Satisfiability of Non-Linear Arithmetic over Algebraic Numbers

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Overview

- Introduction
- Non-Linear (Ir)rational Arithmetic
  - Extension by $\sqrt{2}$
  - Extension by $\sqrt[n]{m}$
- Evaluation
- Conclusion

Satisfiability of Non-Linear (Ir)rational Arithmetic
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Question

is following TRS of Lucas (AAECC 2006) polynomially terminating ?

$$\begin{aligned} \mathsf{k}(x, x, \mathsf{b}_1) &\to \mathsf{k}(\mathsf{g}(x), \mathsf{b}_2, \mathsf{b}_2) \\ \mathsf{k}(x, \mathsf{a}_2, \mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_1, x, \mathsf{b}_1) \\ \mathsf{k}(\mathsf{a}_4, x, \mathsf{b}_1) &\to \mathsf{k}(x, \mathsf{a}_3, \mathsf{b}_1) \\ \mathsf{k}(\mathsf{g}(x), \mathsf{b}_3, \mathsf{b}_3) &\to \mathsf{k}(x, x, \mathsf{b}_4) \end{aligned}$$

$$\begin{aligned} \mathsf{g}(\mathsf{c}(x)) &\to \mathsf{f}(\mathsf{c}(\mathsf{f}(x))) \\ \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) &\to \mathsf{k}(x, x, x) \end{aligned}$$

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$$\begin{aligned} k(x, x, b_1) &\to k(g(x), b_2, b_2) \\ k(x, a_2, b_1) &\to k(a_1, x, b_1) \\ k(a_4, x, b_1) &\to k(x, a_3, b_1) \\ k(g(x), b_3, b_3) &\to k(x, x, b_4) \end{aligned} \qquad \begin{aligned} g(c(x)) &\to f(c(f(x))) \\ f(f(x)) &\to g(x) \\ f(f(f(f(x)))) &\to k(x, x, x) \end{aligned}$$

Answer: Yes

$$\begin{aligned} a_{1\mathbb{R}} &= 3\sqrt{2} & b_{1\mathbb{R}} &= 5 & f_{\mathbb{R}}(x) &= 2\sqrt{2}x + \sqrt{2} \\ a_{2\mathbb{R}} &= 1 + 3\sqrt{2} & b_{2\mathbb{R}} &= 1 & g_{\mathbb{R}}(x) &= 4\sqrt{2}x + 2 \\ a_{3\mathbb{R}} &= 2 & b_{3\mathbb{R}} &= 3 + \sqrt{2} & c_{\mathbb{R}}(x) &= (6 + 4\sqrt{2})x + \sqrt{2} \\ a_{4\mathbb{R}} &= 4 & b_{4\mathbb{R}} &= 2 + 4\sqrt{2} & k_{\mathbb{R}}(x, y, z) &= x + y + z + 2 \end{aligned}$$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$b + b$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$b + b$ $3 \times b$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$(b + b) > (3 \times b)$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$(b + b) > (3 \times b)$ $\neg x$

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Example

$((\mathsf{b} + \mathsf{b}) > (3 \times \mathsf{b})) \vee (\neg x)$

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Example

$((b + b) > (3 \times b)) \vee (\neg x)$ $\quad$ $a \times a = 2$

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Example

$(((\mathsf{b} + \mathsf{b}) > (3 \times \mathsf{b})) \vee (\neg x)) \wedge (\mathsf{a} \times \mathsf{a} = 2)$

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Example

$(((b + b) > (3 \times b)) \vee (\neg x)) \wedge (a \times a = 2)$

Facts

- undecidable over $\mathbb{N}$, $\mathbb{Z}$ (Hilbert's 10th problem, Matiyasevich 1970)

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Remark

such constraints appear in hard-/software verification

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SMT-COMP

- competition since 2005
- non-linear integer arithmetic (QF_NIA) since 2009
- non-linear real arithmetic (QF_NRA) since 2010

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State-of-the-Art Solvers ($\mathbb{N}$, $\mathbb{Z}$)

no solver handles non-linear arithmetic directly

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- AProVE-NIA : bit-blasts to SAT

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Arithmetic over $\mathbb{R}$

Arithmetic over $\mathbb{R} \supset \mathbb{Q}(\sqrt{2})$

$(\mathbf{c}, \mathbf{d})$ in **R** with $\mathbf{c}, \mathbf{d}$ from **Q** representing $c + d\sqrt{2}$

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$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6})$

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Examples

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2}$

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$$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2}$$

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Definitions

$$(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) := (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f})$$

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$$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6})$$

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$$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = 66 + 28\sqrt{2}$$

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$$(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) := \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \ \wedge \ \mathbf{d} =_{\mathbf{Q}} \mathbf{f}$$

Arithmetic over $\mathbb{R}$ (cont'd)

Problem

how to test $(\mathbf{6}, \mathbf{3}) >_{\mathsf{R}} (\mathbf{2}, \mathbf{5})$?

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$6 + 3\sqrt{2} >_{\mathbb{R}} 2 + 5\sqrt{2}$

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Problem

how to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$? $\frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

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Approximating $m\sqrt{2}$

$$\text{under}(m\sqrt{2}) = \left((m \geqslant 0)\,?\,\frac{5}{4} : \frac{3}{2}\right) \times m$$

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Definition (LPAR-16)

$(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) \;:=\; \mathbf{c} +_{\mathbf{Q}} \text{under}(\mathbf{d}) >_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \text{over}(\mathbf{f})$

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative Solution

how to test $(\mathbf{6}, \mathbf{3}) >_{\mathsf{R}} (\mathbf{2}, \mathbf{5})$?

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$6 + 3\sqrt{2} > 2 + 5\sqrt{2}$

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Alternative Solution

how to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \iff 4 > 2\sqrt{2}$

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how to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$?

$$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \iff 4 > 2\sqrt{2} \iff 16 > 4 \times 2$$

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Definition (SCSS 2010)

$$(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} 0 \ :=$$

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Arithmetic over $\mathbb{R}$ (cont'd)

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$$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \iff 4 > 2\sqrt{2} \iff 16 > 4 \times 2 \iff \top$$

Definition (SCSS 2010)

$$\begin{aligned}(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} 0 \ := \ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0) \vee (\mathbf{c} >_{\mathbf{Q}} 0 \wedge \mathbf{d} \geqslant_{\mathbf{Q}} 0) \vee \\ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} <_{\mathbf{Q}} 0 \wedge \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2) \vee \\ & (\mathbf{c} \leqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0 \wedge \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} <_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2)\end{aligned}$$

Pros

exact

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative Solution

how to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \iff 4 > 2\sqrt{2} \iff 16 > 4 \times 2 \iff \top$

Definition (SCSS 2010)

$$
\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} 0 \ :=\ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0) \vee (\mathbf{c} >_{\mathbf{Q}} 0 \wedge \mathbf{d} \geqslant_{\mathbf{Q}} 0) \vee \\
& (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} <_{\mathbf{Q}} 0 \wedge \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2) \vee \\
& (\mathbf{c} \leqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0 \wedge \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} <_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2)
\end{aligned}
$$

Pros

exact, arbitrary base

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative Solution

how to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \iff 4 > 2\sqrt{2} \iff 16 > 4 \times 2 \iff \top$$

Definition (SCSS 2010)

$$\begin{aligned}(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} 0 \ := \ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0) \vee (\mathbf{c} >_{\mathbf{Q}} 0 \wedge \mathbf{d} \geqslant_{\mathbf{Q}} 0) \vee \\ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} <_{\mathbf{Q}} 0 \wedge \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2) \vee \\ & (\mathbf{c} \leqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0 \wedge \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} <_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2)\end{aligned}$$

Pros & Cons

exact, arbitrary base, auxiliary multiplications

Examples

- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 2)$

Examples

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Examples

- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 2)$ ✓
- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 3)$ ✗

Examples

- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 2)$ ✓
- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 3)$ ✓

Examples

- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 2)$ ✓
- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 3)$ ✓
- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a \times a = 2)$

Examples

- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 2)$ ✓
- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 3)$ ✓
- $((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a \times a = 2)$ ✗

Examples

- $((\mathsf{b}+\mathsf{b} > 3 \times \mathsf{b}) \vee (\mathsf{b} > -\mathsf{a})) \wedge (\mathsf{a} \times \mathsf{a} = 2)$ ✓
- $((\mathsf{b}+\mathsf{b} > 3 \times \mathsf{b}) \vee (\mathsf{b} > -\mathsf{a})) \wedge (\mathsf{a} \times \mathsf{a} = 3)$ ✓
- $((\mathsf{b}+\mathsf{b} > 3 \times \mathsf{b}) \vee (\mathsf{b} > -\mathsf{a})) \wedge (\mathsf{a} \times \mathsf{a} \times \mathsf{a} = 2)$ ✗

Solution

allow $\sqrt[n]{m}$

Overview

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \dots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Examples

$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3})$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Examples

$$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2}$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \dots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Examples

$$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$$

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$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$

$([\mathbf{0}, \mathbf{1}], \mathbf{4})$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Examples

$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$

$([\mathbf{0}, \mathbf{1}], \mathbf{4}) \equiv 0\sqrt[2]{4^0} + 1\sqrt[2]{4^1}$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1\sqrt[n]{m^0} + \cdots + c_n\sqrt[n]{m^{n-1}}$$

Examples

$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$

$([\mathbf{0}, \mathbf{1}], \mathbf{4}) \equiv 0\sqrt[2]{4^0} + 1\sqrt[2]{4^1} = 2$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Examples

$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$

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$([\mathbf{2}, \mathbf{0}], \mathbf{4})$

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Examples

$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$

$([\mathbf{0}, \mathbf{1}], \mathbf{4}) \equiv 0\sqrt[2]{4^0} + 1\sqrt[2]{4^1} = 2$

$([\mathbf{2}, \mathbf{0}], \mathbf{4}) \equiv 2\sqrt[2]{4^0} + 0\sqrt[2]{4^1}$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Examples

$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$

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$([\mathbf{2}, \mathbf{0}], \mathbf{4}) \equiv 2\sqrt[2]{4^0} + 0\sqrt[2]{4^1} = 2$

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Examples

$$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$$

$$([\mathbf{0}, \mathbf{1}], \mathbf{4}) \equiv 0\sqrt[2]{4^0} + 1\sqrt[2]{4^1} = 2$$

$$([\mathbf{2}, \mathbf{0}], \mathbf{4}) \equiv 2\sqrt[2]{4^0} + 0\sqrt[2]{4^1} = 2$$

Remark

representation is not canonical

Example (Addition)

$([1, 2], 3) +_{\mathsf{R}} ([5, 3], 3)$

Example (Addition)

$$([\mathbf{1}, \mathbf{2}], \mathbf{3}) +_{\mathsf{R}} ([\mathbf{5}, \mathbf{3}], \mathbf{3}) \equiv (1 + 2\sqrt[2]{3}) + (5 + 3\sqrt[2]{3})$$

Example (Addition)

$$([\mathbf{1},\mathbf{2}],\mathbf{3}) +_{\mathsf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{3}) \equiv (1 + 2\sqrt[2]{3}) + (5 + 3\sqrt[2]{3}) = 6 + 5\sqrt[2]{3}$$

Example (Addition)

$$([\mathbf{1},\mathbf{2}],\mathbf{3}) +_{\mathsf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{3}) \equiv (1+2\sqrt[2]{3}) + (5+3\sqrt[2]{3}) = 6+5\sqrt[2]{3} \equiv ([\mathbf{6},\mathbf{5}],\mathbf{3})$$

Example (Addition)

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Definitions

$$(\mathbf{c}_n, m) +_{\mathbf{R}} (\mathbf{d}_n, m) := ([c_1 +_{\mathbf{Q}} d_1, \ldots, c_n +_{\mathbf{Q}} d_n], m)$$

Example (Addition)

$$([\mathbf{1},\mathbf{2}],\mathbf{3}) +_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{3}) \equiv (1+2\sqrt[2]{3}) + (5+3\sqrt[2]{3}) = 6+5\sqrt[2]{3} \equiv ([\mathbf{6},\mathbf{5}],\mathbf{3})$$

Definitions

$$\begin{aligned}(\mathbf{c}_n, m) +_{\mathbf{R}} (\mathbf{d}_n, m) &:= ([c_1 +_{\mathbf{Q}} d_1, \ldots, c_n +_{\mathbf{Q}} d_n], m)\\ (\mathbf{c}_n, m) -_{\mathbf{R}} (\mathbf{d}_n, m) &:= ([c_1 -_{\mathbf{Q}} d_1, \ldots, c_n -_{\mathbf{Q}} d_n], m)\end{aligned}$$

Example (Multiplication)

$$([1,2],2) \times_{\mathsf{R}} ([5,3],2)$$

Example (Multiplication)

$$([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathsf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) = ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathsf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1)$$

Example (Multiplication)

$$\begin{aligned}([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathsf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathsf{R}} (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\ &= ((([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathsf{R}} ((([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1)\end{aligned}$$

Example (Multiplication)

$$
\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathsf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathsf{R}} (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= ((([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathsf{R}} ((([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\
&= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathsf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2})
\end{aligned}
$$

Example (Multiplication)

$$
\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathsf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathsf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= (([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathsf{R}} (([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\
&= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathsf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})
\end{aligned}
$$

Example (Multiplication)

$$
\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= (([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} (([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\
&= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})
\end{aligned}
$$

Definition (Multiplication)

$$(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; ((\mathbf{c}_n, m) \cdot d_1) \gg 0$$

Example (Multiplication)

$$\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= ((([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} ((([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\
&= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})
\end{aligned}$$

Definition (Multiplication)

$$(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) := ((\mathbf{c}_n, m) \cdot d_1) \gg 0 +_{\mathbf{R}} \cdots +_{\mathbf{R}} ((\mathbf{c}_n, m) \cdot d_n) \gg (n-1)$$

Example (Multiplication)

$$
\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= (([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} (([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\
&= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})
\end{aligned}
$$

Definition (Multiplication)

$$
\begin{aligned}
(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) &:= ((\mathbf{c}_n, m) \cdot d_1) \gg 0 +_{\mathbf{R}} \cdots +_{\mathbf{R}} ((\mathbf{c}_n, m) \cdot d_n) \gg (n-1) \\
(\mathbf{c}_n, m) \cdot d &:= ([c_1 \times_{\mathbf{Q}} d, \ldots, c_n \times_{\mathbf{Q}} d], m)
\end{aligned}
$$

Example (Multiplication)

$$
\begin{aligned}
(\mathbf{[1,2]},2) \times_{\mathbf{R}} (\mathbf{[5,3]},2) &= ((((\mathbf{[1,2]},2)\cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} ((((\mathbf{[1,2]},2)\cdot \mathbf{3}) \gg 1) \\
&= (((\mathbf{[5,10]},2) \gg 0) +_{\mathbf{R}} (((\mathbf{[3,6]},2) \gg 1) \\
&= (\mathbf{[5,10]},2) +_{\mathbf{R}} (\mathbf{[12,3]},2) = (\mathbf{[17,13]},2)
\end{aligned}
$$

Definition (Multiplication)

$$
\begin{aligned}
(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) &:= ((\mathbf{c}_n, m)\cdot d_1) \gg 0 +_{\mathbf{R}} \cdots +_{\mathbf{R}} ((\mathbf{c}_n, m)\cdot d_n) \gg (n-1) \\
(\mathbf{c}_n, m)\cdot d &:= ([c_1 \times_{\mathbf{Q}} d, \ldots, c_n \times_{\mathbf{Q}} d], m) \\
(\mathbf{c}_n, m) \gg 0 &:= (\mathbf{c}_n, m) \\
(\mathbf{c}_n, m) \gg (i+1) &:= ([m \times_{\mathbf{Q}} c_n, c_1, \ldots, c_{n-1}], m) \gg i
\end{aligned}
$$

Definitions

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

Definitions

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

$$(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m)$$

Definitions

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

$$(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m)$$

$$(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) \;:=\; a >_{\mathbf{Q}} 0$$

Definitions

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

$$(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m)$$

$$(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) \;:=\; a >_{\mathbf{Q}} 0$$

$$(\mathbf{c}_{k+1}, m) >^a_{\mathbf{R}} (\mathbf{d}_{k+1}, m) \;:=\; c_{k+1} +_{\mathbf{Q}} a \geqslant_{\mathbf{Q}} d_{k+1}$$

Definitions

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

$$(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (\mathbf{c}_n, m) >_{\mathbf{R}}^{0} (\mathbf{d}_n, m)$$

$$(\mathbf{c}_0, m) >_{\mathbf{R}}^{a} (\mathbf{d}_0, m) \;:=\; a >_{\mathbf{Q}} 0$$

$$(\mathbf{c}_{k+1}, m) >_{\mathbf{R}}^{a} (\mathbf{d}_{k+1}, m) \;:=\; c_{k+1} +_{\mathbf{Q}} a \geqslant_{\mathbf{Q}} d_{k+1} \wedge (\mathbf{c}_k, m) >_{\mathbf{R}}^{a+(c_{k+1}-d_{k+1})} (\mathbf{d}_k, m)$$

Definitions

$$
\begin{aligned}
(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) &:= (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n) \\
(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) &:= (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m) \\
(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) &:= a >_{\mathbf{Q}} 0 \\
(\mathbf{c}_{k+1}, m) >^a_{\mathbf{R}} (\mathbf{d}_{k+1}, m) &:= c_{k+1} +_{\mathbf{Q}} a \geqslant_{\mathbf{Q}} d_{k+1} \wedge (\mathbf{c}_k, m) >^{a+(c_{k+1}-d_{k+1})}_{\mathbf{R}} (\mathbf{d}_k, m)
\end{aligned}
$$

Examples

$([2, 0], 4) \neq_{\mathbf{R}} ([0, 1], 4)$

Definitions

$$
\begin{aligned}
(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) &:= (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n) \\
(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) &:= (\mathbf{c}_n, m) >_{\mathbf{R}}^{0} (\mathbf{d}_n, m) \\
(\mathbf{c}_0, m) >_{\mathbf{R}}^{a} (\mathbf{d}_0, m) &:= a >_{\mathbf{Q}} 0 \\
(\mathbf{c}_{k+1}, m) >_{\mathbf{R}}^{a} (\mathbf{d}_{k+1}, m) &:= c_{k+1} +_{\mathbf{Q}} a \geqslant_{\mathbf{Q}} d_{k+1} \wedge (\mathbf{c}_k, m) >_{\mathbf{R}}^{a+(c_{k+1}-d_{k+1})} (\mathbf{d}_k, m)
\end{aligned}
$$

Examples

$([\mathbf{2}, \mathbf{0}], \mathbf{4}) \neq_{\mathbf{R}} ([\mathbf{0}, \mathbf{1}], \mathbf{4})$ but $([\mathbf{2}, \mathbf{0}], \mathbf{4}) \equiv ([\mathbf{0}, \mathbf{1}], \mathbf{4})$

Definitions

$$\begin{aligned}
(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) &:= (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n) \\
(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) &:= (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m) \\
(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) &:= a >_{\mathbf{Q}} 0 \\
(\mathbf{c}_{k+1}, m) >^a_{\mathbf{R}} (\mathbf{d}_{k+1}, m) &:= c_{k+1} +_{\mathbf{Q}} a \geqslant_{\mathbf{Q}} d_{k+1} \wedge (\mathbf{c}_k, m) >^{a+(c_{k+1}-d_{k+1})}_{\mathbf{R}} (\mathbf{d}_k, m)
\end{aligned}$$

Examples

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$$

Examples

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$([\mathbf{5}, \mathbf{1}], \mathbf{2}) \not>_{\mathbf{R}} ([\mathbf{2}, \mathbf{3}], \mathbf{2})$

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$([\mathbf{5}, \mathbf{1}], \mathbf{2}) \not>_{\mathbf{R}} ([\mathbf{2}, \mathbf{3}], \mathbf{2})$ but $5 + \sqrt{2} > 6.41 > 6.24 > 2 + 3\sqrt{2}$

Definitions

$$(\mathbf{c}_n, m) =_\mathbf{R} (\mathbf{d}_n, m) \;:=\; (c_1 =_\mathbf{Q} d_1) \wedge \cdots \wedge (c_n =_\mathbf{Q} d_n)$$

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Examples

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Problem

$=_\mathbf{R}$ and $>_\mathbf{R}$ may not appear at "negative" positions

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\end{aligned}
$$

Examples

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$([5,1],2) \not>_{\mathbf{R}} ([2,3],2)$ but $5 + \sqrt{2} > 6.41 > 6.24 > 2 + 3\sqrt{2}$

Problem

$=_{\mathbf{R}}$ and $>_{\mathbf{R}}$ may not appear at "negative" positions

Solution for $=_{\mathbf{R}}$

ensure canonical representation

Definitions

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\end{aligned}
$$

Examples

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Problem

$=_{\mathbf{R}}$ and $>_{\mathbf{R}}$ may not appear at "negative" positions

Solution for $=_{\mathbf{R}}$

ensure canonical representation: $\sqrt[n]{m^p} \notin \mathbb{Z} \quad (1 \leqslant p < n)$

Definitions

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

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Problem

$=_{\mathbf{R}}$ and $>_{\mathbf{R}}$ may not appear at “negative” positions

Solution for $>_{\mathbf{R}}$

replace $>_{\mathbf{R}}$ at negative positions by approximation of $\leqslant_{\mathbf{R}}$

Overview

- Introduction
- Non-Linear (Ir)rational Arithmetic
  - Extension by $\sqrt{2}$
  - Extension by $\sqrt[n]{m}$
- Evaluation
- Conclusion

MiniSmt

`http://cl-informatik.uibk.ac.at/software/minismt`

MiniSmt

http://cl-informatik.uibk.ac.at/software/minismt

Comparison (2×1391 Problems from Termination Analysis)

	$(2,4)$		$(3,4)$		$(3,5)$	
	lpar	scss	lpar	scss	lpar	scss

MiniSmt

http://cl-informatik.uibk.ac.at/software/minismt

Comparison (2×1391 Problems from Termination Analysis)

	$(2,4)$		$(3,4)$		$(3,5)$	
	lpar	scss	lpar	scss	lpar	scss
	sat	sat	sat	sat	sat	sat
matrix 1	308	308	306	303	311	294

MiniSmt

http://cl-informatik.uibk.ac.at/software/minismt

Comparison (2×1391 Problems from Termination Analysis)

	$(2,4)$				$(3,4)$				$(3,5)$			
	lpar		scss		lpar		scss		lpar		scss	
	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg
matrix 1	308	1.5	308	3.0	306	2.0	303	4.3	311	3.8	294	5.5

MiniSmt

http://cl-informatik.uibk.ac.at/software/minismt

Comparison (2×1391 Problems from Termination Analysis)

	(2, 4)				(3, 4)				(3, 5)			
	lpar		scss		lpar		scss		lpar		scss	
	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg
matrix 1	308	1.5	308	3.0	306	2.0	303	4.3	311	3.8	294	5.5
matrix 2	296		276		285		264		237		221	

MiniSmt

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Comparison (2×1391 Problems from Termination Analysis)

	(2, 4)				(3, 4)				(3, 5)			
	lpar		scss		lpar		scss		lpar		scss	
	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg
matrix 1	308	1.5	308	3.0	306	2.0	303	4.3	311	3.8	294	5.5
matrix 2	296	7.0	276	8.4	285	7.8	264	11.6	237	11.0	221	13.7

SMT-COMP 2010

http://www.smtcomp.org/2010/

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QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time
AProVE-NIA	118	2949
CVC3	65	7
MiniSmt	136	1322

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SMT-COMP 2010

http://www.smtcomp.org/2010/

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

SMT-COMP 2010

http://www.smtcomp.org/2010/

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

QF_NRA category (60 problems, $\mathbb{Q}$, $\mathbb{R}$)

tool	score	time
CVC3	3	1
MiniSmt	44	52

SMT-COMP 2010

http://www.smtcomp.org/2010/

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

QF_NRA category (60 problems, $\mathbb{Q}$, $\mathbb{R}$)

tool	score	time	sat	time
CVC3	3	1	–	–
MiniSmt	44	52	44	52

Examples

"Hard" SMT Problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((a Real))
:formula (= (* a a) 2)
)
```

Examples

"Hard" SMT Problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((a Real))
:formula (= (* a a) 2)
)
```

Corresponding SAT Problem (5 bits for variables)

1,207 clauses   417 variables $<$ 0.05 seconds solving time

“Hard” Termination Problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((x17 Real) ...  (x0 Real))
:assumption (>= x17 0)
...
:assumption (>= x0 0)
:formula (and (and (and (and (and (and (and (and (and (and (>= (+ x0 (* x3 x4)) (+ (+ (+ x0 (* x1 x5)) (* x2
x7)) (* x3 x7))) (>= (+ x1 x2) (* x1 x6))) (and (> (+ x0 (* x3 x4)) (+ (+ (+ x0 (* x1 x5)) (* x2 x7)) (* x3
 x7))) (>= (+ x0 (* x3 x4)) (+ (+ (+ x0 (* x1 x5)) (* x2 x7)) (* x3 x7))))) (and (and (>= (+ (+ x0 (* x2 x8
)) (* x3 x4)) (+ (+ x0 (* x1 x9)) (* x3 x4))) (>= x1 x2)) (and (> (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0 (
* x1 x9)) (* x3 x4))) (>= (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0 (* x1 x9)) (* x3 x4)))))) (and (and (>= (
+ (+ x0 (* x1 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4))) (>= x2 x1)) (and (> (+ (+ x0 (* x1 x10)) (*
 x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4))) (>= (+ (+ x0 (* x1 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4
)))))) (and (and (>= (+ (+ (+ x0 (* x1 x5)) (* x2 x12)) (* x3 x12)) (+ x0 (* x3 x13))) (>= (* x1 x6) (+ x1
x2))) (and (> (+ (+ (+ x0 (* x1 x5)) (* x2 x12)) (* x3 x12)) (+ x0 (* x3 x13))) (>= (+ (+ (+ x0 (* x1 x5))
(* x2 x12)) (* x3 x12)) (+ x0 (* x3 x13)))))) (and (and (>= (+ x5 (* x6 x14)) (+ x16 (* x17 (+ x14 (* x15 x
16))))) (>= (* x6 x15) (* x17 (* x15 x17)))) (and (> (+ x5 (* x6 x14)) (+ x16 (* x17 (+ x14 (* x15 x16)))))
 (>= (+ x5 (* x6 x14)) (+ x16 (* x17 (+ x14 (* x15 x16)))))))) (and (and (>= (+ x16 (* x17 x16)) x5) (>= (*
 x17 x17) x6)) (and (> (+ x16 (* x17 x16)) x5) (>= (+ x16 (* x17 x16)) x5)))) (and (and (>= (+ x16 (* x17 (
+ x16 (* x17 (+ x16 (* x17 x16)))))) x0) (>= (* x17 (* x17 (* x17 x17))) (+ (+ x1 x2) x3))) (and (> (+ x16
(* x17 (+ x16 (* x17 (+ x16 (* x17 x16)))))) x0) (>= (+ x16 (* x17 (+ x16 (* x17 (+ x16 (* x17 x16)))))) x0
)))) (and (and (and (and (and (and (and (and (>= (+ x16 (* x17 (+ x16 (* x17 (+ x16 (* x17 x16)))))) x0) (>
= (* x17 (* x17 (* x17 x17))) (+ (+ x1 x2) x3))) (and (> (+ x16 (* x17 (+ x16 (* x17 (+ x16 (* x17 x16)))))
) x0) (>= (+ x16 (* x17 (+ x16 (* x17 (+ x16 (* x17 x16)))))) x0))) (and (and (>= (+ x16 (* x17 x16)) x5) (
>= (* x17 x17) x6)) (and (> (+ x16 (* x17 x16)) x5) (>= (+ x16 (* x17 x16)) x5)))) (and (and (>= (+ x5 (* x
6 x14)) (+ x16 (* x17 (+ x14 (* x15 x16))))) (>= (* x6 x15) (* x17 (* x15 x17)))) (and (> (+ x5 (* x6 x14))
 (+ x16 (* x17 (+ x14 (* x15 x16))))) (>= (+ x5 (* x6 x14)) (+ x16 (* x17 (+ x14 (* x15 x16)))))))) (and (a
nd (>= (+ (+ (+ x0 (* x1 x5)) (* x2 x12)) (* x3 x12)) (+ x0 (* x3 x13))) (>= (* x1 x6) (+ x1 x2))) (and (>
```

"Hard" Termination Problem

```
(+ (+ (+ x0 (* x1 x5)) (* x2 x12)) (* x3 x12)) (+ x0 (* x3 x13))) (>= (+ (+ (+ x0 (* x1 x5)) (* x2 x12)) (*
 x3 x12)) (+ x0 (* x3 x13)))))) (and (and (>= (+ (+ x0 (* x1 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4
))) (>= x2 x1)) (and (> (+ (+ x0 (* x1 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4))) (>= (+ (+ x0 (* x1
 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4)))))) (and (and (>= (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0
 (* x1 x9)) (* x3 x4))) (>= x1 x2)) (and (> (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0 (* x1 x9)) (* x3 x4)))
(>= (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0 (* x1 x9)) (* x3 x4)))))) (and (and (>= (+ x0 (* x3 x4)) (+ (+
(+ x0 (* x1 x5)) (* x2 x7)) (* x3 x7))) (>= (+ x1 x2) (* x1 x6))) (and (> (+ x0 (* x3 x4)) (+ (+ (+ x0 (* x
1 x5)) (* x2 x7)) (* x3 x7))) (>= (+ x0 (* x3 x4)) (+ (+ (+ x0 (* x1 x5)) (* x2 x7)) (* x3 x7))))))) (and (
and (and (and (and (>= x1 1) (>= x2 1)) (>= x3 1)) (>= x6 1)) (>= x15 1)) (>= x17 1))))
```

"Hard" Termination Problem

```
(+ (+ (+ x0 (* x1 x5)) (* x2 x12)) (* x3 x12)) (+ x0 (* x3 x13))) (>= (+ (+ (+ x0 (* x1 x5)) (* x2 x12)) (*
 x3 x12)) (+ x0 (* x3 x13)))))) (and (and (>= (+ (+ x0 (* x1 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4
))) (>= x2 x1)) (and (> (+ (+ x0 (* x1 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4))) (>= (+ (+ x0 (* x1
 x10)) (* x3 x4)) (+ (+ x0 (* x2 x11)) (* x3 x4)))))) (and (and (>= (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0
 (* x1 x9)) (* x3 x4))) (>= x1 x2)) (and (> (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0 (* x1 x9)) (* x3 x4)))
(>= (+ (+ x0 (* x2 x8)) (* x3 x4)) (+ (+ x0 (* x1 x9)) (* x3 x4)))))) (and (and (>= (+ x0 (* x3 x4)) (+ (+
(+ x0 (* x1 x5)) (* x2 x7)) (* x3 x7))) (>= (+ x1 x2) (* x1 x6))) (and (> (+ x0 (* x3 x4)) (+ (+ (+ x0 (* x
1 x5)) (* x2 x7)) (* x3 x7))) (>= (+ x0 (* x3 x4)) (+ (+ (+ x0 (* x1 x5)) (* x2 x7)) (* x3 x7))))))) (and (
and (and (and (and (>= x1 1) (>= x2 1)) (>= x3 1)) (>= x6 1)) (>= x15 1)) (>= x17 1))))
```

Corresponding SAT Problem (5 bits for variables)

65,420 clauses   149,755 variables < 1 second solving time

Underlying TRS (Lucas 2006)

$$
\begin{aligned}
\mathsf{k}(x, x, \mathsf{b}_1) &\to \mathsf{k}(\mathsf{g}(x), \mathsf{b}_2, \mathsf{b}_2) \\
\mathsf{k}(x, \mathsf{a}_2, \mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_1, x, \mathsf{b}_1) \\
\mathsf{k}(\mathsf{a}_4, x, \mathsf{b}_1) &\to \mathsf{k}(x, \mathsf{a}_3, \mathsf{b}_1) \\
\mathsf{k}(\mathsf{g}(x), \mathsf{b}_3, \mathsf{b}_3) &\to \mathsf{k}(x, x, \mathsf{b}_4)
\end{aligned}
\qquad
\begin{aligned}
\mathsf{g}(\mathsf{c}(x)) &\to \mathsf{f}(\mathsf{c}(\mathsf{f}(x))) \\
\mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\
\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) &\to \mathsf{k}(x, x, x)
\end{aligned}
$$

Underlying TRS (Lucas 2006)

$$\begin{aligned}
\mathsf{k}(x,x,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{g}(x),\mathsf{b}_2,\mathsf{b}_2) & \mathsf{g}(\mathsf{c}(x)) &\to \mathsf{f}(\mathsf{c}(\mathsf{f}(x)))\\
\mathsf{k}(x,\mathsf{a}_2,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_1,x,\mathsf{b}_1) & \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x)\\
\mathsf{k}(\mathsf{a}_4,x,\mathsf{b}_1) &\to \mathsf{k}(x,\mathsf{a}_3,\mathsf{b}_1) & \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) &\to \mathsf{k}(x,x,x)\\
\mathsf{k}(\mathsf{g}(x),\mathsf{b}_3,\mathsf{b}_3) &\to \mathsf{k}(x,x,\mathsf{b}_4)
\end{aligned}$$

using MiniSmt as backend, $\mathsf{T_TT_2}$ finds following compatible interpretation:

$$\begin{aligned}
\mathsf{a}_{1\mathbb{R}} &= 3\sqrt{2} & \mathsf{b}_{1\mathbb{R}} &= 5 & \mathsf{f}_{\mathbb{R}}(x) &= 2\sqrt{2}x+\sqrt{2}\\
\mathsf{a}_{2\mathbb{R}} &= 1+3\sqrt{2} & \mathsf{b}_{2\mathbb{R}} &= 1 & \mathsf{g}_{\mathbb{R}}(x) &= 4\sqrt{2}x+2\\
\mathsf{a}_{3\mathbb{R}} &= 2 & \mathsf{b}_{3\mathbb{R}} &= 3+\sqrt{2} & \mathsf{c}_{\mathbb{R}}(x) &= (6+4\sqrt{2})x+\sqrt{2}\\
\mathsf{a}_{4\mathbb{R}} &= 4 & \mathsf{b}_{4\mathbb{R}} &= 2+4\sqrt{2} & \mathsf{k}_{\mathbb{R}}(x,y,z) &= x+y+z+2
\end{aligned}$$

Overview

- Introduction
- Non-Linear (Ir)rational Arithmetic
  - Extension by $\sqrt{2}$
  - Extension by $\sqrt[n]{m}$
- Evaluation
- Conclusion

SMT Solving is Suitable for Termination Analysis

Arctic Interpretations
Multiset Path Order
Knuth-Bendix Order
predictive labeling
usable rules
Finding Loops
dependency pairs
dependency graphs
semantic labeling
recursive
Matrix Interpretations
argument filterings
below zero
SCC
usable rules
Lexicographic Path Order
match-bounds
Polynomial Interpretations
Increasing Interpretations
Monotonic Semantic Path Order

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Future Work

- preprocessing

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Future Work

- preprocessing
- unsatisfiability

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Future Work

- preprocessing
- unsatisfiability
- certification