Triangulation

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HZ: Technische Universiteit Eindhoven and Radboud Universiteit Nijmegen

TeReSe, Eindhoven, June 17, 2011

Triangulation

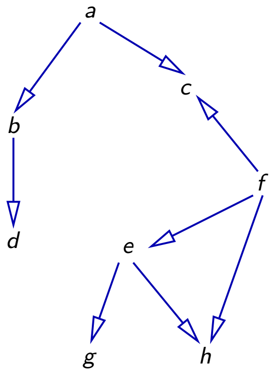
Completion

Triangulated

Completed

Co-nclusion

Triangulation example



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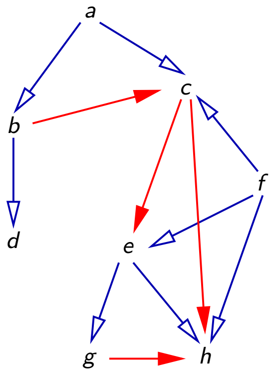

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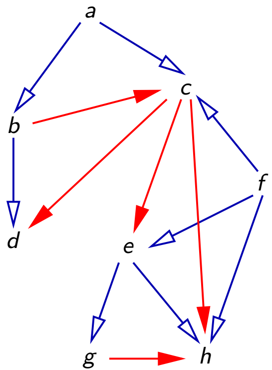

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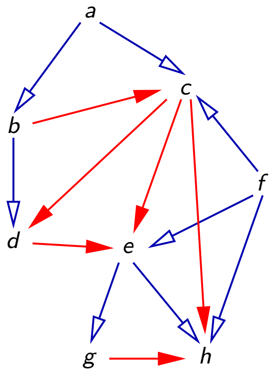

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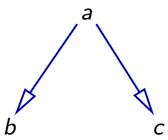

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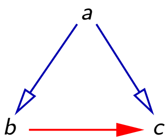

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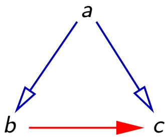

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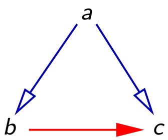

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Definition

triangulation of $\triangleright$ with respect to R is $\rightarrow = \bigcup_{n \geq 1} \rightarrow_n$ with

- $\rightarrow_1 = \triangleright$
- $b \rightarrow_{n+m+1} c$ if $b \leftarrow_n a \rightarrow_m c$ and $b \; R \; c$
  but no triangle yet: b not $\bigcup_{1 \leq k \leq n+m} \leftrightarrow_k^=$-related to c

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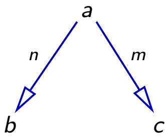

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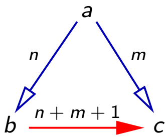

depends on a total relation R to determine direction of $\blacktriangleright$

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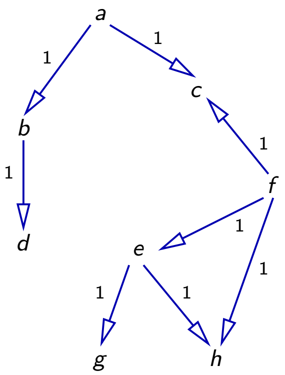

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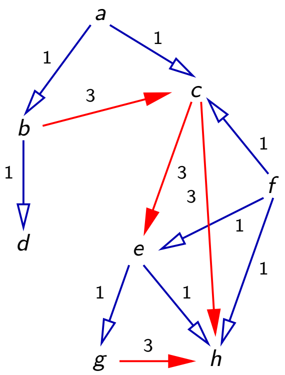

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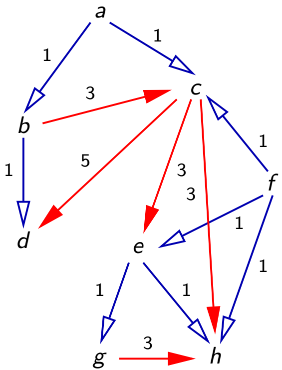

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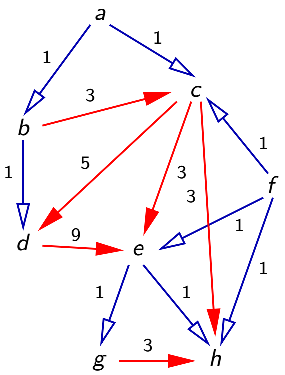

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Lemma

triangulation yields confluification

Proof.

suppose triangulating $\triangleright$ with respect to R yields $\to$

then $\to^=$ has the diamond property $\square$

Confluification as Completion?

Definition

completion is confluification that preserves termination

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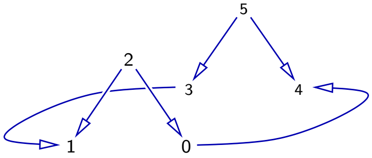
Counterexample

triangulating $\triangleright$ *with respect to* $>$ *loses termination*



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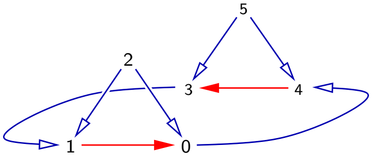

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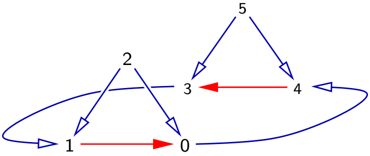

need benign interaction between $\triangleright$ and R

Benign interaction 1: $\triangleright \cup R$ terminating

Theorem

triangulation is completion if $\triangleright \cup R$ *terminating*

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Theorem

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Proof.

$\rightarrow = \bigcup_{n \geq 1} \rightarrow_n \subseteq \triangleright \cup R$
(by construction $\rightarrow_1 = \triangleright$ and $\rightarrow_{n+1} \subseteq R$ for $n \geq 1$)
hence termination of $\rightarrow$ follows from termination of $\triangleright \cup R$ □

Benign interaction 2: $\triangleright$ co-deterministic, R terminating

Definition

- $\triangleright$ is co-P if its converse $\triangleleft$ is P
- $\triangleright$ is deterministic if $a \triangleright b$ and $a \triangleright c$ implies $b = c$

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Example

- β-reduction in λ-calculus is confluent but not co-confluent
- rewrite relation on a finite set is terminating iff co-terminating
- trees with steps towards root are deterministic
  trees with steps towards leaves are co-determistic

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Δ property

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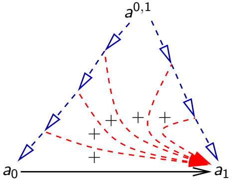
if triangulating co-determinstic $\triangleright$ *yields* $\rightarrow$ *by adjoining* $\blacktriangleright$*-steps then property* Δ *holds:*



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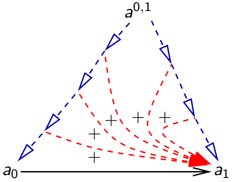

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By well-founded induction on n for $\rightarrow_n$ and gluing Δs □

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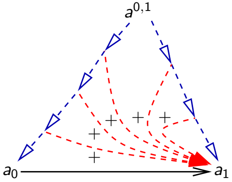

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under the same assumptions

Corollary

$\triangleright^{+} \cdot \blacktriangleright \subseteq \triangleright^{+} \cup (\blacktriangleright \cdot \twoheadrightarrow)$

Lazy Commutation

Theorem (Doornbos & von Karger)

if $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup (\blacktriangleright \cdot \twoheadrightarrow)$ *with* $\rightarrow = \triangleright \cup \blacktriangleright$
then termination of $\triangleright$ *and* $\blacktriangleright$ *implies termination of* $\rightarrow$

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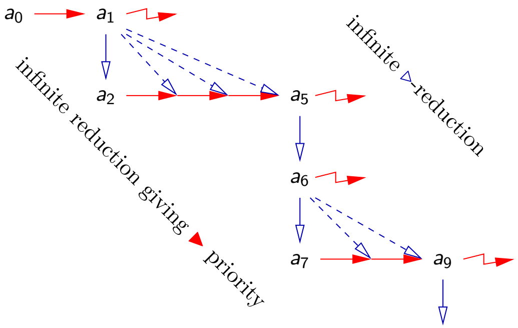
Proof.

Ramsey-like construction of infinite $\triangleright$-reduction from $\rightarrow$-reduction



Doing away with induction in triangulation?

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- $\blacktriangleright \subseteq \leftarrow \cdot \rightarrow$ (triangle creation)

Confluence by Triangulatedness

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Corollary (Total Triangle)

$\twoheadrightarrow$ *is total on reductions peaks*
if $\leftarrow \cdot \rightarrow \subseteq \leftrightarrow^{=}$ *(triangulated)*

Termination by Finiteness

Lemma

$\to$ *is terminating if set of objects finite,* $\triangleright$ *and* $\blacktriangleright$ *terminating, and*

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Proof.

because of finitenes, termination equivalent to acyclicity

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by contradiction: assume a cycle with minimal weight (multiset of objects on cycle ordered my multiset extension of $\blacktriangleleft$)

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$$a \triangleright b \blacktriangleright c \twoheadrightarrow a$$

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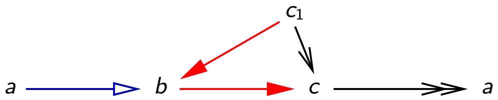

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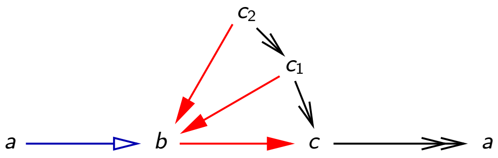

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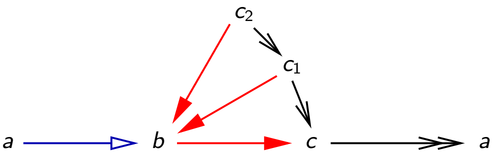

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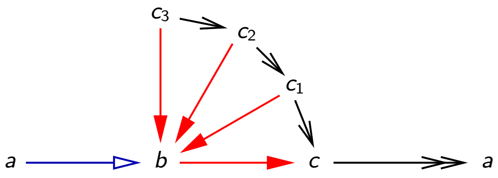

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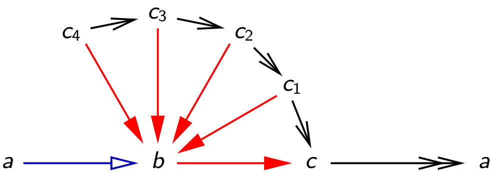

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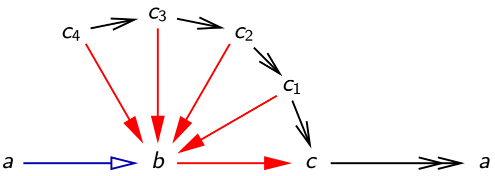

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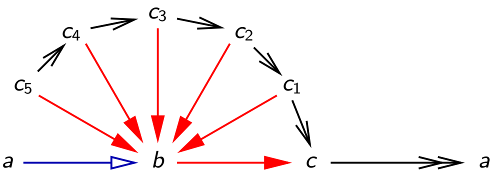

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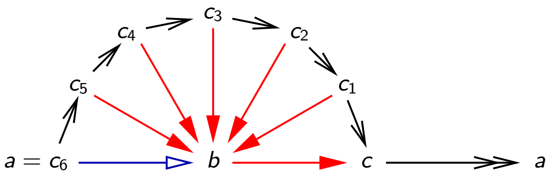

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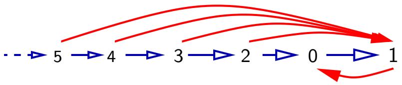
Counterexample

Loss of termination by infinite $\triangleright$-expansion



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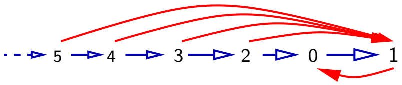

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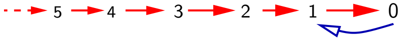

Termination by Shallow Triangles

Observations on triangulation of co-deterministic $\triangleright$:

Termination by Shallow Triangles

Observations on triangulation of co-deterministic $\triangleright$:

- $\blacktriangleright \subseteq (\triangleleft \cdot \triangleright) \cup (\blacktriangleleft \cdot \triangleright) \cup (\triangleleft \cdot \blacktriangleright)$ (shallow triangle)

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- $\blacktriangleright \subseteq (\triangleleft \cdot \triangleright) \cup (\blacktriangleleft \cdot \triangleright) \cup (\triangleleft \cdot \blacktriangleright)$ (shallow triangle)
- $\blacktriangleright \subseteq \triangleleft\!\!\triangleleft \cdot ((\triangleleft \cdot \triangleright) - \mathrm{id}) \cdot \triangleright\!\!\triangleright$ (bifurcation)

Lemma

$\rightarrow$ is terminating if $\triangleright$ and $\blacktriangleright$ terminating, and

- *$\rightarrow = \triangleright \cup \blacktriangleright$ (adjoin)*
- *$\triangleright$ co-deterministic (co-determinism)*
- $\blacktriangleright \subseteq ((\triangleleft \cdot \triangleright) \cup (\blacktriangleleft \cdot \triangleright) \cup (\triangleleft \cdot \blacktriangleright)) \cap (\triangleleft\!\!\triangleleft \cdot ((\triangleleft \cdot \triangleright) - \mathrm{id}) \cdot \triangleright\!\!\triangleright)$

Termination by co-conditions

Lemma

$\rightarrow$ is terminating if $\triangleright$ and $\blacktriangleright$ terminating, and

- *$\rightarrow = \triangleright \cup \blacktriangleright$ (adjoin)*
- *$\triangleright$ deterministic (determinism)*
- *$\blacktriangleright \subseteq \rightarrow \cdot \leftarrow$ (triangle creation)*

Termination by co-conditions

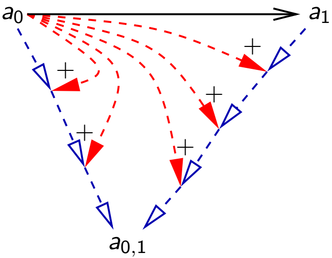
Lemma

$\rightarrow$ is terminating if $\triangleright$ and $\blacktriangleright$ terminating, and

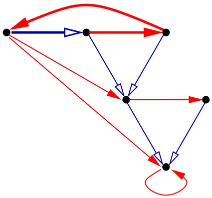
- $\rightarrow = \triangleright \cup \blacktriangleright$ *(adjoin)*
- $\triangleright$ *deterministic (determinism)*
- $\blacktriangleright \subseteq \rightarrow \cdot \leftarrow$ *(triangle creation)*

Proof.

based essentially on ∇-property:



Puzzle



Consider a city with Red (▶) and Blue (▷) buslines

- Blue buses are *deterministic*, i.e. the next stop of a Blue bus (if it can go anywhere at all) is completely determined by the stop it's currently at;
- Red buses can be *triangulated*, i.e. if a Red bus can go directly from stop a to stop b, then there is a stop c such that one can go directly from both a and b to c, in each case by either taking a Red or a Blue bus.

Show that if one can make an infinite trip using buses of either company, then one can make an infinite trip using buses of one and the same company only.

- triangles vs squares
- applications??