Home Page

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

1. Vicious circles in rewriting

Jeroen Ketema
VU

Jan-Willem Klop
VU, CWI, RU

Vincent van Oostrom
UU

2. Kind of results

$\ldots \Rightarrow$ acyclic

3. Why acyclicity?

Complete 'results': no infinite reductions

normalisation $+$? $\Rightarrow$ termination

3. Why acyclicity?

Complete 'results': no infinite reductions
normalisation $+$? $\Rightarrow$ termination

- self-delimiting (this morning)
- typed $\lambda\beta$ (BGK-conjecture)

3. Why acyclicity?

Complete 'results': no infinite reductions
normalisation $+$? $\Rightarrow$ termination

- self-delimiting (this morning)
- typed $\lambda\beta$ (BGK-conjecture)

normalisation $+$? $\Rightarrow$ acyclic $\Rightarrow$ unbounded growth

3. Why acyclicity?

Complete 'results': no infinite reductions
normalisation + ? $\Rightarrow$ termination

- self-delimiting (this morning)
- typed $\lambda\beta$ (BGK-conjecture)

normalisation + ? $\Rightarrow$ acyclic $\Rightarrow$ unbounded growth

Partial 'results': no cyclic reductions
head normalisation + ? $\Rightarrow$ acyclic

4. WRS’04

Theorem.
normalising orthogonal first-order TRS $\Rightarrow$ acyclic

4. WRS'04

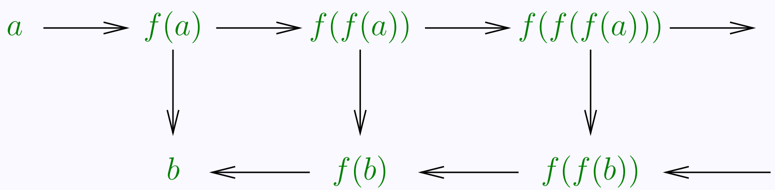
Theorem.
normalising orthogonal first-order TRS $\Rightarrow$ acyclic

- normalising: object reducible to normal form (WN)
- orthogonal: left-linear, no critical pairs (ORTH)
- first-order: no bound variables (F-O)
- acyclic: no reduction cycles (AC)

Home Page
Title Page
Contents
Go Back
Full Screen
Close
Quit

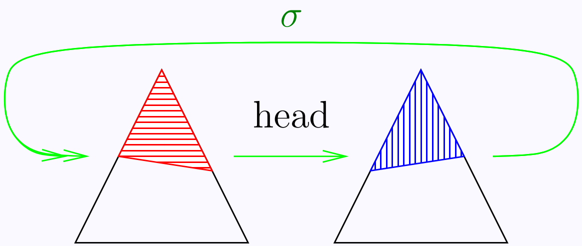
4.1. Typical example

$$\begin{array}{rcl} a & \to & f(a) \\ f(x) & \to & b \end{array}$$



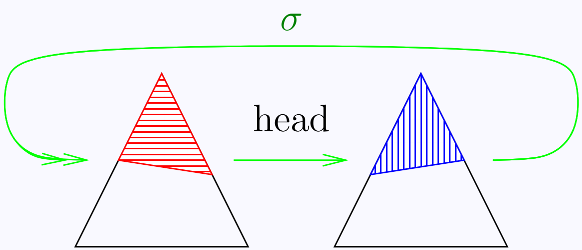
- WN: all terms reducible to a
- ORTH: ≤ 1 variable in lhs, distinct defined symbols
- F-O: no bound variables
- AC: yes
- SN: no

4.2. Proof by minimal counterexample



$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

4.2. Proof by minimal counterexample

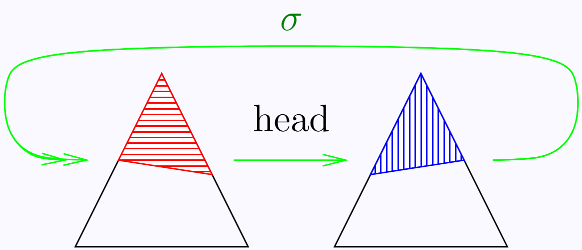


$$\sigma^{\omega} = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

Theorem (Head Normalisation).
Eventually a head step $\Rightarrow$ strategy is normalising

Proof. By O'Donnell (77): outermost-fair strategies are normalising for first-order orthogonal TRSs. □

4.2. Proof by minimal counterexample



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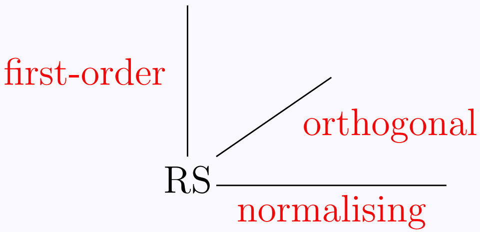
Theorem (Head Normalisation).
Eventually a head step $\Rightarrow$ strategy is normalising

Proof. By O'Donnell (77): outermost-fair strategies are normalising for first-order orthogonal TRSs. □

So σ^ω would be normalising. Contradiction

5. WRS'05

Generalisation into 3 directions



6. Generalising Normalisation

WN———————— ?

Home Page

Title Page

Contents

Page 9 of 41

Go Back

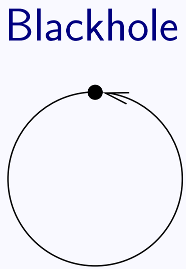
Full Screen

Close

Quit

6.1. Just omit?

6.1. Just omit?



6.2. Other restriction?

6.2. Other restriction?

head normalisation

WN ⟶ WHN

Home Page

Title Page

Contents

Go Back

Full Screen

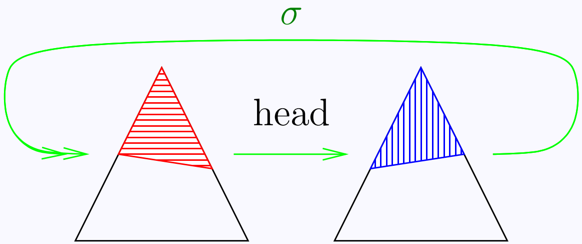
Close

Quit

6.3. Does WHN work?

6.3. Does WHN work?

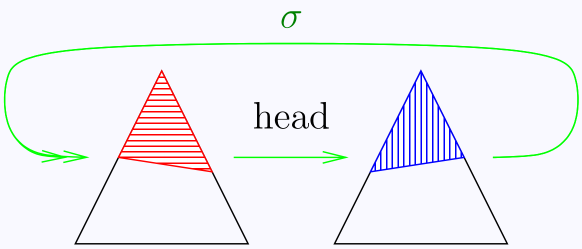
Yes.



$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

6.3. Does WHN work?

Yes.



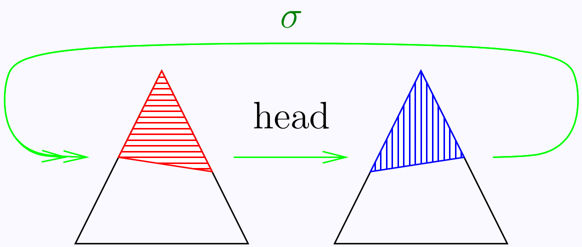
$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

Theorem (Head Head Normalisation).
Eventually a head step $\Rightarrow$ strategy is head normalising

Proof. By Middeldorp (97): hyper head needed strategies are head normalising for first-order orthogonal TRSs. □

6.3. Does WHN work?

Yes.



$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

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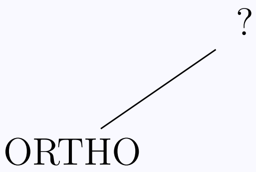
Eventually a head step $\Rightarrow$ strategy is head normalising

Proof. By Middeldorp (97): hyper head needed strategies are head normalising for first-order orthogonal TRSs. □

So σ^ω would be head normalising. Contradiction

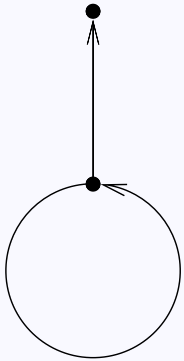
7. Generalising Orthogonality



7.1. Just omit?

7.1. Just omit?

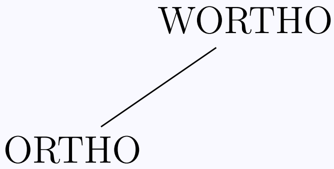
Weak blackhole



7.2. Other restriction?

7.2. Other restriction?

weak orthogonality



7.3. WORTHO

WORTHO: only trivial critical pairs $\langle t, t \rangle$

7.3. WORTHO

WORTHO: only trivial critical pairs $\langle t, t \rangle$

$$\begin{aligned} P(S(x)) &\rightarrow x \\ S(P(x)) &\rightarrow x \end{aligned}$$

$$P(x) \leftarrow \underline{P(S}\overline{S(P}(x)) \rightarrow P(x)$$

$$S(x) \leftarrow \underline{S(P}\overline{P(S}(x)) \rightarrow S(x)$$

7.3. WORTHO

WORTHO: only trivial critical pairs $\langle t, t \rangle$

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\begin{aligned}
P(S(x)) &\rightarrow x \\
S(P(x)) &\rightarrow x
\end{aligned}
$$

$$P(x) \leftarrow \underline{P(\overline{S}}\overline{(P}(x)) \rightarrow P(x)$$

$$S(x) \leftarrow \underline{S(\overline{P}}\overline{(S}(x)) \rightarrow S(x)$$

Other examples:

- Parallel or
- $\lambda\beta\eta$-calculus

7.4. Does WORTHO work?

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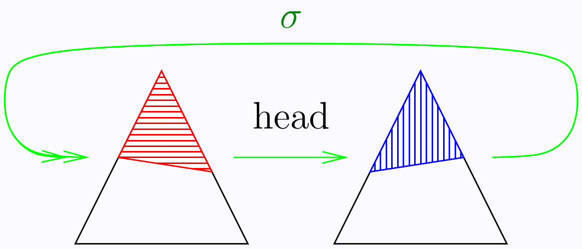
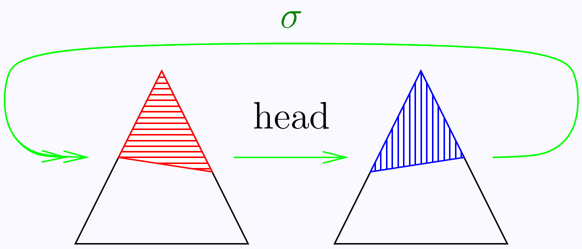
Yes.



$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

7.4. Does WORTHO work?

Yes.



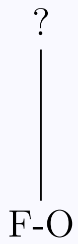
$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

Theorem (Head Normalisation). Eventually a head step $\Rightarrow$ strategy is normalising

Proof. By vO (99): outermost fair strategies are normalising for first-order weakly orthogonal TRSs. □

So σ^ω would be normalising. Contradiction

8. Generalising First-Orderness



8.1. Just omit?

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What does orthogonality mean?

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What does orthogonality mean?

Proof needs closure under sub-structures
What are they?

8.1. Just omit?

What does orthogonality mean?

Proof needs closure under sub-structures
What are they?

Thm does not hold (without more) for residual systems

Home Page

Title Page

Contents

Go Back

Full Screen

Close

Quit

8.2. Other restriction?

Home Page

Title Page

Contents

Page 19 of 41

Go Back

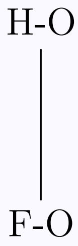
Full Screen

Close

Quit

8.2. Other restriction?

higher-order



8.3. H-O

H-O: rewriting of λ-terms $\bmod \alpha\beta\eta$

$$
\begin{aligned}
(\lambda x.M(x))N &\rightarrow_{\mathsf{beta}} M(N) \\
\lambda x.Mx &\rightarrow_{\mathsf{eta}} M
\end{aligned}
$$

8.3. H-O

H-O: rewriting of λ-terms $\bmod \alpha\beta\eta$

$$
\begin{aligned}
(\lambda x.M(x))N &\rightarrow_{\text{beta}} M(N) \\
\lambda x.Mx &\rightarrow_{\text{eta}} M
\end{aligned}
$$

Other examples:

- $\mu x.Z(x) \rightarrow Z(\mu x.Z(x))$
- Any transformation on terms with bound variables

Home Page

Title Page

Contents

Page 21 of 41

Go Back

Full Screen

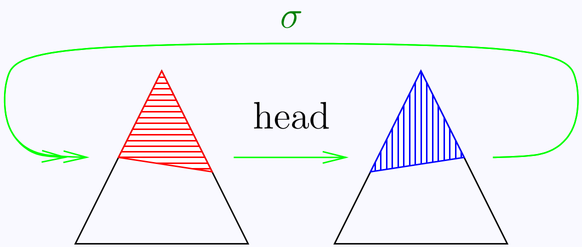
Close

Quit

8.4. Does H-O work?

8.4. Does H-O work?

Yes.



$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

8.4. Does H-O work?

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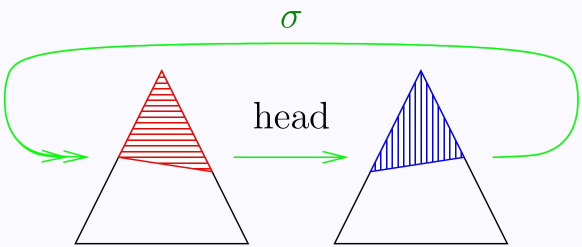


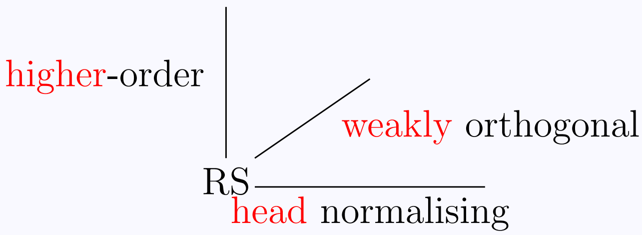
$$\sigma^\omega = \sigma \cdot \sigma \cdot \sigma \cdot \ldots$$

Theorem (Head Normalisation).
Eventually a head step $\Rightarrow$ strategy is normalising

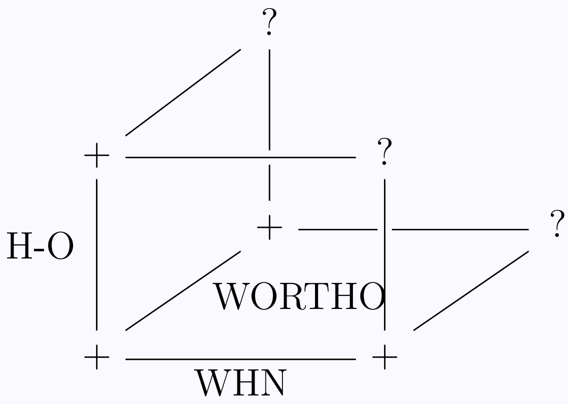
Proof. By Van Raamsdonk (96): outermost fair strategies are normalising for orthogonal higher-order TRSs. □

So σ^ω would be normalising. Contradiction

9. Three directions

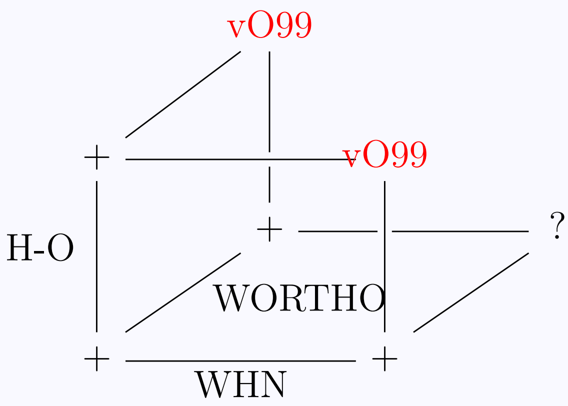


10. Pair-wise combinations



Home Page
Title Page
Contents
Page 24 of 41
Go Back
Full Screen
Close
Quit

10.1. Two pairs



11. Weakly ortho and head normalising

Head Head Normalisation Theorem fails for WORTHO!

$$
\begin{aligned}
a &\rightarrow b \\
g(b, x) &\rightarrow g(x, x) \\
f(g(a, x)) &\rightarrow f(g(b, x))
\end{aligned}
$$

11. Weakly ortho and head normalising

Head Head Normalisation Theorem fails for WORTHO!

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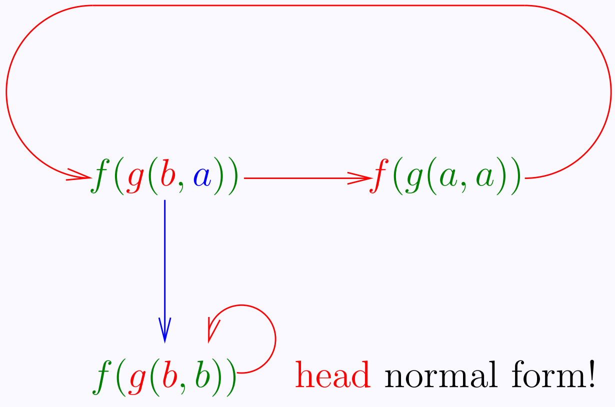

11.1. Alternative proof method

Commutation of (head) cycles

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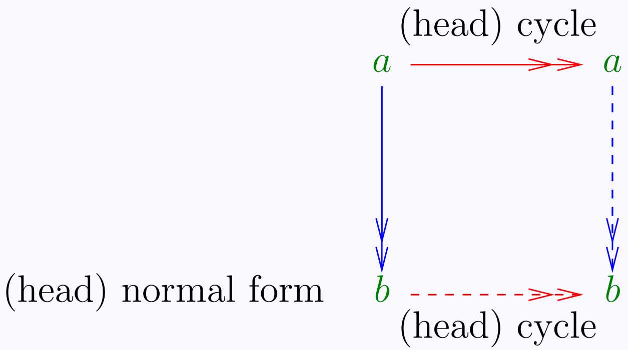
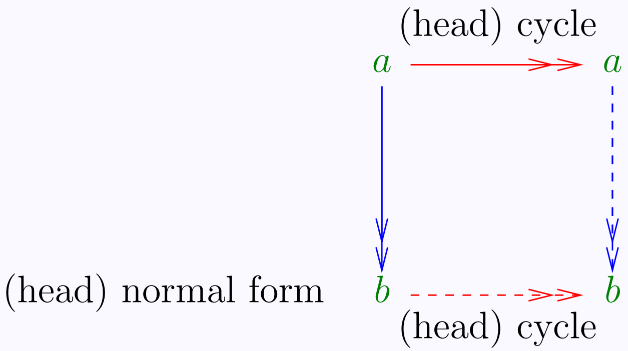
Commutation of (head) cycles



11.1. Alternative proof method

Commutation of (head) cycles



Contradiction

Home Page

Title Page

Contents

Page 27 of 41

Go Back

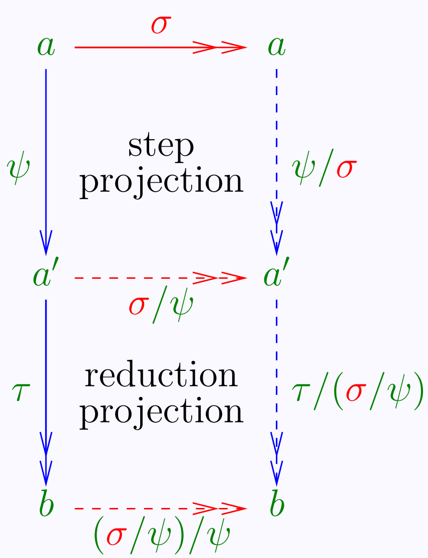
Full Screen

Close

Quit

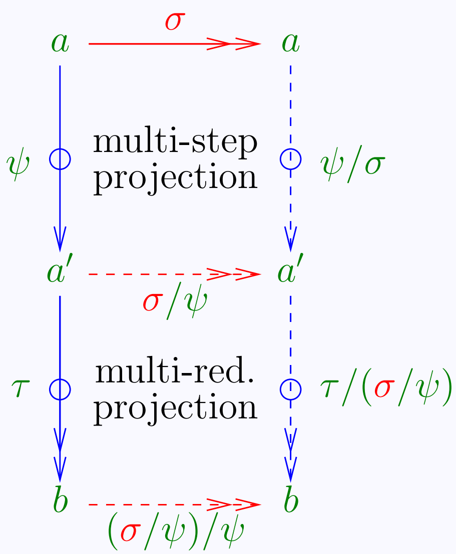
11.2. Commutation via projection

Klop projection



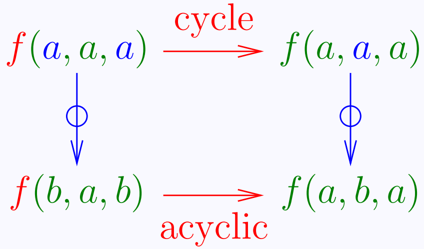
11.3. Commutation via multi-projection

Canonical projection



11.4. Cycles do **not** project

$$
\begin{array}{rcl}
a & \longrightarrow & b \\
f(x, y, z) & \longrightarrow & f(y, x, a)
\end{array}
$$



Home Page

Title Page

Contents

Page 30 of 41

Go Back

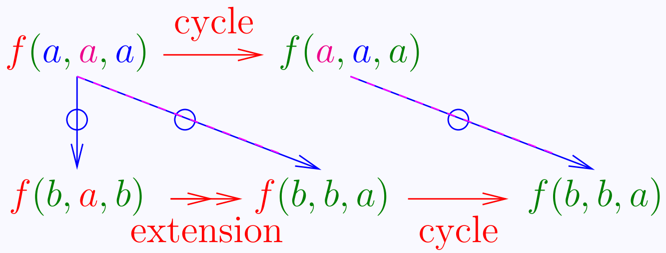
Full Screen

Close

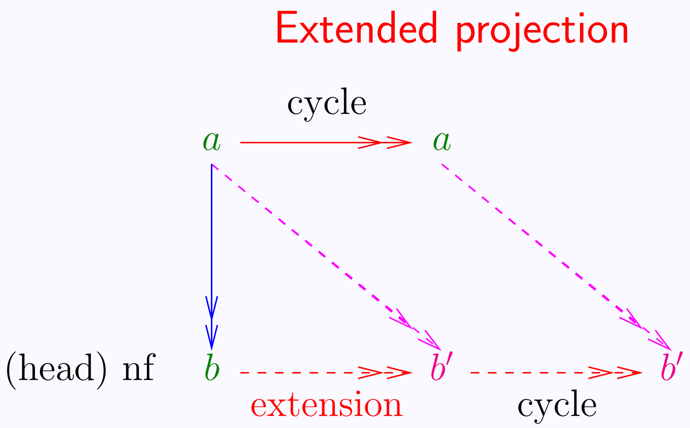
Quit

11.5. Cycles do project over extensions

$$a \;\longrightarrow\; b$$
$$f(x,y,z) \;\longrightarrow\; f(y,x,a)$$

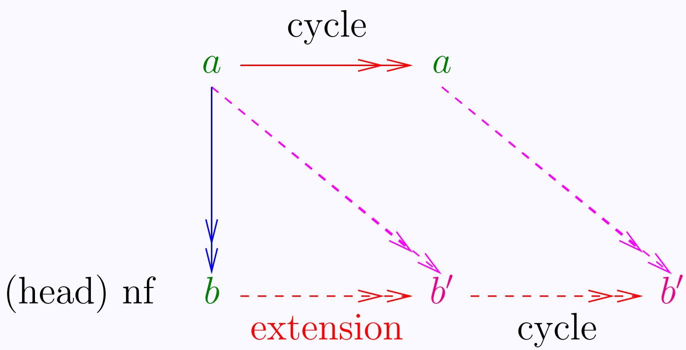


11.6. Commutation via extended projection



11.6. Commutation via extended projection

Extended projection



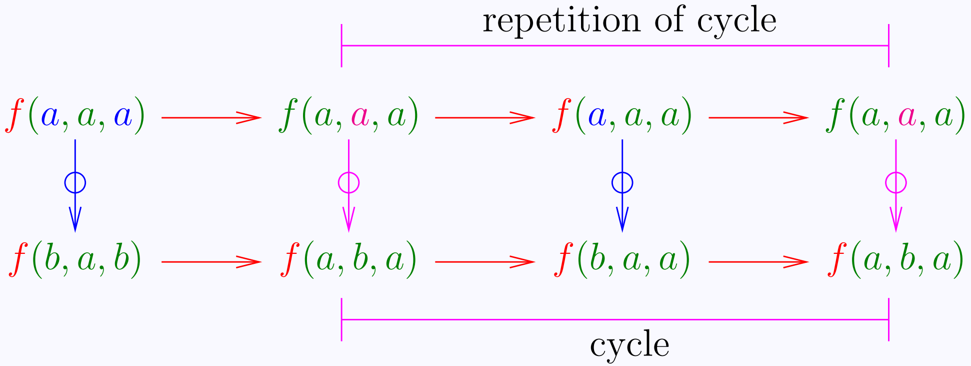
Contradiction

Home Page
Title Page
Contents
Page 32 of 41
Go Back
Full Screen
Close
Quit

11.7. Construction of extension 1

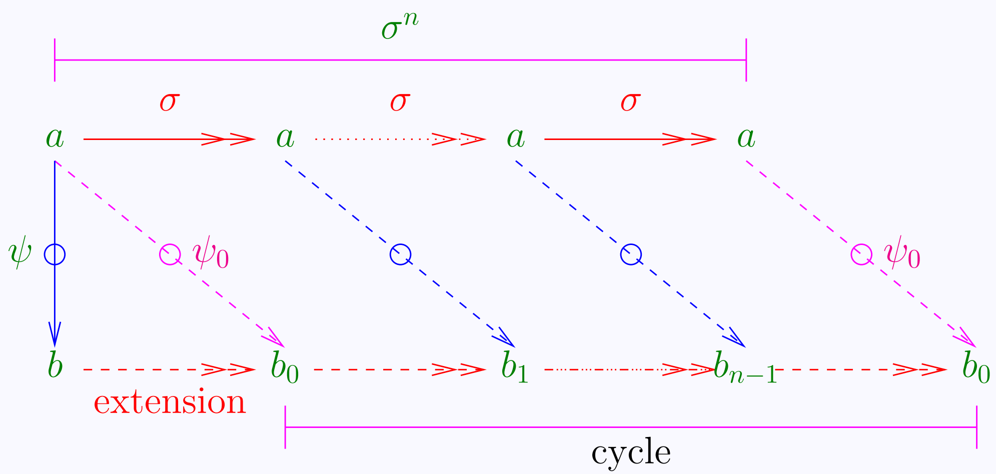
Repetition of cycle does project

$$a \;\to\; b$$
$$f(x,y,z) \;\to\; f(y,x,a)$$



11.8. Repetition Lemma

Lemma (Repetition).
$\exists$ extension ψ_0 of ψ,
$\exists$ positive natural number n such that $\psi_0/\sigma^n = \psi_0$



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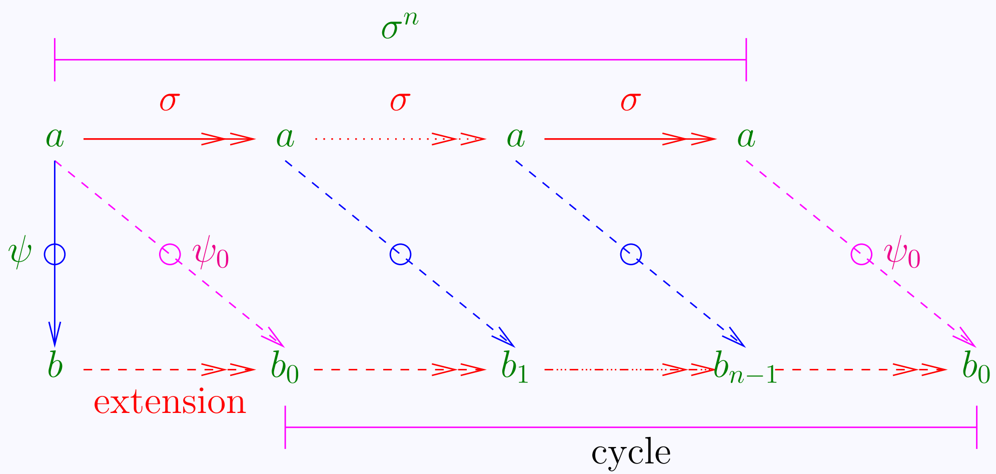
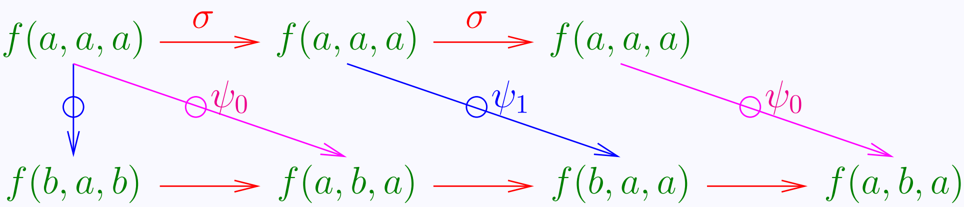


Proof. By Pigeon Hole Principle (cf. division). □

11.9. Construction of extension 2

Repetition of cycles can be compressed

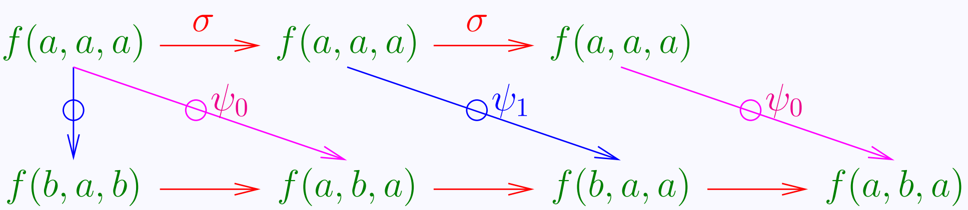
$$a \rightarrow b$$
$$f(x,y,z) \rightarrow f(y,x,a)$$



11.9. Construction of extension 2

Repetition of cycles can be compressed

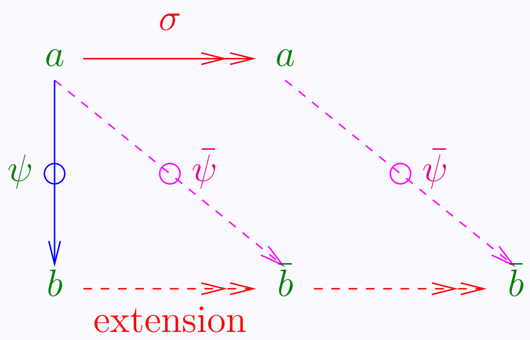
$$a \;\rightarrow\; b$$
$$f(x, y, z) \;\rightarrow\; f(y, x, a)$$



$$(\psi_0 \cup \psi_1)/\sigma = (\psi_0/\sigma) \cup (\psi_1/\sigma) = \psi_1 \cup \psi_0 = \psi_0 \cup \psi_1$$

11.10. Compression Lemma

Lemma (Compression).
If $\bar{\psi} = \bigcup_{0 \le i < n} \psi_0/\sigma^i$, then $\bar{\psi}/\sigma = \bar{\psi}$ and $\bar{\psi}$ extends ψ.



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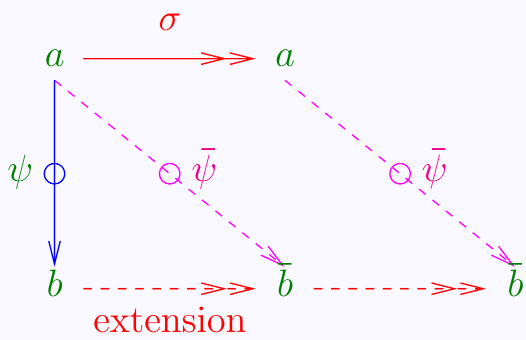

Proof.

$$(\bigcup_{0 \le i < n} \psi_i)/\sigma = \bigcup_{0 \le i < n} (\psi_i/\sigma) = \bigcup_{0 \le i < n} \psi_{i+1 \bmod n} = \bigcup_{0 \le i < n} \psi_i$$

(distributivity) □

11.11. Extension preserves non-emptyness

Lemma (Non-head). If σ is a non-empty parallel head cycle of minimal length, then $\sigma/\bar{\psi}$ is so as well.

Proof. Could only fail if $\sigma/\bar{\psi} = \emptyset$. Then $|\bar{\psi}| > |\bar{\psi}|$. □

11.11. Extension preserves non-emptyness

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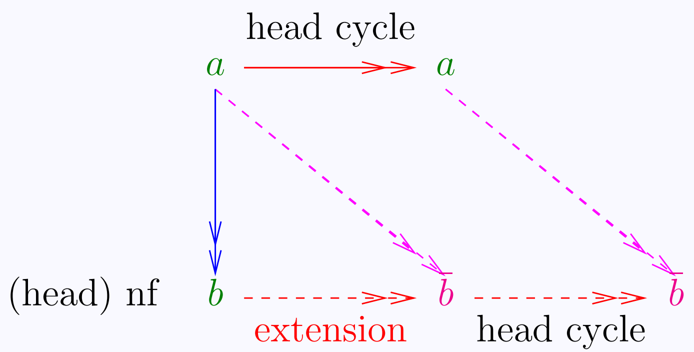

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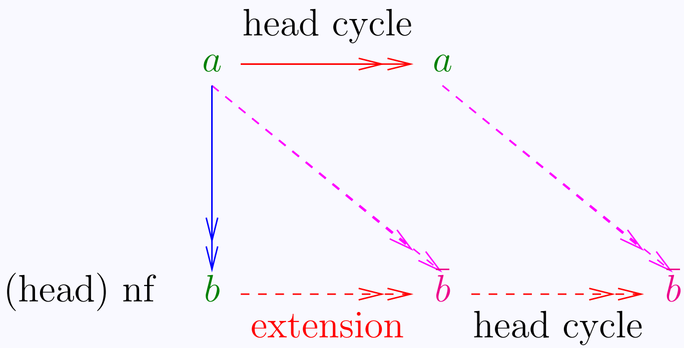
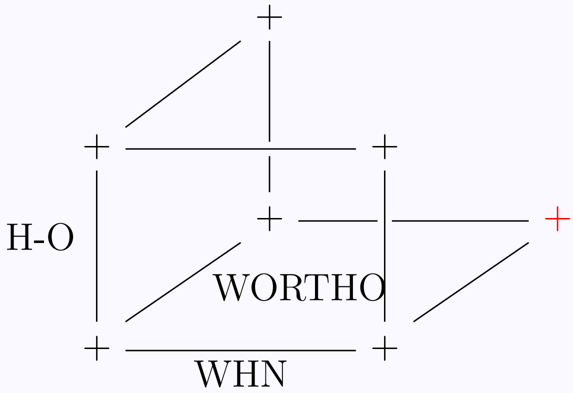
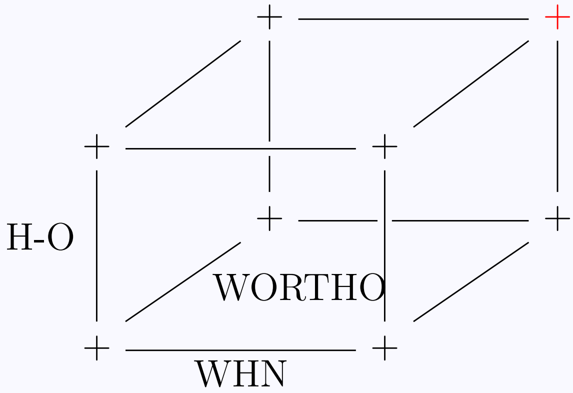


Contradiction

11.12. Three pairs



12. Conjecture



13. λ-calculi

- $\lambda\beta$-calculus is W(H)N $\Rightarrow$ acyclic.
- $\lambda\beta\eta$-calculus is WN $\Rightarrow$ acyclic.
- $\lambda\mathbf{x}^-$ is W(H)N $\Rightarrow$ acyclic.
- $\lambda\sigma$ is W(H)N $\Rightarrow$ acyclic.

13. λ-calculi

- $\lambda\beta$-calculus is W(H)N $\Rightarrow$ acyclic.
- $\lambda\beta\eta$-calculus is WN $\Rightarrow$ acyclic.
- $\lambda\mathbf{x}^{-}$ is W(H)N $\Rightarrow$ acyclic.
- $\lambda\sigma$ is W(H)N $\Rightarrow$ acyclic.

Extends to sub-calculi (e.g. typed)

13. λ-calculi

- $\lambda\beta$-calculus is W(H)N $\Rightarrow$ acyclic.
- $\lambda\beta\eta$-calculus is WN $\Rightarrow$ acyclic.
- $\lambda\mathrm{x}^{-}$ is W(H)N $\Rightarrow$ acyclic.
- $\lambda\sigma$ is W(H)N $\Rightarrow$ acyclic.

Extends to sub-calculi (e.g. typed)

Corollary: Typed $\lambda\sigma$ is acyclic
Melliès' counterexample couldn't have been bounded!

14. Full version

- See www.
- Fully-applied/extendedness (variable conditions)
- WORTHO projection.
- WORTHO $\Rightarrow$ redex-clusters coverable by redex-chains.
- Etc. (50+ pages)

15. Conclusion

1. Extended WN & ORTHO & F-O TRS $\Rightarrow$ acyclic to WHN, WORTHO, H-O, pair-wise combinations.
2. Conjecture: WHN & WORTHO & H-O $\Rightarrow$ acyclic.
3. WN λ-calculus (with explicit substitutions) $\Rightarrow$ acyclic.

15. Conclusion

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