Conditional via Unconditional Rewriting – Some Recent Developments

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Outline

- a unifying approach to transformations from CTRSs to TRSs [WADT'08/LNCS'09]
- (dis)proving properties of CTRSs via transformations (unravelings) and context-sensitivity
  [JLAP'09, to appear]
- unsoundness of unravelings: new sufficient criteria and new counterexamples
  [RTA'10]

(joint work with Karl Gmeiner and Felix Schernhammer)

Transformations – A General View

transformation T from (class of) CTRSs to (class of) TRSs:

$T\colon \mathcal{R} = (\mathcal{F}, R) \mapsto ((\mathcal{F}', R', \rightarrow_{\mathcal{R}'}), \phi, \psi)$ with $\mathcal{R}' = (\mathcal{F}', R')$ TRS, $\rightarrow_{\mathcal{R}'}$ subset of rewrite relation induced by $\mathcal{R}'$ satisfying:

- $\mathcal{R} = (\mathcal{F}, R)$ finite $\Rightarrow \mathcal{R}' = (\mathcal{F}', R')$ finite
- T restricted to finite systems effectively constructible
- $\phi\colon \mathcal{T} \rightarrow \mathcal{T}'$ injective and total (initialization)
- $\psi\colon \mathcal{T}' \hookrightarrow \mathcal{T}$ (partial backtranslation) defined at least on the set $\mathcal{T}'_{\mathcal{R}}$ of *reachable terms*: $\mathcal{T}'_r = \{t' \in \mathcal{T}' \mid \phi(s) \rightarrow^*_{\mathcal{R}'} t' \text{ for some } t \in \mathcal{T}\}$
- $\psi(\phi(s)) = s$ for all $s \in \mathcal{T}$

Properties of Transformations

Two versions of *soundness* and *completeness* properties

- *T sound (for reduction)* (or *simulation sound*) *[w.r.t. reachable terms]* if for every $\mathcal{R} = (\mathcal{F}, R)$ with $T(\mathcal{R}) = ((\mathcal{F}', R', \rightarrow_{\mathcal{R}'}), \phi, \psi)$ we have:
  $\forall s, t \in \mathcal{T}\colon \phi(s) \rightarrow^*_{\mathcal{R}'} \phi(t) \Rightarrow s \rightarrow^*_{\mathcal{R}} t$
  $[\forall s', t' \in \mathcal{T}'_r\colon s' \rightarrow^*_{\mathcal{R}'} t' \Rightarrow \psi(s') \rightarrow^*_{\mathcal{R}} \psi(t'))\ ]$.
- *T complete (for reduction)* (or *simulation complete*) *[w.r.t. reachable terms]* if for every $\mathcal{R} = (\mathcal{F}, R)$ with $T(\mathcal{R}) = ((\mathcal{F}', R', \rightarrow_{\mathcal{R}'}), \phi, \psi)$ we have:
  $\forall s, t \in \mathcal{T}\colon s \rightarrow^*_{\mathcal{R}} t \Rightarrow \phi(s) \rightarrow^*_{\mathcal{R}'} \phi(t)$
  $[\forall s' \in \mathcal{T}'_r, t \in \mathcal{T}\colon \psi(s') \rightarrow^*_{\mathcal{R}} t \Rightarrow \exists t' \in \mathcal{T}'_r\colon s' \rightarrow^*_{\mathcal{R}'} t', \psi(t') = t\ ]$.
- analogously *soundness/completeness for P [w.r.t. reachable terms]* for *P* termination, confluence ...

How to transform?

Initialize, verify, eliminate

$$\rho :: l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \Longrightarrow \begin{cases} l \to C_\rho[s_1, \ldots, s_n, D[l/r]] \\ C_\rho[t_1, \ldots, t_n, D[l/r]] \to r \end{cases}$$

How to preserve information for reproducing rhs's?

- $D[l/r] = l$: trivially leads to non-termination
- $D[l/r] = r$: ok for simulation completeness, but violates soundness
- avoid trivial non-termination by extending the arity of function symbols (e.g. [Viry'99], [Antoy/Brassl/Hanus'03], [Serbanuta/Rosu'06])
- avoid trivial non-termination and further interference by just storing variable bindings (*unravelings*, [Marchiori'96], [Ohlebusch'02])

Unraveling for normal 1-CTRSs [Marchiori'96]

just keep variable bindings

$$\rho : l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$$

is transformed into

$$\rho' : l \to U_\rho(s_1, \ldots, s_n, \overrightarrow{Var(l)})$$
$$\rho'' : U_\rho(t_1, \ldots, t_n, \overrightarrow{Var(l)}) \to r$$

Example

$$\mathsf{or}(x, y) \to y \Leftarrow x \to^* \mathsf{false} \quad \Longrightarrow \begin{cases} \mathsf{or}(x, y) \to U_1(x, x, y) \\ U_1(\mathsf{false}, x, y) \to y \end{cases}$$

$$\mathsf{or}(x, y) \to x \Leftarrow x \to^* \mathsf{true} \quad \Longrightarrow \begin{cases} \mathsf{or}(x, y) \to U_2(x, x, y) \\ U_2(\mathsf{true}, x, y) \to x \end{cases}$$

Properties of Unravelings

- CTRS $\mathcal{R} = (\mathcal{F}, R)$ over terms $\mathcal{T} = \mathcal{T}(\mathcal{F}, \mathcal{V})$
- Unraveling $\mathcal{R}' = (\mathcal{F}', R')$ over terms $\mathcal{T}' = \mathcal{T}(\mathcal{F}', \mathcal{V})$ where $\mathcal{F} \subseteq \mathcal{F}'$

Completeness

$s \rightarrow^*_{\mathcal{R}} t \Longrightarrow s \rightarrow^*_{\mathcal{R}'} t$ for all $s, t \in \mathcal{T}$.

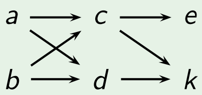
Soundness

$s \rightarrow^*_{\mathcal{R}'} t \Longrightarrow s \rightarrow^*_{\mathcal{R}} t$ for all $s, t \in \mathcal{T}$.

- Only known result: Unravelings sound for left-linear normal 1-CTRSs [Marchiori'96], [Ohlebusch'02]
- Potential problem: Reductions in variable positions in *mixed terms*.

Unsoundness: Simplified example of [Marchiori'96]

Example



$$
\begin{array}{rcl}
h(x,x) & \to & g(x,x,f(k)) \\
g(d,x,x) & \to & A \\
f(x) & \to & x \Leftarrow x \to^* e
\end{array}
\quad \Longrightarrow \quad
\left\{ \begin{array}{r} f(x) \to U(x,x) \\ U(e,x) \to x \end{array} \right.
$$

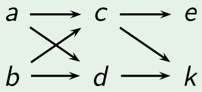
Unraveling $\mathcal{R}'$ is unsound:

$$
h(f(a),f(b)) \to^* h(U(a,a),U(b,b)) \to^* h(U(c,d),U(c,d))
$$
$$
\to g(U(c,d),U(c,d),f(k)) \to^* g(d,U(k,k),U(k,k)) \to A
$$

Yet, $h(f(a),f(b)) \not\to^*_{\mathcal{R}} A$.

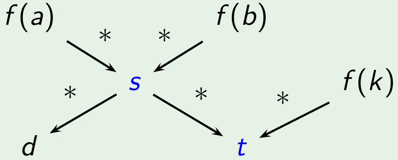
Unsoundness: Simplified example of [Marchiori'96] (2)

Example



$$f(x) \;\to\; x \Leftarrow x \to^* e \qquad \Longrightarrow \qquad \begin{cases} f(x) \to U(x,x) \\ U(e,x) \to x \end{cases}$$

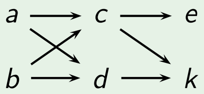
For the following diagram, there are terms s, t in $\mathcal{R}'$ but not in $\mathcal{R}$



- In $\mathcal{R}'$, $s = U(c,d), t = U(k,k)$.
- In $\mathcal{R}$, $t = f(k)$, therefore $s \in \{f(c), f(d), f(k)\}$, yet $s \not\to^*_{\mathcal{R}} d$.

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



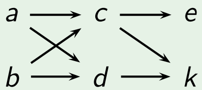
$$\begin{array}{rcl} h(x,x) & \to & g(x,x,f(k)) \\ g(\color{red}{d},x,x) & \to & A \\ f(x) & \to & x \Leftarrow x \to^* e \end{array} \quad \Longrightarrow \quad \left\{ \begin{array}{r} f(x) \to U(x,x) \\ U(e,x) \to x \end{array} \right.$$

Example is *not*

- a constructor system

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



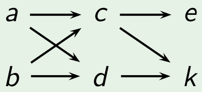
$$
\begin{array}{rcl}
h(x,x) & \to & g(x,x,f(k)) \\
g(d,x,x) & \to & A \\
f(x) & \to & {\color{red}x} \Leftarrow x \to^* e
\end{array}
\quad\Longrightarrow\quad
\begin{cases}
f(x) \to U(x,x) \\
U(e,x) \to {\color{red}x}
\end{cases}
$$

Example is *not*

- a constructor system
- non-collapsing

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



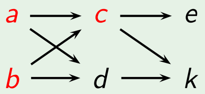
$$
\begin{aligned}
h(x,x) &\rightarrow g(x,x,f(k)) \\
g(d,x,x) &\rightarrow A \\
f(x) &\rightarrow x \Leftarrow x \rightarrow^* e
\end{aligned}
\quad\Longrightarrow\quad
\begin{cases}
f(x) \rightarrow U(x,x) \\
U(e,x) \rightarrow x
\end{cases}
$$

Example is *not*

- a constructor system
- non-collapsing
- an overlay system

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



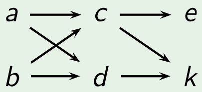
$$
\begin{aligned}
h(x,x) &\rightarrow g(x,x,f(k)) \\
g(d,x,x) &\rightarrow A \\
f(x) &\rightarrow x \Leftarrow x \rightarrow^* e
\end{aligned}
\quad \Longrightarrow \quad
\begin{cases}
f(x) \rightarrow U(x,x) \\
U(e,x) \rightarrow x
\end{cases}
$$

Example is *not*

- a constructor system
- non-collapsing
- an overlay system
- non-overlapping

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



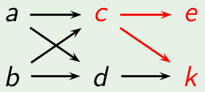
$$
\begin{aligned}
h(x,x) &\to g(x,x,f(k)) \\
g(d,x,x) &\to A \\
f(x) &\to x \Leftarrow x \to^* e
\end{aligned}
\qquad \Longrightarrow \qquad
\begin{cases} f(x) \to U(x,x) \\ U(e,x) \to x \end{cases}
$$

Example is *not*

- a constructor system
- non-collapsing
- an overlay system
- non-overlapping
- a system with only one left-linear rule

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



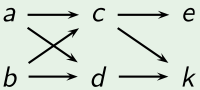
$$
\begin{aligned}
h(x,x) &\rightarrow g(x,x,f(k)) \\
g(d,x,x) &\rightarrow A \\
f(x) &\rightarrow x \Leftarrow x \rightarrow^* e
\end{aligned}
\quad \Longrightarrow \quad
\begin{cases} f(x) \rightarrow U(x,x) \\ U(e,x) \rightarrow x \end{cases}
$$

Example is *not*

- a constructor system
- non-collapsing
- an overlay system
- non-overlapping
- a system with only one left-linear rule
- confluent

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



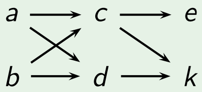
$$\begin{aligned} h(x,x) &\to g(x,x,f(k)) \\ \color{red}{g(d,x,x)} &\color{red}{\to A} \\ f(x) &\to x \Leftarrow x \to^* e \end{aligned} \quad \Longrightarrow \quad \begin{cases} f(x) \to U(x,x) \\ U(e,x) \to x \end{cases}$$

Example is *not*

- a constructor system
- non-collapsing
- an overlay system
- non-overlapping
- a system with only one left-linear rule
- confluent
- non-erasing

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



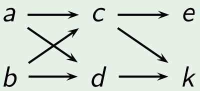
$$\begin{aligned} h(x,x) &\to g(x,x,f(k)) \\ g(d,x,x) &\to A \\ f(x) &\to x \Leftarrow x \to^* e \end{aligned} \quad \Longrightarrow \quad \begin{cases} f(x) \to U(x,x) \\ U(e,x) \to x \end{cases}$$

Example is *not*

- a constructor system
- non-collapsing
- an overlay system
- non-overlapping
- a system with only one left-linear rule
- confluent
- non-erasing
- right-linear

Unsoundness: Simplified example of [Marchiori'96] (3)

Example



$$
\begin{aligned}
h(x,x) &\to g(x,x,f(k)) \\
g(d,x,x) &\to A \\
f(x) &\to x \Leftarrow x \to^* e
\end{aligned}
\quad \Longrightarrow \quad
\begin{cases}
f(x) \to U(x,x) \\
U(e,x) \to x
\end{cases}
$$

Example is *not*

- a constructor system
- non-collapsing
- an overlay system
- non-overlapping
- a system with only one left-linear rule
- confluent
- non-erasing
- right-linear
- left-linear ([Marchiori'96])

Proving soundness

Soundness

- CTRS $\mathcal{R}$ with terms $\mathcal{T}$, unraveling $\mathcal{R}'$ with terms $\mathcal{T}'$
- To prove: $s \to^*_{\mathcal{R}'} t \Longrightarrow s \to^*_{\mathcal{R}} t$ for all $s, t \in \mathcal{T}$

Proof structure

- Derivation $D : u_1 \to_{p_1} u_2 \to_{p_2} \cdots \to_{p_{n-1}} u_n$ in $\mathcal{R}'$,
- Translation $\mathbf{t} : \mathcal{T}' \mapsto \mathcal{T}$,
- Monotony of $\mathbf{t}$: If $\mathbf{t}(u_i|_{p_i}) \to^*_{\mathcal{R}} \mathbf{t}(u_{i+1}|_{p_i})$, then $\mathbf{t}(u_i|_q) \to^*_{\mathcal{R}} \mathbf{t}(u_{i+1}|_{q'})$ for all $q \in Pos(u_i)$ and descendants q' of q,
- Technical key result: $\mathbf{t}(u_i|_{p_i}) \to^*_{\mathcal{R}} \mathbf{t}(u_{i+1}|_{p_i})$.
- Projecting reductions:

$$\begin{array}{ccccccc} u_1 & \to_{\mathcal{R}'} & u_2 & \cdots & \to_{\mathcal{R}'} & u_n \\ \downarrow & & \downarrow & & & \downarrow \\ \mathbf{t}(u_1) & \to^*_{\mathcal{R}} & \mathbf{t}(u_2) & \cdots & \to^*_{\mathcal{R}} & \mathbf{t}(u_n) \end{array}$$

Translation of U-terms

Translate backwards tb

$$\mathrm{tb}(t) = \begin{cases} t & \text{if } t \in \mathcal{T} \\ f(\mathrm{tb}(t_1), \ldots, \mathrm{tb}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } f \in \mathcal{F} \\ l\,\mathrm{tb}(\sigma) & \text{if } t = U_\rho(u_1, \ldots, u_n, \overrightarrow{Var(l)}\sigma) \\ & \text{where } \rho : l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \end{cases}$$

Translate forward tf

$$\mathrm{tf}(t) = \begin{cases} t & \text{if } t \in \mathcal{T} \\ f(\mathrm{tf}(t_1), \ldots, \mathrm{tf}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } f \in \mathcal{F} \\ r\,\mathrm{tf}(\sigma) & \text{if } t = U_\rho(u_1, \ldots, u_n, \overrightarrow{Var(l)}\sigma) \\ & \text{where } \rho : l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \end{cases}$$

Translations of U-terms - cont'd

Example

$$a \to c \qquad a \to d$$

$$f(x) \to x \Leftarrow x \to^* c \quad \Longrightarrow \quad \begin{cases} f(x) \to U(x,x) \\ U(c,x) \to x \end{cases}$$

$$D_1 : f(a) \to U(a,a) \to^2 U(c,d) \to d : \begin{cases} \mathrm{tb}(U(c,d)) = f(d) \\ \mathrm{tf}(U(c,d)) = d \end{cases}$$

$$D_2 : f(a) \to U(a,a) \to^2 U(d,d) \not\to d : \begin{cases} \mathrm{tb}(U(d,d)) = f(d) \\ \mathrm{tf}(U(d,d)) = d \end{cases}$$

Soundness for confluent CTRSs

Soundness result for confluent CTRSs

$D : u_1 \rightarrow_{\mathcal{R}',p_1} u_2 \rightarrow_{\mathcal{R}',p_2} \cdots u_n$ where $u_1 \in \mathcal{T}$.

- If $\mathcal{R}$ is confluent, then $\mathrm{tb}(u_i|_{p_i}) \rightarrow_{\mathcal{R}}^{\leq 1} \mathrm{tb}(u_{i+1}|_{p_i})$.
- If $\mathcal{R}$ is confluent, then $\mathcal{R}'$ is sound.

Soundness for non-erasing CTRSs

Soundness result for non-erasing CTRSs

$D : u_1 \rightarrow_{\mathcal{R}',p_1} u_2 \rightarrow_{\mathcal{R}',p_2} \cdots u_n$ where $u_n \in \mathcal{T}$.

- If $\mathcal{R}$ is non-erasing, then $\mathsf{tf}(u_i|_{p_i}) \rightarrow_{\mathcal{R}}^{\leq 1} \mathsf{tf}(u_{i+1}|_{p_i})$.
- If $\mathcal{R}$ is non-erasing, then $\mathcal{R}'$ is sound.

Soundness for CTRSs with ground conditions

Right-linearity?

Right-linearity of $\mathcal{R}$ not sufficient.

Right-linear $\mathcal{R}'$

If $\mathcal{R}'$ is right-linear, all conditions are ground:

$$l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \quad \Longrightarrow \begin{cases} l \to U_\rho(s_1, \ldots, s_n, \overrightarrow{Var(l)}) \\ U_\rho(t_1, \ldots, t_n, \overrightarrow{Var(l)}) \to r \end{cases}$$

Soundness result for CTRSs with ground conditions

- If $\mathcal{R}$ has only ground conditions, $\mathcal{R}'$ is sound.

Weakly left-linear normal 1-CTRSs

Motivation

- Join 1-CTRSs consist of rules $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$.
- Join 1-CTRSs can be simulated by normal 1-CTRSs:

$$l \to r \Leftarrow s \downarrow t \quad \Longrightarrow \begin{cases} l \to r \Leftarrow eq(x,x) \to^* true \\ \mathrm{eq}(x,x) \to \mathrm{true} \end{cases}$$

Weakly left-linear normal 1-CTRSs

- $\mathcal{R}$ is *weakly left-linear*, if every rule $l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ in $\mathcal{R}$ is left-linear, or unconditional and all non-linear variables in l do not occur in r.
- $\mathrm{eq}(x,x) \to \mathrm{true}$ is weakly left-linear,
- $h(x,x) \to x$ is not weakly left-linear.

Soundness for weakly left-linear normal 1-CTRSs

Idea: Translate backwards dynamically, depending on

$D: u_1 \rightarrow_{\mathcal{R}',p_1} u_2 \rightarrow_{\mathcal{R}',p_2} \cdots u_n$ where $u_1 \in \mathcal{T}$.

Translate backwards w.r.t. actual derivation

$$
\mathrm{tb}_D(i,p) = \begin{cases} u_{i,p} & \text{if } u_{i,p} \in \mathcal{T} \\ f(\mathrm{tb}_D(i,p.1),\ldots,\mathrm{tb}_D(i,p.n)) & \text{if } u_i|_p = f(t_1,\ldots,t_n) \\ & \text{and } f \in \mathcal{F} \\ \mathrm{tb}_D(i-1,p') & \text{if } \mathrm{root}(u_i|_p) \notin \mathcal{F} \\ & \text{and } u_{i-1}|_{p'} \text{ ancestor of } u_i|_p \end{cases}
$$

- For weakly left-linear $\mathcal{R}$, $\mathrm{tb_D}$ is well-defined (ancestors of U-terms are unique).
- If $\mathcal{R}$ is weakly left-linear, $\mathrm{tb}_D(i,p_i) \rightarrow^*_{\mathcal{R}} \mathrm{tb}(u_i|_{p_i})$.

Soundness for weakly left-linear normal 1-CTRSs (2)

Example

$$a \to c \qquad a \to d$$

$$f(x) \to x \Leftarrow x \to^* c \quad \Longrightarrow \quad \begin{cases} f(x) \to U(x,x) \\ U(c,x) \to x \end{cases}$$

$\mathrm{tb_D}$ moves backwards in derivation. $\mathrm{tb}_D(5,\epsilon) = f(f(a))$:

$$D : f(f(a)) \to f(U(a,a)) \to^2 f(U(c,d)) \to U(U(c,d),U(c,d))$$

Soundness for weakly left-linear normal 1-CTRSs (3)

Soundness result for weakly left-linear CTRSs

- If $\mathcal{R}$ is weakly left-linear, $\mathsf{tb}_D(i, p_i) \rightarrow^*_{\mathcal{R}} \mathsf{tb}_D(i+1, p_i)$.
- If $\mathcal{R}$ is weakly left-linear, $\mathcal{R}'$ is sound.
- Left-linear join 1-CTRSs can be soundly unraveled.

Transforming CTRSs into unconditional TRSs

Some history (incomplete)

- [Dershowitz/Plaisted'85]
- [Bergstra/Klop'86]
- [Marchiori'96] unravelings
- [Ohlebusch'99/02]
- [Viry'99] different class of transformation
- [Antoy/Brassel/Hanus'03]
- [Serbanuta/Rosu'06]
- [Gmeiner/Gramlich'08/09]
- [Duran et al 04/08]
- [Nishida et al 04/05/07/09]
- [Schernhammer/Gramlich'09]
- [Gmeiner/Gramlich/Schernhammer'10]

Conclusion and perspective

Positive results

- Soundness for non-erasing normal 1-CTRSs
- Soundness for confluent normal 1-CTRSs
- Soundness for normal 1-CTRSs with ground conditions
- Soundness for weakly left-linear normal 1-CTRSs (implies soundness for left-linear join 1-CTRSs)

Perspective

- Extension to more general classes of CTRSs (deterministic CTRSs): Ongoing work.