Z

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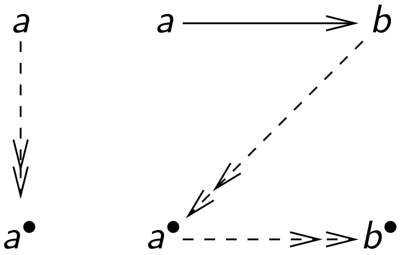
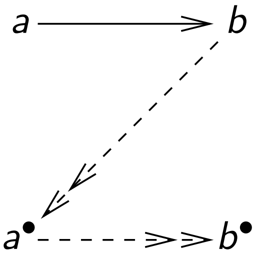
Dehornoy's I Z



$\exists \bullet : A \to A, \forall a, b \in A$:

$a \twoheadrightarrow a^\bullet \& a \to b \Rightarrow b \twoheadrightarrow a^\bullet, a^\bullet \twoheadrightarrow b^\bullet$

Z



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$a \to b \;\Rightarrow\; b \twoheadrightarrow a^\bullet, a^\bullet \twoheadrightarrow b^\bullet$

Z $\Leftrightarrow$ I Z

Consider the reflexive closure of $\rightarrow$

Self-distributivity

$$xyz \longrightarrow xz(yz)$$

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$$\begin{aligned} x^{\bullet} &= x \\ (ts)^{\bullet} &= t^{\bullet}[x_1{:=}x_1 s^{\bullet}, x_2{:=}x_2 s^{\bullet}, \ldots] \end{aligned}$$

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$$(xy)^\bullet = xy$$

$$(xyz)^\bullet = xz(yz)$$

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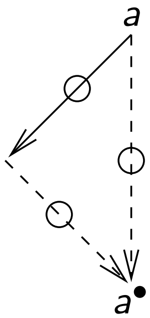
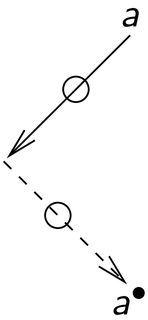
See: Braids and Self-distributivity (Dehornoy, 2000)

Triangle



$$\exists \bullet : A \to A, \exists \mathrel{-\!\!\circ\!\!\to} \; \to \; \subseteq \; \mathrel{-\!\!\circ\!\!\to} \; \subseteq \; \twoheadrightarrow, \forall a \in A :$$
$$a \mathrel{-\!\!\circ\!\!\to} a^{\bullet} \& a \mathrel{-\!\!\circ\!\!\to} b \Rightarrow b \mathrel{-\!\!\circ\!\!\to} a^{\bullet}$$

<



$$\exists \bullet : A \longrightarrow A, \exists \mathrel{\circ\!\!\!\!\longrightarrow}\ \longrightarrow\ \subseteq\ \mathrel{\circ\!\!\!\!\longrightarrow}\ \subseteq\ \twoheadrightarrow, \forall a \in A :$$
$$a \mathrel{\circ\!\!\!\!\longrightarrow} b \;\Rightarrow\; b \mathrel{\circ\!\!\!\!\longrightarrow} a^{\bullet}$$

$< \Leftrightarrow$ Triangle

Consider the reflexive closure of $\rightarrow$

λ-calculus

$$(\lambda x.M)N \to M[x{:=}N]$$

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$$(\lambda x.M)N \rightarrow M[x{:=}N]$$

$$\begin{array}{rcll}
x^\bullet & = & x & \\
(\lambda x.M)^\bullet & = & \lambda x.M^\bullet & \\
(MN)^\bullet & = & M'[x{:=}N^\bullet] & \text{if } M \text{ is an abstraction, } M^\bullet = \lambda x.M' \\
 & = & M^\bullet N^\bullet & \text{otherwise}
\end{array}$$

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$$I^\bullet = I$$

$$((\lambda xy.Iyx)zI)^\bullet = (\lambda y.yz)I$$

λ-calculus

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See Barendregt: The Lambda Calculus, Its Syntax and Semantics (1985)

$\mathsf{Z} \Leftrightarrow <$

(If) Suppose $a \to b$

$\to \subseteq \mathrel{-\!\!\ominus\!\!\to} \;\Rightarrow\; a \mathrel{-\!\!\ominus\!\!\to} b$

The triangle property $\Rightarrow\; b \mathrel{-\!\!\ominus\!\!\to} a^\bullet$ hence also $a^\bullet \mathrel{-\!\!\ominus\!\!\to} b^\bullet$.

$\mathrel{-\!\!\ominus\!\!\to} \subseteq \twoheadrightarrow \;\Rightarrow\; b \twoheadrightarrow a^\bullet \twoheadrightarrow b^\bullet$

$Z \Leftrightarrow <$

(only if) Def. $a \mathrel{-\!\!\ominus\!\!\!\rightarrow} b$ if b *between* a and $a^\bullet$, i.e. $(a \twoheadrightarrow b \twoheadrightarrow a^\bullet)$:

- $a \longrightarrow b \;\Rightarrow\; b \twoheadrightarrow a^\bullet \;\Rightarrow\; \longrightarrow \subseteq \mathrel{-\!\!\ominus\!\!\!\rightarrow}$.
- $a \mathrel{-\!\!\ominus\!\!\!\rightarrow} b \;\Rightarrow\; a \twoheadrightarrow b \;\Rightarrow\; \mathrel{-\!\!\ominus\!\!\!\rightarrow} \subseteq \twoheadrightarrow$.
- Suppose $a \mathrel{-\!\!\ominus\!\!\!\rightarrow} b$.
  $a \twoheadrightarrow b \twoheadrightarrow a^\bullet$ by definition of $\mathrel{-\!\!\ominus\!\!\!\rightarrow}$.
  $a \twoheadrightarrow b \;\Rightarrow\; a^\bullet \twoheadrightarrow b^\bullet$.
  Hence $b \mathrel{-\!\!\ominus\!\!\!\rightarrow} a^\bullet$.

(Hyper)Cofinality

Definition

$\triangleright$ is a *many-step strategy* for $\longrightarrow$, if $\triangleright \subseteq \longrightarrow^{+}$ and both have the same normal forms.

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If $\bullet$ has the Z-property for $\rightarrow$, the many-step $\rightarrow$-strategy $\multimap\!\!\!\rightarrow$ is:

$$a \multimap\!\!\!\rightarrow b \text{ if } a \text{ is not a normal form and } b = a^\bullet.$$

(Hyper)Normalization

Theorem

$\rightarrow\!\!\bullet\!\!\rightarrow$ *is hyper-cofinal, if* $\bullet$ *has the Z-property.*

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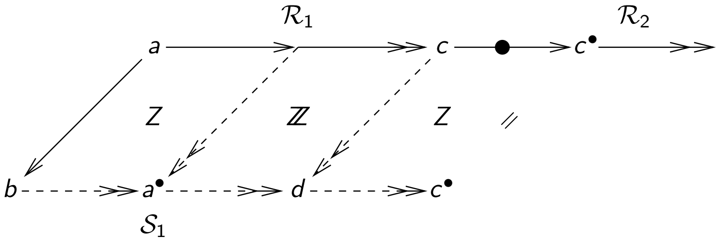

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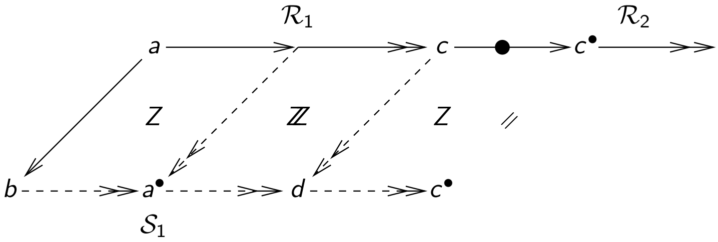
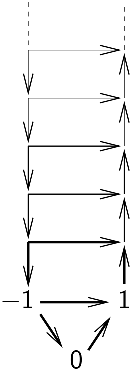


$\Rightarrow$ confluent

Confluence $\not\Rightarrow$ Z



Composition

If $\bullet_1$, $\bullet_2$ have the Z-property for $\longrightarrow$, so does their composition $\bullet_1 \circ \bullet_2$.

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If $\bullet_1$, $\bullet_2$ have the Z-property for $\rightarrow$, so does their composition $\bullet_1 \circ \bullet_2$. Moreover, $a^{\bullet_i} \twoheadrightarrow (a^{\bullet_2})^{\bullet_1}$

The Z-property for λ-calculus

(Self) $M \twoheadrightarrow M^\bullet$;

(Rhs) $M^\bullet[x{:=}N^\bullet] \twoheadrightarrow M[x{:=}N]^\bullet$; and

(Z) $M \rightarrow N \Rightarrow N \twoheadrightarrow M^\bullet \twoheadrightarrow N^\bullet$;

each by induction and cases on M.

$\lambda\sigma$

Calculi with explicit substitutions

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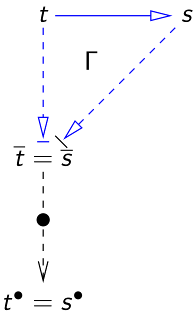

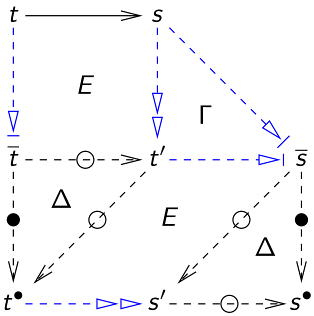

Weakly orthogonal rewriting

rewrite systems only having trivial critical pairs ($\lambda\beta\eta$).

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$$c(x) \rightarrow x$$

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$$g(f(f(f(x)))) \rightarrow g(f(f(x)))$$

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Contract maximal set of non-overlapping redexes inside-out

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$$g(f(f(c(f(f(x))))))^{\bullet} = g(f(f(x))) = g(f(f(f(f(x)))))^{\bullet}$$

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outside-in does not work!

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Conclusions

Surprising input from outside (Dehornoy): simple notion not known

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Does the Z-property hold for β-reduction with restricted η-expansion.