The Polynomial Path Order and the Rules of Predicative Recursion with Parameter Substitution

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WST '09

Automatic Complexity Analysis

Goal

- purely automatic complexity analysis

Approach

- employ term rewriting as model of computation
- translate termination proofs into complexity certificates

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Approach

- employ term rewriting as model of computation
- translate termination proofs into complexity certificates
- to detect feasible computation, restrictions on termination technique usually inevitable

Outline

2. Complexity Analysis and Polynomial Path Orders

M. Avanzini and G. Moser
Complexity Analysis by Rewriting
In Proc. of FLOPS'08, LNCS vol. 4989, pp. 130–146, 2008

Outline

1. Predicative Recursion

   S. Bellantoni and S. Cook
   A new Recursion-Theoretic Characterization of the Polytime Functions
   Computation and Complexity, 2(2), pp. 97–110, 1992

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3. POP^* and Parameter Substitution

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The Primitive Recursive Functions

- initial functions 0, s, Π_i^k
- composition

$$f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$$

- primitive recursion

$$f(z+1, \vec{x}) = g(\vec{x})$$
$$f(z+1, \vec{x}) = h(z, \vec{x}, f(z, \vec{x}))$$

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- initial functions ε, s_0, s_1, Π_i^k
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$$f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$$

- primitive recursion on notation

$$f(\varepsilon, \vec{x}) = g(\vec{x})$$
$$f(z{\cdot}i, \vec{x}) = h_i(z, \vec{x}, f(z, \vec{x})), \quad i \in \{0, 1\}$$

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$$\begin{aligned} dup(\varepsilon) &= \varepsilon \\ dup(x{\cdot}i) &= s_i(s_i(dup(x)) \end{aligned} \qquad \begin{aligned} exp(\varepsilon) &= s_1(\varepsilon) \\ exp(x{\cdot}i) &= dup(exp(x)) \end{aligned}$$

Bellantoni and Cook's Definition of $\mathcal{BC}$

- initial functions ε, s_0, s_1, Π^k_i, ...
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$$f(\varepsilon, \vec{x}; \vec{y}) = g(\vec{x}; \vec{y})$$
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$$f(\underbrace{x_1, \ldots, x_m}_{\text{normal}}; \underbrace{y_1, \ldots, y_n}_{\text{safe}})$$

Bellantoni and Cook's Definition of $\mathcal{BC}$

- initial functions $\varepsilon,\ s_0,\ s_1,\ \Pi^k_i,\ \dots$
- predicative composition

$$f(\vec{x};\vec{y}) = g(n_1(\vec{x};),\dots,n_j(\vec{x};);\ s_1(\vec{x};\vec{y}),\dots,s_k(\vec{x};\vec{y}))$$

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Theorem

$$\mathcal{BC} = \mathbf{FP}$$

polytime computable functions

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$$dup(\varepsilon;) = \varepsilon \qquad exp(\varepsilon) = s_1(;\varepsilon)$$
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$$\begin{aligned} dup(\varepsilon;) &= \varepsilon \\ dup(x{\cdot}i;) &= s_i(;s_i(;dup(x;))) \end{aligned}$$

APPROVED

$$\begin{aligned} exp(\varepsilon) &= s_1(;\varepsilon) \\ exp(x{\cdot}i) &= dup(exp(x);) \end{aligned}$$

FAILED

Automatic Complexity Analysis by Rewriting

$$dup(\varepsilon) = \varepsilon \qquad exp(\varepsilon) = s_1(\varepsilon)$$

$$dup(x \cdot i) = s_i(s_i(dup(x))) \qquad exp(x \cdot i) = dup(exp(x))$$

$$\Downarrow$$

$$\mathsf{dup}(\epsilon) \to \epsilon \qquad \mathsf{exp}(\epsilon) \to \mathsf{s}_1(\epsilon)$$

$$\mathsf{dup}(\mathsf{s}_i(x)) \to \mathsf{s}_i(\mathsf{s}_i(\mathsf{dup}(x))) \qquad \mathsf{exp}(\mathsf{s}_i(x)) \to \mathsf{dup}(\mathsf{exp}(x))$$

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$$f(w_1, \ldots, w_n) = v \iff \mathsf{f}(\ulcorner w_1 \urcorner, \ldots, \ulcorner w_n \urcorner) \to^{!}_{\mathcal{R}} \ulcorner v \urcorner$$

computation

Runtime Complexity of TRSs

- derivation length

$$\mathsf{dl}(t, \to) = \max\{n \mid \exists s.\ t \to^n s\}$$

$$\mathsf{dl}(n, T, \to) = \max\{\mathsf{dl}(t, \to) \mid t \in T \text{ and } |t| \leqslant n\}$$

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- derivational complexity

$$\mathsf{dc}_{\mathcal{R}}(n) = \mathsf{dl}(n, \mathcal{T}(\mathcal{F}, \mathcal{V}), \rightarrow_{\mathcal{R}})$$

- runtime complexity

$$\mathsf{rc}_{\mathcal{R}}(n) = \mathsf{dl}(n, \mathcal{T}_{\mathsf{b}}, \rightarrow_{\mathcal{R}})$$

$$\mathcal{T}_{\mathsf{b}} := \{f(c_1, \dots, c_n) \mid f \in \mathcal{D} \text{ and } c_i \in \mathcal{T}(\mathcal{C}, \mathcal{V})\}$$

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$$\mathsf{dc}_{\mathcal{R}}(n) = \mathsf{dl}(n, \mathcal{T}(\mathcal{F}, \mathcal{V}), \to_{\mathcal{R}})$$

- innermost runtime complexity

$$\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}(n) = \mathsf{dl}(n, \mathcal{T}_{\mathsf{b}}, \xrightarrow{\mathsf{i}}_{\mathcal{R}})$$

$$\mathcal{T}_{\mathsf{b}} := \{f(c_1, \dots, c_n) \mid f \in \mathcal{D} \text{ and } c_i \in \mathcal{T}(\mathcal{C}, \mathcal{V})\}$$

Polynomial Path Orders $>_{\mathsf{pop*}}$

- $>_{\mathsf{pop*}} \subsetneq >_{\mathsf{mpo}}$
- $>_{\mathsf{pop*}}$ induced by precedence $\succsim$ and safe mapping safe
- $>_{\mathsf{pop*}} \approx >_{\mathsf{mpo}} +$ predicative recursion

Theorem

$$\mathcal{R} \subseteq >_{\mathsf{pop*}} \Rightarrow \mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \textit{ polynomially bounded}$$

for constructor TRS $\mathcal{R}$

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for constructor TRS $\mathcal{R}$

$$\mathsf{rev}(xs;\,) \to \mathsf{rev}_{\mathsf{tl}}(xs;\, [\,])$$

$$\mathsf{rev}_{\mathsf{tl}}([\,];\, ys) \to ys$$

$$\mathsf{rev}_{\mathsf{tl}}(x{:}xs;\, ys) \to \mathsf{rev}_{\mathsf{tl}}(xs;\, x{:}ys)$$

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$$\mathsf{rev}_{\mathsf{tl}}(x{:}xs; ys) \to \mathsf{rev}_{\mathsf{tl}}(xs; x{:}ys)$$

$$\mathsf{rev}_{\mathsf{tl}}(x{:}xs; ys) \not>_{\mathsf{mpo}} \mathsf{rev}_{\mathsf{tl}}(xs; x{:}ys) \qquad ✗$$

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$$\mathsf{rev}(xs;) \to \mathsf{rev}_{\mathsf{tl}}(xs; [\,]) \qquad \mathsf{rev}_{\mathsf{tl}}([\,]; ys) \to ys$$

$$\mathsf{rev}_{\mathsf{tl}}(x{:}xs; ys) \to \mathsf{rev}_{\mathsf{tl}}(xs; x{:}ys)$$

$$\mathcal{R}_{\mathsf{rev}} \not\subseteq >_{\mathsf{pop}*}$$

Predicative Recursion with Parameter Substitution

$$f(\varepsilon, \vec{x}; \vec{y}) = g(\vec{x}; \vec{y})$$
$$f(z{\cdot}i, \vec{x}; \vec{y}) = h_i(z, \vec{x}; \vec{y}, f(z, \vec{x}; \vec{y})),\ \ i \in \{0, 1\}$$

$$\Downarrow$$

$$f(\varepsilon, \vec{x}; \vec{y}) = g(\vec{x}; \vec{y})$$
$$f(z{\cdot}i, \vec{x}; \vec{y}) = h_i(z, \vec{x}; \vec{y}, f(z, \vec{x}; \vec{p}(z, \vec{x}; \vec{y}))),\ \ i \in \{0, 1\}$$

predicative recursion with parameter substitution

Theorem (Bellantoni,1993)

$\mathcal{BC}$ is closed under predicative recursion with parameter substitution scheme.

Polynomial Path Order with Parameter Substitution $>_{\mathsf{pps*}}$

$s = f(s_1, \dots, s_o; s_{o+1}, \dots, s_m) >_{\mathsf{pps*}} t$ if

1. $s_i \gtrsim_{\mathsf{pps*}} t$ for some $i \in \{1, \dots, m\}$
2. $t = g(t_1, \dots, t_p; t_{p+1}, \dots, t_n)$, $f \succ g$, $f \in \mathcal{D}$
   - $s >_{\mathsf{pps}} t_j$ for all $j \in \{1, \dots, p\}$, $s >_{\mathsf{pps*}} t_j$ for all $j \in \{p+1, \dots, n\}$
   - $t_j \in \mathcal{T}(\mathcal{F}^{\prec f}, \mathcal{V})$ for all but one $j \in \{p+1, \dots, n\}$
3. $t = g(t_1, \dots, t_p; t_{p+1}, \dots, t_n)$, $f \approx g$, $f \in \mathcal{D}$
   - $\{\!\{s_1, \dots, s_o\}\!\} >^{\mathsf{mul}}_{\mathsf{pps*}} \{\!\{t_1, \dots, t_p\}\!\}$
   - for all $j \in \{p+1, \dots, n\}$, $s >_{\mathsf{pps*}} t_j$ and $t_j \in \mathcal{T}(\mathcal{F}^{\prec f}, \mathcal{V})$

RPOs $\rightsquigarrow$ $\mathrm{POP}^*_{\mathsf{ps}}$

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   - for all $j \in \{p+1, \ldots, n\}$, $s >_{\mathsf{pps}*} t_j$

$\mathsf{f}(\mathsf{s}(;x); y) >_{\mathsf{pps}*} \mathsf{f}(x; \mathsf{f}(x; y))$ ✗

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$$\mathsf{f}(\mathsf{s}(;x); y) \not>_{\mathsf{pps}*} \mathsf{f}(x; \mathsf{f}(x; y)) \quad \checkmark$$

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$$\mathsf{f}(\mathsf{s}(;x); y) >_{\mathsf{pps}*} \mathsf{g}(\mathsf{f}(x; y), \mathsf{f}(x; y))$$ ✗

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$$f(\vec{x}; \vec{y}) = g(n_1(\vec{x};), \dots, n_j(\vec{x};); s_1(\vec{x}; \vec{y}), \dots, s_k(\vec{x}; \vec{y}))$$

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$$f(\vec{x}; \vec{y}) >_{\mathsf{pps}*} g(n(\vec{x}; \vec{y}); s(\vec{x}; \vec{y})) \quad ✗$$

Auxiliary Order $>_{\mathsf{pps}}$

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$$\mathsf{f}(x; y) \not>_{\mathsf{pps}} \mathsf{g}(x; y)$$

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2. $t = g(t_1, \ldots, t_p; t_{p+1}, \ldots, t_n)$, $f \succ g$, $f \in \mathcal{D}$
   - $s >_{\mathsf{pps}} t_j$ for all $j \in \{1, \ldots, n\}$

$$\mathsf{f}(x; y) \not>_{\mathsf{pps}} y$$

Auxiliary Order $>_{\mathsf{pps}}$

$s = f(s_1, \dots, s_o; s_{o+1}, \dots, s_m) >_{\mathsf{pps}} t$ if

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$$\mathsf{f}(x; y) >_{\mathsf{pps}} x$$

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$$\mathsf{f}(x; y) \ >_{\mathsf{pps}} \mathsf{g}(x;)$$

RPOs $\rightsquigarrow$ $\mathrm{POP}^*_{\mathsf{ps}}$

$s = f(s_1, \ldots, s_o; s_{o+1}, \ldots, s_m) >_{\mathsf{pps}*} t$ if

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   - $s >_{\mathsf{pps}*} t_j$ for all $j \in \{1, \ldots, n\}$
   - $t_j \in \mathcal{T}(\mathcal{F}^{\prec f}, \mathcal{V})$ for all but one $j \in \{p+1, \ldots, n\}$
3. $t = g(t_1, \ldots, t_p; t_{p+1}, \ldots, t_n)$, $f \approx g$
   - $\{\!\{s_1, \ldots, s_o\}\!\} >^{\mathsf{mul}}_{\mathsf{pps}*} \{\!\{t_1, \ldots, t_p\}\!\}$
   - for all $j \in \{p+1, \ldots, n\}$, $s >_{\mathsf{pps}*} t_j$ and $t_j \in \mathcal{T}(\mathcal{F}^{\prec f}, \mathcal{V})$

$$f(\vec{x}; \vec{y}) >_{\mathsf{pps}*} g(n(\vec{x}; \vec{y}); s(\vec{x}; \vec{y})) \quad ✗$$

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Polynomial Path Order with Parameter Substitution $>_{\mathsf{pps}*}$

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Polynomial Path Order with Parameter Substitution $>_{\mathsf{pps*}}$

Theorem

$$\mathcal{R} \subseteq >_{\mathsf{pps*}} \Rightarrow \mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \textit{ polynomially bounded}$$

for constructor TRS $\mathcal{R}$

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$$\mathcal{R} \subseteq >_{\mathsf{pps*}} \Rightarrow \mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \textit{ polynomially bounded}$$

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Observation

$$>_{\mathsf{pop*}} \subsetneq >_{\mathsf{pps*}} \not\subseteq >_{\mathsf{mpo}}$$

$$\mathsf{rev}(xs;\,) \rightarrow \mathsf{rev_{tl}}(xs;\,[\,]) \qquad \mathsf{rev_{tl}}([\,];\,ys) \rightarrow ys$$

$$\mathsf{rev_{tl}}(x{:}xs;\,ys) \rightarrow \mathsf{rev_{tl}}(xs;\,x{:}ys)$$

Polynomial Path Order with Parameter Substitution $>_{\mathsf{pps}*}$

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Polytime Computability and $>_{\mathsf{pps}*}$

Theorem

If $\mathcal{R} \subseteq >_{\mathsf{pps}*}$ then the functions computed by $\mathcal{R}$ are polytime computable.

Polytime Computability and $>_{\mathsf{pps}*}$

terms grow only
polynomial in size

Theorem

- *Let $\mathcal{R}$ be an S-sorted constructor TRS based on a simple signature. If $\mathcal{R} \subseteq >_{\mathsf{pps}*}$ then the functions computed by $\mathcal{R}$ are polytime computable.*

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Example

- $s_1 : \mathsf{Word} \to \mathsf{Word}$ ✓
- $(:) : \mathsf{a} \times \mathsf{List(a)} \to \mathsf{List(a)}$ ✓
- $\mathsf{node} : \mathsf{Tree} \times \mathsf{Tree} \to \mathsf{Tree}$ ✗

Polytime Computability and $>_{\mathsf{pps}*}$

terms grow only polynomial in size

Theorem

- *Let $\mathcal{R}$ be an S-sorted constructor TRS based on a simple signature. If $\mathcal{R} \subseteq >_{\mathsf{pps}*}$ then the functions computed by $\mathcal{R}$ are polytime computable.*
- *Each polytime computable function is computable by an S-sorted, orthogonal, constructor TRS compatible with $>_{\mathsf{pps}*}$.*

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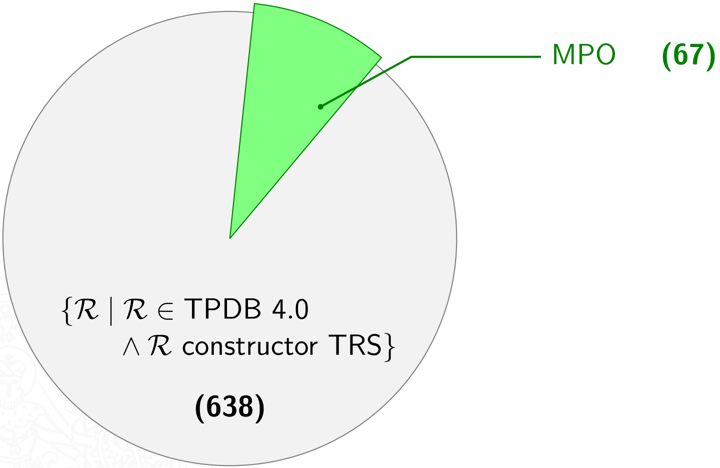
Experimental Results

$\{\mathcal{R} \mid \mathcal{R} \in \text{TPDB 4.0}$
$\quad \wedge \mathcal{R} \text{ constructor TRS}\}$

(638)

Experimental Results



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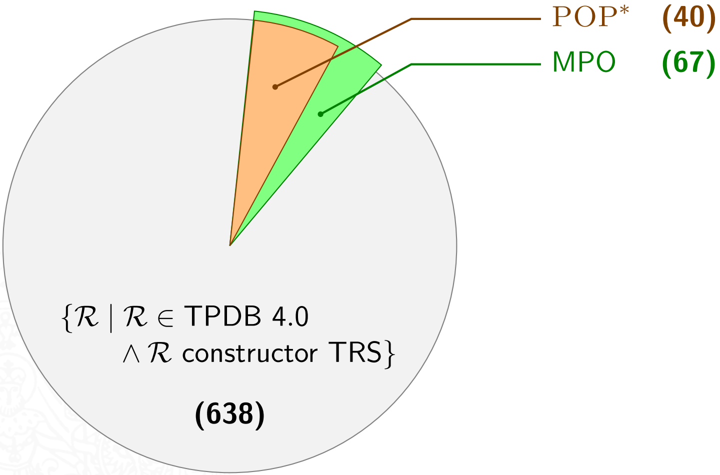

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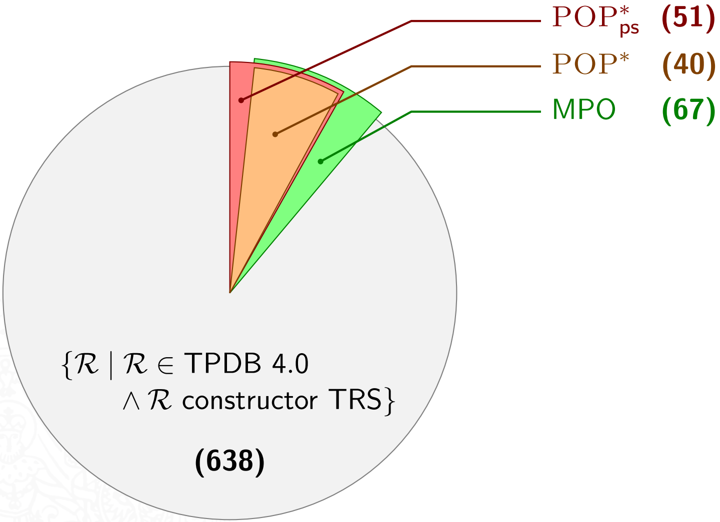

Conclusion

- integrating the rules of predicative recursion with parameter substitution increases the power of polynomial path orders considerably

☺

- term rewriting allows us to combine different techniques for complexity analysis
  - $>_{\mathsf{pps}*}$ + argument filterings used with weak dependency pairs
  - $>_{\mathsf{pps}*}$ + (finite) semantic labeling
  - ...

☺

- $>_{\mathsf{pps}*}$ extensible to non-constructor TRS by putting additional constraints on the precedence

hack ☹

Polynomial Runtime Complexity $\not\Rightarrow$ Polytime Computability

$$\begin{aligned} \mathsf{f}(\epsilon) &\to \mathsf{tip} & \qquad \mathsf{mkTree}(x) &\to \mathsf{node}(x, x) \\ \mathsf{f}(\mathsf{s}_i(x)) &\to \mathsf{mkTree}(\mathsf{f}(x)) \end{aligned}$$

Polynomial Runtime Complexity $\not\Rightarrow$ Polytime Computability

$$\begin{aligned} \mathsf{f}(\epsilon;) &\to \mathsf{tip} & \mathsf{mkTree}(;x) &\to \mathsf{node}(;x,x) \\ \mathsf{f}(\mathsf{s}_i(;x);) &\to \mathsf{mkTree}(;\mathsf{f}(x;)) \end{aligned}$$

$$\mathsf{f} > \mathsf{mkTree} > \mathsf{node} > \mathsf{tip}$$

Polynomial Runtime Complexity $\not\Rightarrow$ Polytime Computability

$$\mathsf{f}(\epsilon;) >_{\mathsf{pps}*} \mathsf{tip} \qquad \mathsf{mkTree}(;x) >_{\mathsf{pps}*} \mathsf{node}(;x,x)$$

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Observations

- $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of above TRS $\mathcal{R}$ polynomial
- $\xrightarrow{\mathsf{i}}_{\mathcal{R}}$ does not give a (direct) polytime algorithm for computing f
  - terms grow exponentially in size

Controlling Size Growth

Definition

an S-sorted signature is called simple if for each constructor $c : s_1 \times \cdots \times s_n \to s$

- $\mathrm{rk}(s_i) \leqslant \mathrm{rk}(s)$ for all $i \in \{1, \ldots, n\}$
- $\mathrm{rk}(s_i) = \mathrm{rk}(s)$ for at most one $i \in \{1, \ldots, n\}$

$\mathrm{rk} : S \to \mathbb{N}$

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Polytime Computability and $>_{\mathsf{pps}*}$

Theorem

let $\mathcal{R}$ be an S-sorted constructor TRS based on a simple signature

- *if $\mathcal{R}$ is compatible with $>_{\mathsf{pps}*}$ then the functions computed by $\mathcal{R}$ are polytime computable*
- *each polytime computable function is computable by such a TRS compatible with $>_{\mathsf{pps}*}$*

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$$[\,] \colon \mathsf{List(a)} \qquad \mathsf{rev} \colon \mathsf{List(a)} \to \mathsf{List(a)}$$

$$(:) \colon \mathsf{a} \times \mathsf{List(a)} \to \mathsf{List(a)} \qquad \mathsf{rev}_{\mathsf{tl}} \colon \mathsf{List(a)} \times \mathsf{List(a)} \to \mathsf{List(a)}$$

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$$\begin{aligned} [\,] &: \mathsf{List}(\mathsf{a}) \\ (:) &: \mathsf{a} \times \mathsf{List}(\mathsf{a}) \to \mathsf{List}(\mathsf{a}) \end{aligned} \qquad \begin{aligned} \mathsf{rev} &: \mathsf{List}(\mathsf{a}) \to \mathsf{List}(\mathsf{a}) \\ \mathsf{rev}_{\mathsf{tl}} &: \mathsf{List}(\mathsf{a}) \times \mathsf{List}(\mathsf{a}) \to \mathsf{List}(\mathsf{a}) \end{aligned}$$