Dependency Graphs, Relative Rule Removal, and Derivational Complexity

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Derivational Complexity

(We consider only finite TRS in this talk)

Definition

The derivation height of a (terminating) term t is:

$$\mathsf{dh}(t, \to) = \max\{n \in \mathbb{N} \mid \exists s : t \to^n s\}$$

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The derivational complexity of a (terminating) TRS $\mathcal{R}$ is:

$$\mathsf{dc}_{\mathcal{R}}(n) = \max\{\mathsf{dh}(t, \to_{\mathcal{R}}) \mid |t| \leqslant n\}$$

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- Bounds on $\mathsf{dc}_{\mathcal{R}}$ can be extracted from termination proofs

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- Polynomial Interpretations (double exponential or lower) (Hofbauer and Lautemann 1989)
- Matrix Interpretations (exponential or lower) (Endrullis et al. 2008)
- MPO (primitive recursive, characterisation) (Hofbauer 1992)
- LPO (multiply recursive, characterisation) (Weiermann 1995)
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What about the Dependency Pair Framework?

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A DP problem is a pair $(\mathcal{P}, \mathcal{R})$ ($\mathcal{P}$ a set of dependency pairs, $\mathcal{R}$ a set of rewrite rules)

A DP problem $(\mathcal{P}, \mathcal{R})$ is finite if there exists no infinite derivation

$$t_1^\sharp \to_{\mathcal{P}} t_2^\sharp \to_{\mathcal{R}}^* t_3^\sharp \to_{\mathcal{P}} t_4^\sharp \to_{\mathcal{R}}^* \cdots$$

where all proper subterms of $t_i^\sharp$ are terminating

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We lift derivational complexity to DP problems $(\mathcal{P}, \mathcal{R})$:

$$\mathsf{dc}_{(\mathcal{P},\mathcal{R})}(n) = \max\{\mathsf{dh}(t^\sharp, \rightarrow_{\mathcal{P}/\mathcal{R}}) \mid |t| \leqslant n\}$$

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Obtaining upper bounds on $\mathsf{dc}_{(\mathcal{P},\mathcal{R})}$ from $(\succsim, \succ)$ is usually easy

Example

TRS $\mathcal{R}$:

$$\mathsf{f}(\mathsf{S}(x), y) \to \mathsf{f}(x, \mathsf{f}(x, y)) \qquad \mathsf{f}(0, x) \to \mathsf{c}(x, x)$$

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For instance:

$$\begin{aligned}
\mathsf{f}(\mathsf{S}(\mathsf{S}(0)), 0) &\to \mathsf{f}(\mathsf{S}(0), \mathsf{f}(\mathsf{S}(0), 0)) \\
&\to^2 \mathsf{f}(0, \mathsf{f}(0, \mathsf{f}(0, \mathsf{f}(0, 0)))) \\
&\to \mathsf{c}(\mathsf{f}(0, \mathsf{f}(0, \mathsf{f}(0, 0))), \mathsf{f}(0, \mathsf{f}(0, \mathsf{f}(0, 0)))) \\
&\to^{14} \mathsf{c}(\mathsf{c}(\mathsf{c}(\mathsf{c}(0, 0), \mathsf{c}(0, 0)), \mathsf{c}(\mathsf{c}(0, 0), \mathsf{c}(0, 0))), \\
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Reduction pair based on algebra $\mathcal{A}$:

$$\mathsf{f}^\sharp_{\mathcal{A}}(x, y) = x \qquad \mathsf{S}_{\mathcal{A}}(x) = x + 1 \qquad \mathsf{f}_{\mathcal{A}}(x, y) = 0$$
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Hence, $\mathsf{dc}_{(\mathrm{DP}(\mathcal{R}),\mathcal{R})}$ is linear

Complexity of Reduction Pairs

Theorem (Moser and S. 2009)

$$\mathrm{dc}_{\mathcal{R}}(n) \leqslant 2^{2^{n \cdot 2^{O(\mathrm{dc}_{(\mathrm{DP}(\mathcal{R}),\mathcal{R})}(n))}}}$$

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Example

Termination by DP+polynomial interpretation implies elementary derivational complexity

Dependency Graphs

Definition

The dependency graph $\mathrm{DG}(\mathcal{R})$ of $\mathcal{R}$ is this graph:

- Nodes: $\mathrm{DP}(\mathcal{R})$
- Edges: $\{(s \to t, u \to v) \mid \exists\sigma\exists\tau.\ t\sigma \to_{\mathcal{R}}^{*} u\tau\}$

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A TRS $\mathcal{R}$ is terminating iff for every (nontrivial) SCC $\mathcal{C}$ of $\mathrm{DG}(\mathcal{R})$, the DP *problem $(\mathcal{C}, \mathcal{R})$ is finite*

Primitive Recursion as Lower Bound

TRS $\mathcal{R}$:

$$\mathsf{S}(x) \circ (y \circ z) \to x \circ (\mathsf{S}(\mathsf{S}(y)) \circ z)$$
$$\mathsf{S}(x) \circ (y \circ (z \circ w)) \to x \circ (z \circ (y \circ w))$$

Proposition (Hofbauer 1992)

$$\mathsf{dh}(\underbrace{\mathsf{S}^{2(m+1)}(0) \circ (0 \circ (\ldots (0 \circ 0) \ldots))}_{k+2 \text{ occurrences of } \circ}, \to_{\mathcal{R}}) \geqslant \mathsf{Ack}(k, m)$$

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We can fix k by labelling $\circ$ suitably

Primitive Recursion as Lower Bound

TRS $\mathcal{R}_k$ (over signature $\{\mathsf{S}, 0\} \cup \{\circ_n \mid 1 \leqslant n \leqslant k+2\}$):

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Compute (nontrivial) SCCs of DG($\mathcal{R}_k$)

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Members of SCCs in $\mathsf{DG}(\mathcal{R}_k)$:

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Reduction pair for each (nontrivial) SCC based on algebra $\mathcal{A}$:

$$\circ^\sharp_{n\mathcal{A}}(x, y) = x \quad \circ_{n\mathcal{A}}(x, y) = x \quad \mathsf{S}_{\mathcal{A}}(x) = x + 1 \quad 0_{\mathcal{A}} = 0$$

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For each SCC $\mathcal{C}$, $\mathsf{dc}_{(\mathcal{C},\mathcal{R})}$ is linear

Complexity of Dependency Graphs

Theorem (Arts and Giesl 2000)

A TRS $\mathcal{R}$ is terminating iff for every (nontrivial) SCC $\mathcal{C}$ of $\mathsf{DG}(\mathcal{R})$, *the* DP *problem* $(\mathcal{C}, \mathcal{R})$ *is finite*

Theorem (Moser and S. 2010)

If for every SCC $\mathcal{C}$ of $\mathsf{DG}(\mathcal{R})$, $\mathsf{dc}_{(\mathcal{C},\mathcal{R})}$ *is primitive recursive, then* $\mathsf{dc}_{\mathcal{R}}$ *is primitive recursive*

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Example

Termination by DP+DG+polynomial interpretations implies primitive recursive derivational complexity

Relative Rule Removal

Theorem (Giesl et al. 2004)

A DP *problem* $(\mathcal{P}, \mathcal{R})$ *is finite iff there exist* $\mathcal{P}' \subset \mathcal{P}$ *and a* reduction pair $(\gtrsim, \succ)$ *such that* $\mathcal{P} \setminus \mathcal{P}' \subseteq \succ$, $\mathcal{P}' \cup \mathcal{R} \subseteq \gtrsim$, *and* $(\mathcal{P}', \mathcal{R})$ *is finite*

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Intuition: for each SCC, its dependency pairs are successively removed from the termination problem

Example Proof (Endrullis et al. 2008)

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Algebra $\mathcal{A}$:

$$\circ^\sharp_{\mathcal{A}}(x, y) = y \quad \circ_{\mathcal{A}}(x, y) = y + 1 \quad \mathsf{S}_{\mathcal{A}}(x) = x \quad 0_{\mathcal{A}} = 0$$

Example Proof (Endrullis et al. 2008)

TRS $\mathcal{R}$:

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Both reduction pairs imply linear complexity
$\mathsf{dc}_{\mathcal{R}}$ is not primitive recursive

Complexity of Relative Rule Removal

Theorem (Giesl et al. 2004)

A DP *problem* $(\mathcal{P}, \mathcal{R})$ *is finite iff there exist* $\mathcal{P}' \subset \mathcal{P}$ *and a reduction pair* $(\gtrsim, \succ)$ *such that* $\mathcal{P} \setminus \mathcal{P}' \subseteq \succ$, $\mathcal{P}' \cup \mathcal{R} \subseteq \gtrsim$, *and* $(\mathcal{P}', \mathcal{R})$ *is finite*

Theorem (Moser and S. 2010)

Let $\mathcal{R}$ *be terminating by dependency graphs and the above theorem. If for every reduction pair application,* $\mathsf{dc}_{(\mathcal{P}\setminus\mathcal{P}', \mathcal{P}'\cup\mathcal{R})}$ *is multiply recursive, then* $\mathsf{dc}_{\mathcal{R}}$ *is multiply recursive*

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Example

Termination by DP+DG+multiple applications of polynomial interpretations implies multiply recursive derivational complexity

Conclusions

Summary:

- DP + elementary base method (e.g. poly, matrix) imply elementary complexity
- DP + DG + primitive recursive base method (e.g. poly, matrix, MPO) imply primitive recursive complexity
- DP + DG + multiply recursive base method (e.g. poly, matrix, MPO, LPO) + relative rule removal imply multiply recursive complexity

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- Investigate other processors of DP framework

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Question

If $\mathcal{R}$ admits a sound termination proof by $\mathsf{T_TT_2}$/AProVE/..., is $\mathsf{dc}_{\mathcal{R}}$ then bounded by a multiply recursive function?