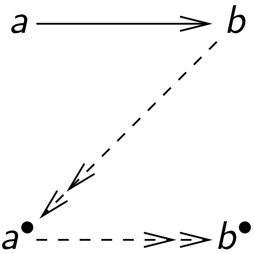


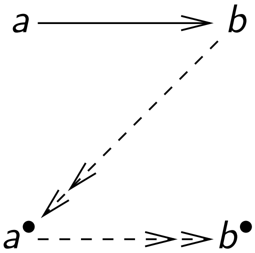
A rewrite relation $\to$ has the Z-property
if there is a map $\bullet$ from objects to objects
such that for any step from a to b
there is a reduction from b to $a^\bullet$
and there is a reduction from $a^\bullet$ to $b^\bullet$

Z



$$\exists \bullet : A \to A, \forall a, b \in A : a \to b \;\Rightarrow\; b \twoheadrightarrow a^\bullet, a^\bullet \twoheadrightarrow b^\bullet$$

Z



This talk: (short) history, interest, and (non-)examples

self-distributivity: $xyz \longrightarrow xz(yz)$

Theorem

self-distributivity has the Z-property

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Map

$$\begin{aligned} x^\bullet &= x \\ (ts)^\bullet &= t^\bullet[x_1{:=}x_1 s^\bullet, x_2{:=}x_2 s^\bullet, \ldots] \end{aligned}$$

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Example

$$(xy)^\bullet = xy$$

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Example

$$(xy)^\bullet = xy$$

$$(xyz)^\bullet = xz(yz)$$

Proof.

This works: Braids and Self-distributivity (Dehornoy 2000) □

Z property for semi-complete rewrite relations

Theorem

Every normalising and confluent rewrite relation has the Z-property

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Let $\bullet$ map every object to its normal form
(exists by normalisation, unique by confluence)

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If $a \to b$, then $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$ since b reduces to its normal form $b^\bullet$ which is the same as the normal form $a^\bullet$ of a. □

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Z-property for β-reduction in typed λ-calculi by using meta-theory

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Z-property for β-reduction in typed λ-calculi by using meta-theory

Here reverse: Z-property to establish meta-theory

Z $\Rightarrow$ confluence

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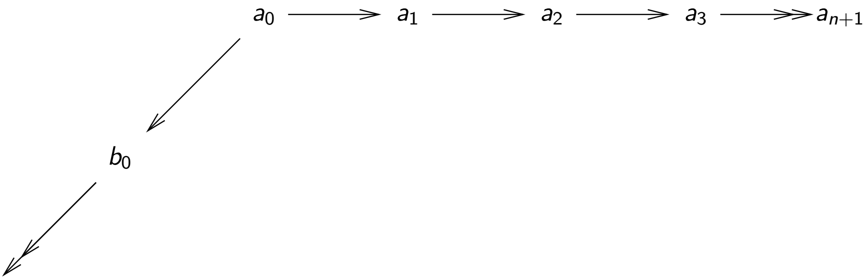

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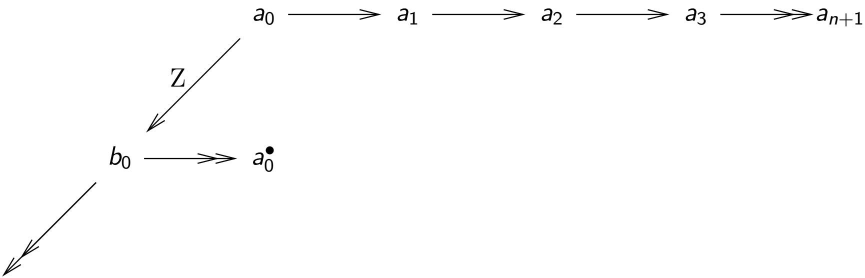

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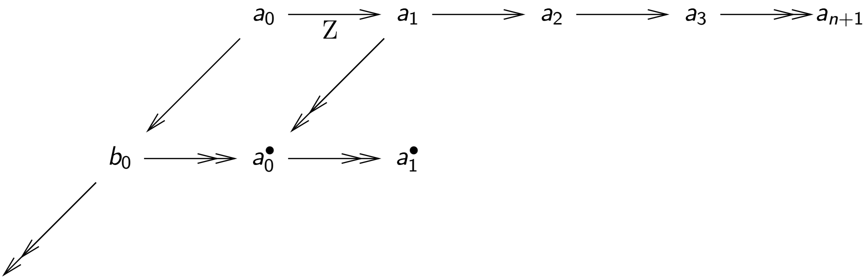

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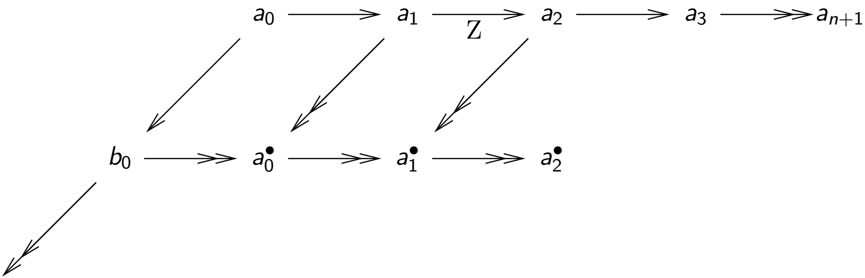

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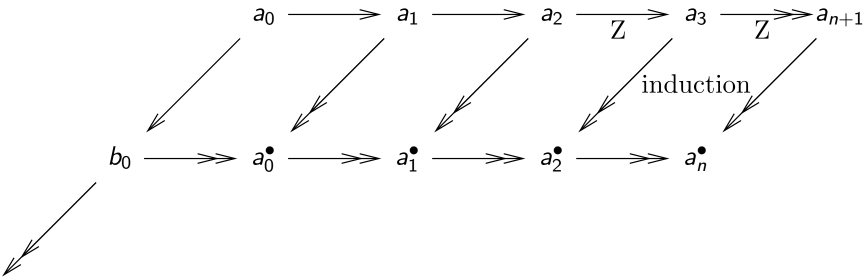

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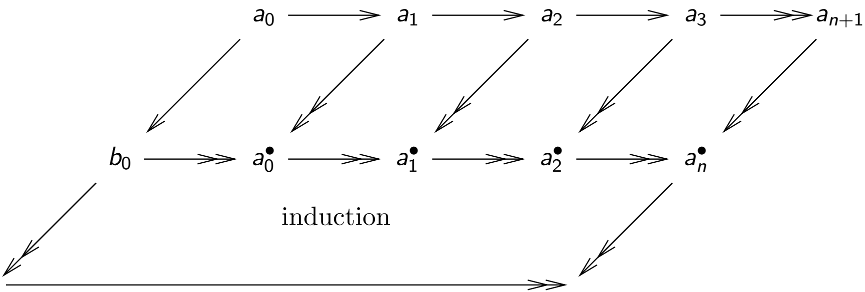


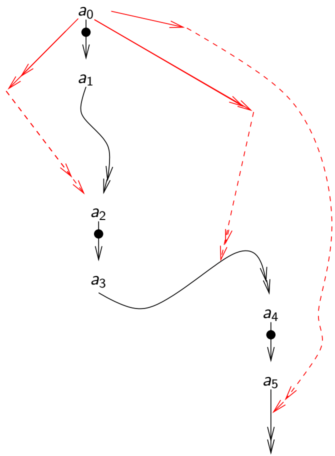
□

Z $\Rightarrow$ hyper-cofinal

Definition (•-strategy)

$a \mathrel{-\!\!\bullet\!\!\!\rightarrow} b$ if a is not a normal form and $b = a^\bullet$

Z $\Rightarrow$ hyper-cofinal



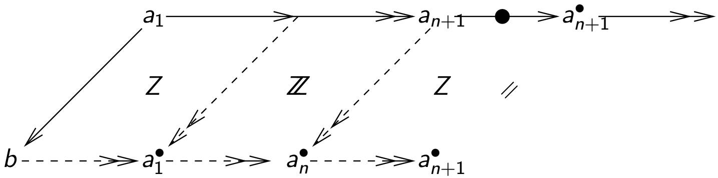
Hyper-cofinality of $\multimap\!\!\!\bullet\!\!\rightarrow$:
for any reduction which eventually always contains $\multimap\!\!\!\bullet\!\!\rightarrow$-step
any co-initial reduction can be extended to reach the first

Z ⇒ hyper-cofinal

Theorem

$\twoheadrightarrow\!\!\!\!\bullet$ *is hyper-cofinal*

Proof.



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$\multimap\!\!\!\to$ *is hyper-cofinal*

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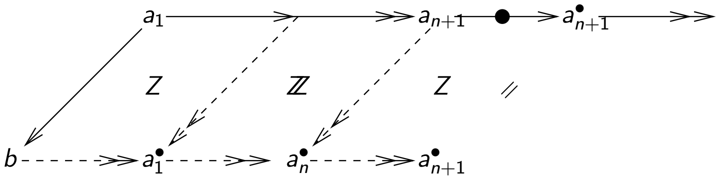

□

Summary: $\multimap\!\!\!\to$ confluent, (hyper-)normalising, bullet-fast,

β has Z

Theorem

$(\lambda x.M)N \to M[x{:=}N]$ *has the Z-property for* λ*-calculus*

β has Z

Theorem

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Proof.

Full-development map (contract all redexes present)

$$
\begin{aligned}
x^\bullet &= x \\
(\lambda x.M)^\bullet &= \lambda x.M^\bullet \\
(MN)^\bullet &= M'[x{:=}N^\bullet] \quad \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x.M' \\
&= M^\bullet N^\bullet \quad \text{otherwise}
\end{aligned}
$$

Example

- $I^\bullet = I$; $(I = \lambda x.x)$
- $I(II)^\bullet = I$, $III^\bullet = II$;
- $(\lambda xy.x)zw^\bullet = (\lambda y.z)w$;
- $((\lambda xy.Iyx)zI)^\bullet = (\lambda y.yz)I$;

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\end{array}
$$

(Self) $M \twoheadrightarrow M^\bullet$;

(Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and

(Z) $M \to N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$.

each by induction and cases on M. □

β has Z

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Full-superdevelopment map (redexes present or upward-created)

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Replace 'is an abstraction' by 'is a term' in development proof. □

β has Z

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Replace 'is an abstraction' by 'is a term' in development proof. □

Moral: possibly more than one witnessing map for Z-property

Comparison

- Dehornoy:
  Z-property of $\to$ for $\bullet$;
- Tait–Martin Löf:
  $\to \subseteq \mathrel{-\!\!\circ\!\!\to} \subseteq \twoheadrightarrow$ and diamond ($\Diamond$) property of $\mathrel{-\!\!\circ\!\!\to}$;
- Takahashi:
  $\to \subseteq \mathrel{-\!\!\circ\!\!\to} \subseteq \twoheadrightarrow$ and angle ($\langle$) property of $\mathrel{-\!\!\circ\!\!\to}$ for $\bullet$.

Comparison

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Mnemonics: $\mathrel{-\!\!\bullet\!\!\!\rightarrow}$ is *full* $\mathrel{-\!\!\circ\!\!\!\rightarrow}$

Comparison

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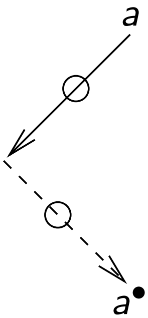
How do Z, $\Diamond$, $\langle$ relate?

⟨ ⇔ Z



Angle property

$\langle \Leftrightarrow Z$

Theorem

for any map $\bullet$, $Z \Leftrightarrow$ *both* $\rightarrow \subseteq \multimap\!\!\rightarrow \subseteq \twoheadrightarrow$ *and* $\langle$

Proof.

$\square$

$\langle \;\Leftrightarrow\; Z$

Theorem

for any map $\bullet$, $Z \Leftrightarrow$ *both* $\rightarrow \subseteq \multimapdotbothvert\!\!\rightarrow \subseteq \twoheadrightarrow$ *and* $\langle$

Proof.

(If)

$$a \longrightarrow b$$

$\square$

$\langle \;\Leftrightarrow\; Z$

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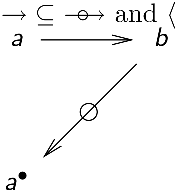
(If)



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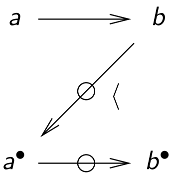

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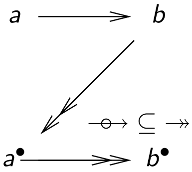

□

$\langle \ \Leftrightarrow \ \mathrm{Z}$

Theorem

for any map $\bullet$, $Z \Leftrightarrow$ both $\longrightarrow \ \subseteq \ \mathrel{-\!\!\ominus\!\!\rightarrow} \ \subseteq \ \twoheadrightarrow$ and $\langle$

Proof.

(only if) Def. $a \mathrel{-\!\!\ominus\!\!\rightarrow} b$ if b between a and $a^\bullet$, i.e. $a \twoheadrightarrow b \twoheadrightarrow a^\bullet$:

- $a \longrightarrow b \ \Rightarrow \ b \twoheadrightarrow a^\bullet \ \Rightarrow \ \longrightarrow \ \subseteq \ \mathrel{-\!\!\ominus\!\!\rightarrow}$.
- $a \mathrel{-\!\!\ominus\!\!\rightarrow} b \ \Rightarrow \ a \twoheadrightarrow b \ \Rightarrow \ \mathrel{-\!\!\ominus\!\!\rightarrow} \ \subseteq \ \twoheadrightarrow$.
- Suppose $a \mathrel{-\!\!\ominus\!\!\rightarrow} b$.
  - $a \twoheadrightarrow b \twoheadrightarrow a^\bullet$ by definition of $\mathrel{-\!\!\ominus\!\!\rightarrow}$.
  - $a \twoheadrightarrow b \ \Rightarrow \ a^\bullet \twoheadrightarrow b^\bullet$ (monotonicity of $\bullet$) by Z
  - $b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$ so $b \mathrel{-\!\!\ominus\!\!\rightarrow} a^\bullet$ by definition of $\mathrel{-\!\!\ominus\!\!\rightarrow}$.

$\square$

$\lambda\sigma$

Theorem

$\lambda\sigma$ has Z property

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Proof.

Map: first σ-normalise ($\triangleright$) then *Beta*-full development ($\multimap\!\!\!\!\bullet\!\!\rightarrow$)

$\lambda\sigma$

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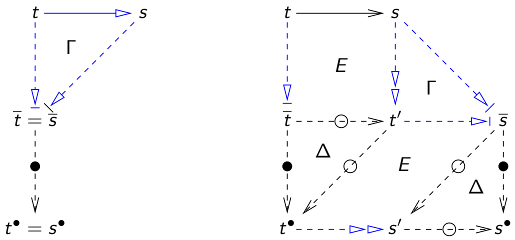

- Δ : angle property of $\multimap\!\!\!\bullet\!\!\rightarrow$
- E: *Beta* commutes with σ-normalisation
- Γ: σ is terminating and confluent

$\lambda\beta\eta$ has Z property

Theorem

Weakly orthogonal rewrite system $\Rightarrow$ *Z property*

Proof.

Map:

Contract maximal set of non-overlapping redexes inside-out

Example

$$c(x) \to x$$

$$f(f(x)) \to f(x)$$

$$g(f(f(f(x)))) \to g(f(f(x)))$$

□

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$$g(f(f(c(f(f(x))))))^\bullet = g(f(f(x))) = g(f(f(f(f(x)))))^\bullet$$

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□

Outside-in (Takahashi) does not give Z (in general)!

$g(f(f(c(f(f(x)))))) \to g(f(f(f(f(x)))))$ holds. . .

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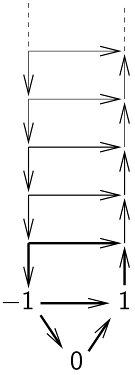
... not $g(f(f(x))) \twoheadrightarrow g(f(f(f(x))))$!

Some more consequences of Z

- if $a \twoheadrightarrow b$ then $a^\bullet \twoheadrightarrow b^\bullet$ (monotonicity)
- $\rightarrow$ has Z-property iff $\rightarrow^=$ has (IZ-property)
- If $\bullet_1$, $\bullet_2$ have the Z-property for $\rightarrow$, so does their composition $\bullet_1 \circ \bullet_2$. Moreover, $a^{\bullet_i} \twoheadrightarrow (a^{\bullet_2})^{\bullet_1}$

May be used to get ideas about systems which do not have Z

Confluence $\not\Rightarrow$ Z



Easy to turn into a finite term rewriting system

Conclusions

- Surprising outsider (Dehornoy) input: simple yet not known

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- Conjecture: β with restricted η-expansion does not have Z

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- Conjecture: β with restricted η-expansion does not have Z
- Problem: characterize systems having Z-property