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1. λ-calculus with end-of-scope

Observations

- λx is like opening an ‘x-bracket’
- no corresponding closing bracket!

Proposal

- Adjoin closing ‘x-brackets’ $\reflectbox{$\lambda$}x$
  (adbmal, unbind, end-of-scope)
- $\reflectbox{$\lambda$}x.M$ closes matching λx: x is ‘free’ in $\lambda x.\reflectbox{$\lambda$}x.x$
- Can be nested $\underbrace{\lambda x.\overbrace{\lambda x.\reflectbox{λ}x}.\reflectbox{λ}x}.x$ (x free again)
- Proper nesting: $\lambda x.\lambda y.\reflectbox{λ}x.\reflectbox{λ}y.M$ not allowed
  (better: $\reflectbox{$\lambda$}x$ implictly closes λy)

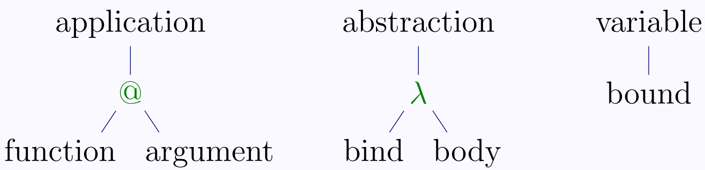
2. α-equivalence via Bourbaki

Represent bound variables as pointers to binder (λ)

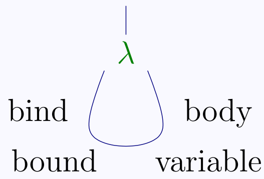
What makes this a good representation?

Thm 1 *λ-terms α-equivalent iff same Bourbaki-graph.*

2.1. Bourbaki λ-graphs

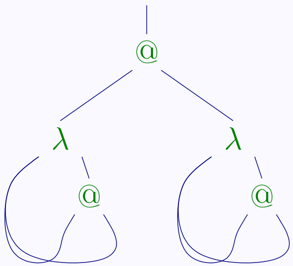


$\lambda x.x =$



$= \lambda y.y$

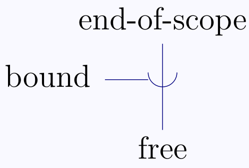
$(\lambda x.xx)(\lambda x.xx) =$



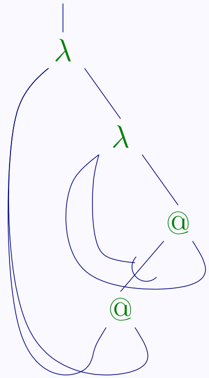
$= \Omega$

2.2. Bourbaki end-of-scope graphs

Represent end-of-scopes as pointers to binder



$\lambda xy.(\not\lambda y.xx)y$



3. β-reduction

Observation: $\reflectbox{$\lambda$}$'s in between $@$ and λ

- $(\lambda x.\reflectbox{$\lambda$}x.M)N$ should reduce to M
- $(\reflectbox{$\lambda$}y.\lambda x.x)y$ should not reduce to $\reflectbox{$\lambda$}y.y$ (but y)
- $(\reflectbox{$\lambda$}y.\lambda x.z)y$ should not reduce to z (but $\reflectbox{$\lambda$}y.z$)

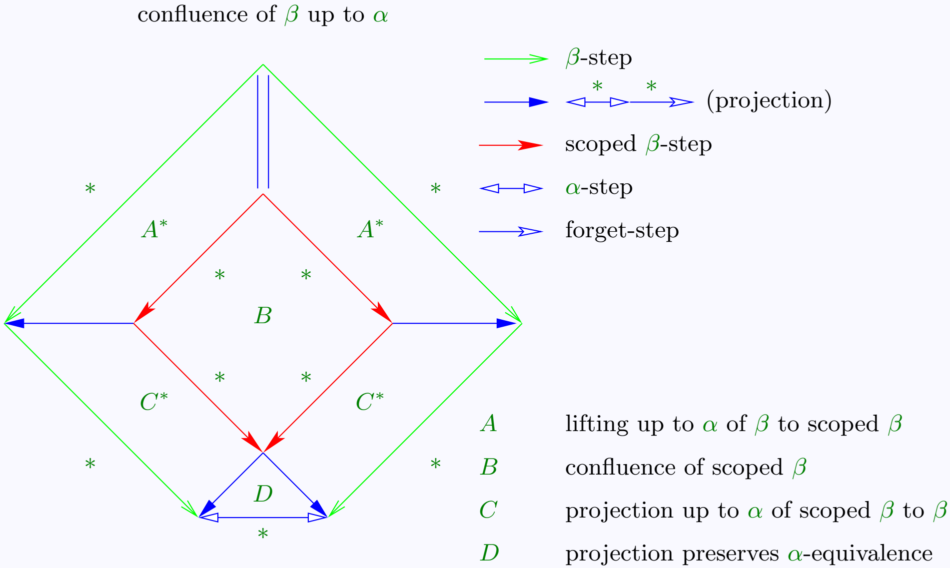
Where should end-of-scopes go?

- search for matching x
- in case of x: remove end-of-scopes, put argument
- in case of $\reflectbox{$\lambda$}x$: put end-of-scopes, remove argument

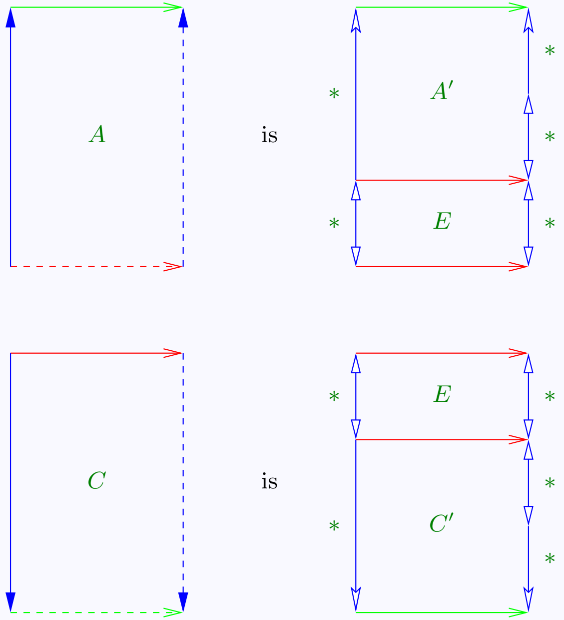
$(\reflectbox{$\lambda$}X.\lambda x.M)N \to M[X, x{:=}N, \Box]$:

- X remembers the end-of-scopes
- third argument: stack used for matching

3.1. Proof of confluence: outline



3.2. Proof of confluence: lifting and projection



A' lifting β to scoped β

E commutation of α and scoped β

C' projecting scoped β to β

4. Formalization in Coq

Axiom 1 *Assume a parameter set $\mathcal{V}$ of infinitely many variable names for which equality is decidable.*

- $x = y \vee x \neq y$ *for all* $x, y : \mathcal{V}$
- $\exists x : \mathcal{V}.\, x \notin X$ *for all* $X : \mathit{list}(\mathcal{V})$

Def 2 *The set Λ of $\lambdabar$-terms is defined by:*

$$\Lambda ::= \mathcal{V} \mid \lambdabar x.\Lambda \mid \lambda x.\Lambda \mid \Lambda\Lambda$$

4.1. Substitution with end-of-scope

Def 3 *Substitution* $M[X, x{:=}N, Y]$ *is defined by:*

$$
\begin{aligned}
y[X, x{:=}N, Y] &= y, \text{ if } y \in Y \\
y[X, x{:=}N, Y] &= \reflectbox{λ} Y.N, \text{ if } y \notin Y, x = y \\
y[X, x{:=}N, Y] &= \reflectbox{λ} Y.\reflectbox{λ} X.y, \text{ if } y \notin Y, x \neq y \\
(\lambda y.M)[X, x{:=}N, Y] &= \lambda y.M[X, x{:=}N, yY] \\
(\reflectbox{λ} y.M)[X, x{:=}N, \Box] &= \reflectbox{λ} X.M, \text{ if } x = y \\
(\reflectbox{λ} y.M)[X, x{:=}N, \Box] &= \reflectbox{λ} X.\reflectbox{λ} y.M, \text{ if } x \neq y \\
(\reflectbox{λ} y.M)[X, x{:=}N, zY] &= \reflectbox{λ} y.M[X, x{:=}N, Y], \text{ if } y = z \\
(\reflectbox{λ} y.M)[X, x{:=}N, zY] &= (\reflectbox{λ} y.M)[X, x{:=}N, Y], \text{ if } y \neq z \\
(M_1 M_2)[X, x{:=}N, Y] &= M_1[X, x{:=}N, Y] M_2[X, x{:=}N, Y]
\end{aligned}
$$

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4.2. β-reduction with end-of-scope

Def 4 *The relation $\rightarrow_\beta$ is defined as the compatible closure of the β-rule.*

$$\frac{}{(\reflectbox{λ}X.\lambda x.M)N \rightarrow_\beta M[X, x{:=}N, \Box]}$$

$$\frac{M \rightarrow_\beta N}{\lambda x.M \rightarrow_\beta \lambda x.N} \qquad \frac{M \rightarrow_\beta N}{\reflectbox{λ}x.M \rightarrow_\beta \reflectbox{λ}x.N}$$

$$\frac{M \rightarrow_\beta M'}{MN \rightarrow_\beta M'N} \qquad \frac{N \rightarrow_\beta N'}{MN \rightarrow_\beta MN'}$$

4.3. Confluence without α

Thm 5 $\rightarrow_\beta$ *is confluent on* Λ.

Proof strategy (Tait, Martin-Löf):

- define (inductively) multi-steps $\multimap$;
- show that multi-steps have the diamond property (substitution lemma(ta));
- then $\multimap$ is confluent;
- show $\rightarrow_\beta \subseteq \multimap \subseteq \rightarrow_\beta^*$;
- conclude $\rightarrow_\beta^* \subseteq \multimap^* \subseteq \rightarrow_\beta^*$.

4.4. Multi-step

Def 6 *Multi-steps* $\multimap\!\!\rightarrow$ *are defined by:*

$$\frac{M_1 \multimap\!\!\rightarrow N_1 \quad M_2 \multimap\!\!\rightarrow N_2}{(\lambda\!\!\!\lambda X.\lambda x.M_1)M_2 \multimap\!\!\rightarrow N_1[X, x{:=}N_2, \Box]}$$

$$\frac{}{x \multimap\!\!\rightarrow x} \qquad \frac{M_1 \multimap\!\!\rightarrow N_1 \quad M_2 \multimap\!\!\rightarrow N_2}{M_1 M_2 \multimap\!\!\rightarrow N_1 N_2}$$

$$\frac{M \multimap\!\!\rightarrow N}{\lambda x.M \multimap\!\!\rightarrow \lambda x.N} \qquad \frac{M \multimap\!\!\rightarrow N}{\lambda\!\!\!\lambda x.M \multimap\!\!\rightarrow \lambda\!\!\!\lambda x.N}$$

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Def 7 *A term* M *is* *scope-balanced* *if* $\langle \square \rangle M$, *where* $\langle X \rangle M$ *is defined by:*

$$\frac{}{\langle X \rangle x} \qquad \frac{\langle Xx \rangle M}{\langle X \rangle \lambda x.M} \qquad \frac{\langle X \rangle M}{\langle Xx \rangle \text{ʎ} x.M} \qquad \frac{\langle X \rangle M \quad \langle X \rangle N}{\langle X \rangle MN}$$

Balancedness is defined as scope-balancedness restricting the first clause to

$$\frac{}{\langle Xx \rangle x}$$

Here $\square$ *is the empty stack and* Xx *is the result of pushing* x *on the stack* X.

4.5. Substitution Lemma

Lem 1 *Multi-step substitution lemma:*
if

$$\langle ZxY\rangle M_1 \qquad \langle ZX\rangle M_2$$

and

$$M_1 \multimap\!\!\!\rightarrow N_1 \qquad M_2 \multimap\!\!\!\rightarrow N_2$$

then:

$$M_1[X, x{:=}M_2, Y] \multimap\!\!\!\rightarrow N_1[X, x{:=}N_2, Y]$$

Compute the critical pair from the term P

$$(\lambda y.(\lambda x.M)N)L$$

Inner redex first:

$$\begin{aligned} P &\to_\beta (\lambda y.M[\Box, x{:=}N, \Box])L \\ &\to_\beta M[\Box, x{:=}N, \Box][\Box, y{:=}L, \Box] \end{aligned}$$

Outer redex first:

$$\begin{aligned} P &\to_\beta ((\lambda x.M)N)[\Box, y{:=}L, \Box] \\ &= (\lambda x.M)[\Box, y{:=}L, \Box]N[\Box, y{:=}L, \Box] \\ &= (\lambda x.M[\Box, y{:=}L, x])N[\Box, y{:=}L, \Box] \\ &\to_\beta M[\Box, y{:=}L, x][\Box, x{:=}N[\Box, y{:=}L, \Box], \Box] \end{aligned}$$

Compute the critical pair from the term Q

$$(\lambda y.(\not\lambda y.\lambda x.M)N)L$$

Inner redex first:

$$\begin{aligned} Q &\to_\beta (\lambda y.M[y, x{:=}N, \Box])L \\ &\to_\beta M[y, x{:=}N, \Box][\Box, y{:=}L, \Box] \end{aligned}$$

Outer redex first:

$$\begin{aligned} Q &\to_\beta ((\not\lambda y.\lambda x.M)N)[\Box, y{:=}L, \Box] \\ &= (\not\lambda y.\lambda x.M)[\Box, y{:=}L, \Box]N[\Box, y{:=}L, \Box] \\ &= (\lambda x.M)N[\Box, y{:=}L, \Box] \\ &\to_\beta M[\Box, x{:=}N[\Box, y{:=}L, \Box], \Box] \end{aligned}$$

Lem 2 *Closed substitution lemma: if* $\langle X'xZ\rangle s$, $\langle Y'yZ'Z\rangle t$ *and* $\langle YZ'Z\rangle u$, *then:*

$$s[Y'yZ', x{:=}t, X'][Y, y{:=}u, X'Y'] \\ = s[Y'YZ', x{:=}t[Y, y{:=}u, Y'], X']$$

```
Lemma closed_subst_bal :
 (s,t,u:term;X',Y,Y',Z,Z':(list name);x,y:name)
 (bal (conc X' (cons x Z)) s)
  -> (bal (conc Y' (conc (cons y Z') Z)) t)
  -> (bal (conc Y (conc Z' Z)) u)
   -> (subst Y (conc X' Y') (subst (conc Y'
         (cons y Z')) X' s x t) y u)
      = (subst (conc Y' (conc Y Z')) X' s x
          (subst Y Y' t y u)).
```

Lem 3 *Open substitution lemma: if $\langle X'xY'yZ\rangle s$, $\langle XY'yZ\rangle t$ and $\langle YZ\rangle u$, then:*

$$s[X,x{:=}t,X'][Y,y{:=}u,X'XY'] = s[Y,y{:=}u,X'xY'][X,x{:=}t[Y,y{:=}u,XY'],X']$$

```
Lemma open_subst_bal :
 (s,t,u:term;X,X',Y,Y',Z:(list name);x,y:name)
  (bal (conc X' (conc (cons x Y')(cons y Z))) s)
   -> (bal (conc X (conc Y' (cons y Z))) t)
    -> (bal (conc Y Z) u)
     -> (subst Y (conc X' (conc X Y'))
          (subst X X' s x t) y u)
         = (subst X X' (subst Y
          (conc X' (cons x Y')) s y u) x
          (subst Y (conc X Y') t y u)).
```

Def 8 *α-equality à la Kahrs.* *We define $M =_\alpha N$, if $\langle\Box\rangle M =_\alpha \langle\Box\rangle N$, where for vectors of variables X and Y, $\langle X\rangle M =_\alpha \langle Y\rangle N$ is inductively defined as follows. By $|X|$ we denote the length of vector X.*

$$
\begin{aligned}
\langle\Box\rangle x &=_\alpha \langle\Box\rangle x \\
\langle Xx\rangle x &=_\alpha \langle Yy\rangle y, \text{ if } |X| = |Y| \\
\langle Xx'\rangle x &=_\alpha \langle Yy'\rangle y, \text{ if } \langle X\rangle x =_\alpha \langle Y\rangle y, \\
&\qquad\qquad\qquad x' \neq x,\ y' \neq y \\
\langle X\rangle \lambda x.M &=_\alpha \langle Y\rangle \lambda y.N, \text{ if } \langle Xx\rangle M =_\alpha \langle Yy\rangle N \\
\langle X\rangle M_1 M_2 &=_\alpha \langle Y\rangle N_1 N_2, \text{ if } \langle X\rangle M_1 =_\alpha \langle Y\rangle N_1 \\
&\qquad\qquad\qquad \langle X\rangle M_2 =_\alpha \langle Y\rangle N_2
\end{aligned}
$$

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For λ-terms we add the following clauses.

$$\begin{aligned}
\langle\Box\rangle\lambda x.M &=_\alpha \langle\Box\rangle\lambda x.N, \text{ if } \langle\Box\rangle M =_\alpha \langle\Box\rangle N\\
\langle Xx\rangle\lambda x.M &=_\alpha \langle Yy\rangle\lambda y.N, \text{ if } \langle X\rangle M =_\alpha \langle Y\rangle N\\
\langle Xx'\rangle\lambda x.M &=_\alpha \langle Yy'\rangle\lambda y.N, \text{ if } \langle X\rangle\lambda x.M =_\alpha \langle Y\rangle\lambda y.N\\
&\qquad\qquad\qquad x' \neq x,\ y' \neq y
\end{aligned}$$

4.6. Operational α-equivalence

Def 9 *α-conversion $\leftrightarrow^*_\alpha$ is defined as the reflexive, symmetric, transitive closure of single-step α-renaming $\longrightarrow_\alpha$, which is defined as the compatible closure of the α-rule:*

```
alpha_rule :
 (M:sterm;x,y:name)
  ~(In y (names M))
   ->(alpha_conv (abs x M)(abs y (rename M x y Nil)))
```

Def 10 *α-equality à la Schroer.*

```
Definition alpha_eq2
 := [M,N:sterm](EX Z:(list name)|(alpha_eq2' M N Z)).
```

makes use of an auxiliary stack Z which records the variables chosen thusfar for renaming.

```
alpha_eq2_rule :
 (M,N:sterm;x,y,z:name;Z:(list name))
  ~(In z (names M))
   ->~(In z (names N))
    ->~(In z Z)
     ->(alpha_eq2' (rename M x z Nil)(rename N y z Nil) Z)
      ->(alpha_eq2' (abs x M)(abs y N)(cons z Z))
```

The clause dealing with λ is just a compatibility clause.

Thm 11 *All three notions of α-equivalence are equivalent.*

Note that to prove that λ-terms which are α-equivalent à la Kahrs are α-equivalent according to the other two definitions, one essentially uses the Fresh variable axiom. (It is not needed in the other direction.)

Thm 12 *α-equivalence is a congruent equivalence relation.*

Lem 4 *Lifting of* $\rightarrow_{\lambda\beta}$ *to* $\rightarrow_{⅄\beta}$ *(schema* A *in Figure). If* $M \rightarrow_{\lambda\beta} N$, $\langle X\rangle M' \rightarrow_{\omega} M$, *then there are* N_1, N_2 *such that:*

$$\langle X\rangle N_1 =_{\alpha} \langle X\rangle N_2 \rightarrow_{\omega} N$$

and

$$M' \rightarrow_{⅄\beta} N_1$$

Lem 5 *Projection of* λ*-substitution to* λ*-substitution. If*

- $\langle ZxY_1\rangle M_1 =_\alpha \langle ZxY_2\rangle M_2 \to_\omega M$
- $\langle ZX\rangle N' \to_\omega N$,
- $X \cap \mathsf{FV}(M_1, Y_1x) = \emptyset$,
- $Y_2 \cap (\{x\} \cup \mathsf{FV}(N', \Box)) = \emptyset$,

then there exists a P *such that:*

$$\begin{aligned}\langle ZXY_1\rangle M_1[X, x{:=}N', Y_1] &=_\alpha \langle ZXY_2\rangle P[X, x{:=}N', Y_2] \\ &\to_\omega M[x{:=}N]\end{aligned}$$

5. Results

– Confluence of β without α

– α needed to remove $\lambdabar$'s:
$\lambda x.\lambdabar x.x \rightarrow_{\alpha} \lambda y.\lambdabar y.x \rightarrow_{\text{forget}} \lambda y.x$

– Confluence of ordinary β modulo α

– Proofs in Coq

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6. Related/current work

Related work

– refines Guillaume, David, Bird, Paterson: de Bruijn ($\reflectbox{$\lambda$}$ = Successor, variable = zero)

– Di Cosmo et al.: with labels (commutativity, α)

– Explicit weakening

Current and further research

– Push $\reflectbox{$\lambda$}$'s locally : $\reflectbox{$\lambda$} x.\lambda x.M \to \lambda_1 x.\reflectbox{λ} x.M$ (reopen, reclose scope)

– optimal implementation (scopes, no croissants/brackets)

– explicit substitution (lemmas as rules) (CR, PSN)

7. Explicit Substitution

Explicit substitution lemma ($\_[\_{:=}\_]$ as syntax):

$$P = (\lambda y.(\lambda x.M)N)L$$

$$P \;\rightarrow_{\text{inner}}\; M[x{:=}N][y{:=}L]$$
$$P \;\rightarrow_{\text{outer}}\; M[y{:=}L][x{:=}N[y{:=}L]]$$

How to orient this critical pair?

7.1. left-orientation: non-confluence

Orientation from right to left:

$$M[x{:=}N][y{:=}L] \leftarrow M[y{:=}\underline{L}][x{:=}N[y{:=}\underline{L}]]$$

Problem: repeated variable ($\underline{L}$) in LHS: non-confluence

7.2. right-orientation: loop

Orientation from left to right:

$$M[x{:=}N][y{:=}L] \\ \rightarrow\ M[y{:=}L][x{:=}N[y{:=}L]]$$

Problem: LHS embedded in RHS: non-termination (loop)

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7.3. right-orientation: breaking the loop

Explicit substitution lemma in λ-calculus:

$$(\lambda y.(\lambda x.M)N)L$$

$$\begin{array}{lll} P & \rightarrow_{\text{inner}} & M[\Box, x{:=}N, \Box][\Box, y{:=}L, \Box] \\ P & \rightarrow_{\text{outer}} & M[\Box, y{:=}L, \underline{x}][\Box, \underline{x}{:=}N[\Box, y{:=}L, \Box], \Box] \end{array}$$

How can we break the loop?

7.3. right-orientation: breaking the loop

Explicit substitution lemma in λ-calculus:

$$(\lambda y.(\lambda x.M)N)L$$

$$P \rightarrow_{\text{inner}} M[\square, x{:=}N, \square][\square, y{:=}L, \square]$$
$$P \rightarrow_{\text{outer}} M[\square, y{:=}L, \underline{x}][\square, \underline{x}{:=}N[\square, y{:=}L, \square], \square]$$

How can we break the loop?
Idea: recognise outside-in order (by the $\underline{x}$)
(substitution over λx leaves x on stack)

7.4. Explicit substitution calculus with names

Explicit substitution lemma rules of $\lambda\!\!\!\lambda_{\mathsf{ws}}$:

$$s[Y'yZ', x{:=}t, X'][Y, y{:=}u, X'Y']$$
$$\to_{\mathsf{closed}}\ s[Y'YZ', x{:=}t[Y, y{:=}u, Y'], X']$$
$$s[X, x{:=}t, X'][Y, y{:=}u, X'XY']$$
$$\to_{\mathsf{open}}\ s[Y, y{:=}u, X'xY'][X, x{:=}t[Y, y{:=}u, XY'], X']$$

with side-conditions to break the loop

Conjecture 1 *preservation of strong normalisation*

SN in λ implies SN in $\lambda\!\!\!\lambda$ implies SN in $\lambda\!\!\!\lambda_{\mathsf{ws}}$

Why?

7.4. Explicit substitution calculus with names

Explicit substitution lemma rules of $ƛ_{\mathsf{ws}}$:

$$s[Y'yZ', x{:=}t, X'][Y, y{:=}u, X'Y']$$
$$\quad\rightarrow_{\mathsf{closed}}\ s[Y'YZ', x{:=}t[Y, y{:=}u, Y'], X']$$
$$s[X, x{:=}t, X'][Y, y{:=}u, X'XY']$$
$$\quad\rightarrow_{\mathsf{open}}\ s[Y, y{:=}u, X'xY'][X, x{:=}t[Y, y{:=}u, XY'], X']$$

with side-conditions to break the loop

Conjecture 1 *preservation of strong normalisation*

SN in λ implies SN in $ƛ$ implies SN in $ƛ_{\mathsf{ws}}$

Why?

$ƛ_{\mathsf{ws}}$ is named version of David and Guillaume's λ_{ws}

length bounded by length of outside-in reduction

8. Optimal reduction

Avoid useless work

$$\begin{aligned}(\lambda x.y)\Omega &\to (\lambda x.y)\Omega \\ &\to y\end{aligned}$$

Avoid duplicate work

$$\begin{aligned}(\lambda x.xx)(IK) &\to (IK)(IK) \\ &\to K(IK) \\ &\to KK\end{aligned}$$

How to avoid useless work?

8. Optimal reduction

Avoid useless work

$$\begin{aligned}(\lambda x.y)\Omega &\rightarrow (\lambda x.y)\Omega \\ &\rightarrow y\end{aligned}$$

Avoid duplicate work

$$\begin{aligned}(\lambda x.xx)(IK) &\rightarrow (IK)(IK) \\ &\rightarrow K(IK) \\ &\rightarrow KK\end{aligned}$$

How to avoid useless work?
Leftmost-outermost strategy

8. Optimal reduction

Avoid useless work

$$
\begin{aligned}
(\lambda x.y)\Omega &\rightarrow (\lambda x.y)\Omega \\
&\rightarrow y
\end{aligned}
$$

Avoid duplicate work

$$
\begin{aligned}
(\lambda x.xx)(IK) &\rightarrow (IK)(IK) \\
&\rightarrow K(IK) \\
&\rightarrow KK
\end{aligned}
$$

How to avoid useless work?
Leftmost-outermost strategy
How to avoid duplicate work?

8. Optimal reduction

Avoid useless work

$$\begin{aligned}(\lambda x.y)\Omega &\rightarrow (\lambda x.y)\Omega \\ &\rightarrow y\end{aligned}$$

Avoid duplicate work

$$\begin{aligned}(\lambda x.xx)(IK) &\rightarrow (IK)(IK) \\ &\rightarrow K(IK) \\ &\rightarrow KK\end{aligned}$$

How to avoid useless work?
Leftmost-outermost strategy
How to avoid duplicate work?
Avoid duplication by making it explicit

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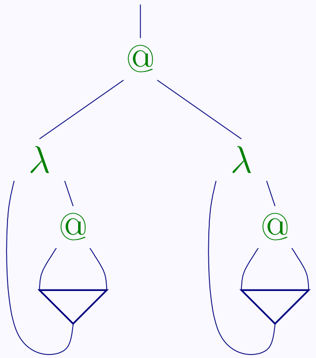
8.1. Explicit duplication

Explicit duplication node: fan

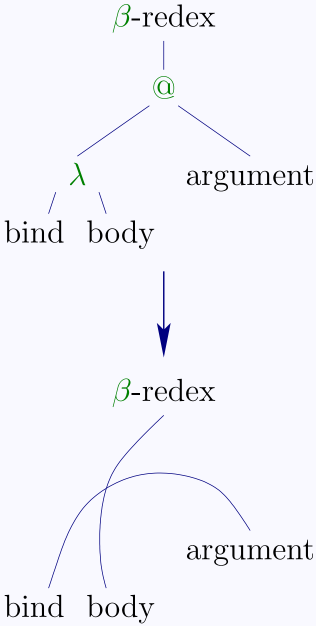
$(\lambda x.xx)(\lambda x.xx) =$

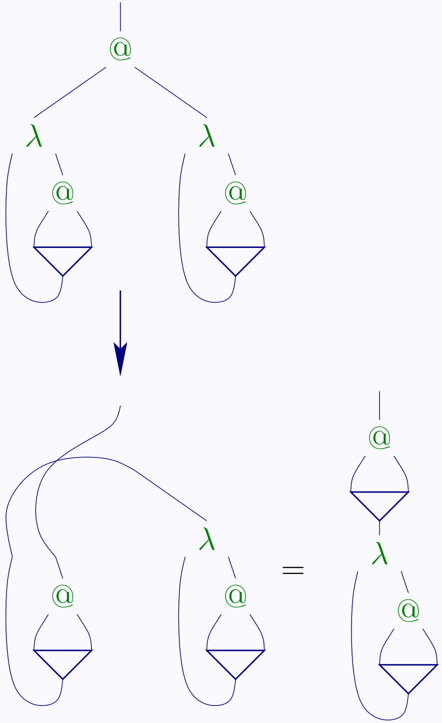


$= \Omega$

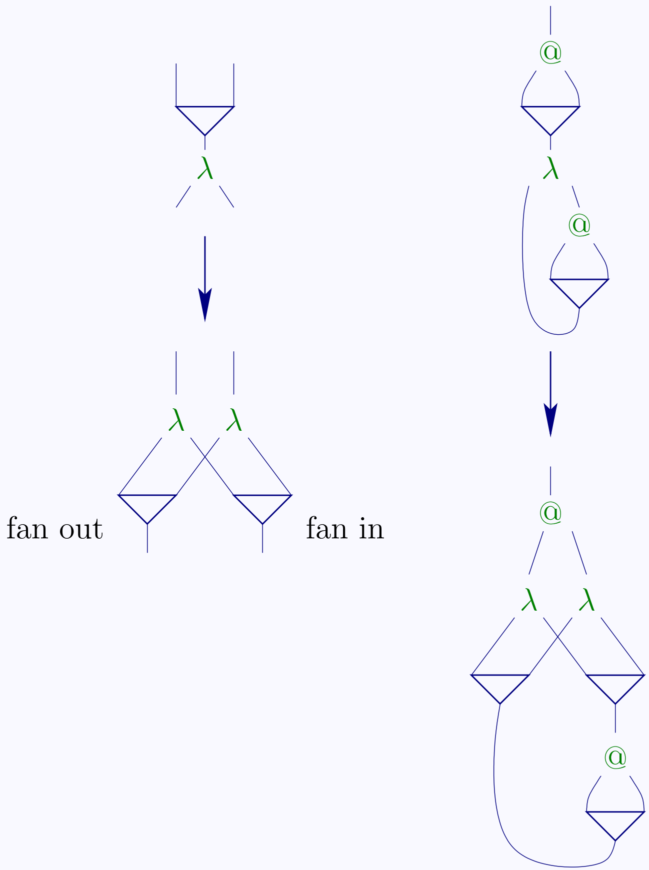
explicite duplicator $=$ ... $=$ fan

8.2. β on graphs

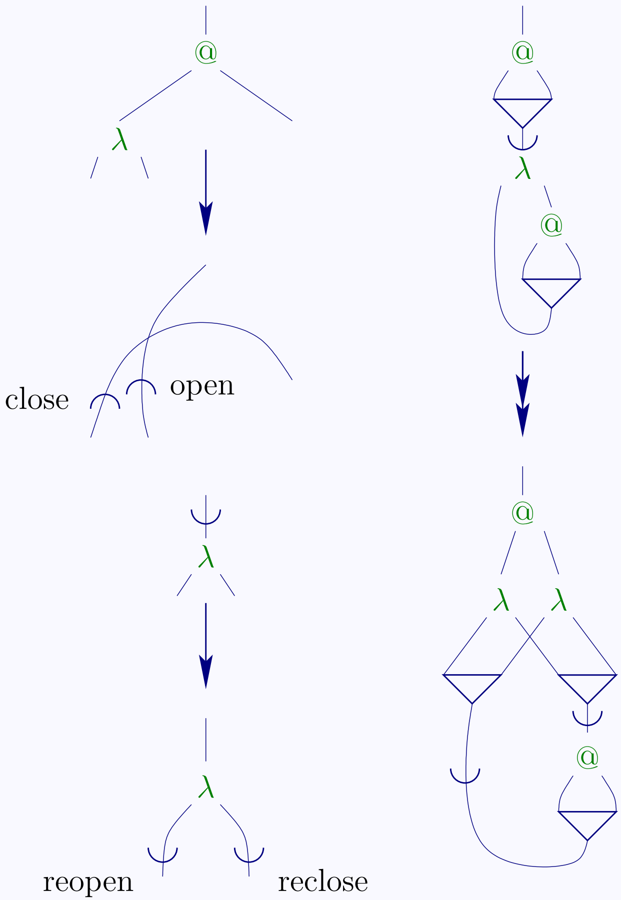




8.3. Localising duplication



8.4. Localising scoping



8.5. Benchmark

$c_{10}c_2c_2II$ where c_n is nth Church-numeral
recall: $c_m c_n \twoheadrightarrow c_{n^m}$

- Caml Light, Haskell, BOHM 1.0 explode
- BOHM 1.1 (Bolgona Optimal Higher-order Machine) 531706 steps (56 β)
- lambdascope : 1781260 steps (56 β)

lambdascope vs BOHM 1.1:

- solves explosion of brackets/croissants, hence
- no heuristics needed (always works)
- atomic steps (no compound nodes, S vs $+n$)

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8.6. Demo of lambdascope