Bounded Completion

Completion

Given an equational system $\mathcal{E}$, find $\mathcal{R}$ such that

- $\mathsf{SN}(\mathcal{R})$
- $\mathcal{R} \subseteq \leftrightarrow_{\mathcal{E}}^{*}$
- $\mathcal{E} \cup \mathsf{CP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$

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Knuth-Bendix completion:

- $\mathsf{SN}(\mathcal{R})$: use reduction order
- $\mathcal{R} \subseteq \leftrightarrow^*_{\mathcal{E}}$: through inference rules
- $\mathcal{E} \cup \mathsf{CP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$: search: fix strategy

Completion

Multi-Completion

- $SN(\mathcal{R})$: construct order iteratively "on-the-fly"
- $\mathcal{R} \subseteq \leftrightarrow^*_{\mathcal{E}}$: through inference rules
- $\mathcal{E} \cup CP(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$: search: fix strategy

Bounded Completion

- $SN(\mathcal{R})$: encode to termination constraints
- $\mathcal{R} \subseteq \leftrightarrow^*_{\mathcal{E}}$: use candidate set $\mathcal{A}$ s.t. $\mathcal{R} \subseteq \mathcal{A} \subseteq \leftrightarrow^*_{\mathcal{E}}$
- $\mathcal{E} \cup CP(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$: encode to termination constraints

Termination Constraints

Definition

Termination constraints:

$$C ::= s \to t \mid \top \mid \bot \mid \neg C \mid C \vee C \mid C \wedge C$$

$\mathcal{R} \models C$ if

- $\mathsf{SN}(\mathcal{R})$ and
- C is true where $s \to t$ is true whenever $s \to t \in \mathcal{R}$

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Example

Consider $C = (1 \vee 2) \wedge (1 \vee 3) \wedge (\neg 1 \vee 4)$ with

$1\colon\ \mathsf{f}(x) \to \mathsf{g}(x) \quad 2\colon\ \mathsf{g}(x) \to \mathsf{f}(x) \quad 3\colon\ \mathsf{g}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \quad 4\colon\ \mathsf{f}(\mathsf{f}(x)) \to \mathsf{a}$

Then $\{1,4\} \models C$, $\{1,3\} \not\models C$, $\{2,3\} \models C$

Termination Constraints

Conversion to order constraints

Definition

- $c_{l>r}$ denotes the order constraint for $l > r$ (e.g. lpo, MI)
- $\widehat{C}$ denotes C where each $l \to r$ replaced by prop. variable

Then $\mathcal{R} \models C$ if and only if $\mathcal{R}_\alpha = \mathcal{R}$ and α satisfies

$$\widehat{C} \wedge \bigwedge \{\neg x_{l\to r} \vee c_{l>r} \mid l \to r \in \mathcal{R}\text{ules}(C)\}$$

Advantages (here)

- can use deMorgan's laws (Tseitin) to convert to CNF
- fast SAT-solving for order constraints

Induced Sets

Compute candidate set of rules for $\mathcal{R}$

Definition

Given $\mathcal{E}$, the induced TRSs $\mathcal{B}_i$ and ESs $\mathcal{A}_i$ ($i \in \mathbb{N}$) are

$$\mathcal{A}_i = \begin{cases} \mathcal{E} & \text{if } i = 0 \\ \mathcal{A}_{i-1} \cup \mathrm{CP}(\mathcal{B}_{i-1}) \cup \to_{\mathcal{B}_{i-1}}(\mathcal{A}_{i-1}) & \text{if } i > 0 \end{cases}$$

$$\mathcal{B}_i = \{ s \to t \mid s \approx t \in \mathcal{A}_i,\ s \notin \mathcal{V}, \text{ and } \mathcal{V}\mathrm{ar}(s) \supseteq \mathcal{V}\mathrm{ar}(t) \}$$

Lemma

Whenever $s \approx t \in \mathcal{A}_i$ or $s \to t \in \mathcal{B}_i$, the conversion $s \leftrightarrow^*_{\mathcal{E}} t$ holds.

Encoding ℓ-joinability

Definition

$((\gamma_1, \ldots, \gamma_m), (\delta_1, \ldots, \delta_n))$ is a ℓ-join instance of (s, t) if $\gamma_1, \ldots, \gamma_m, \delta_1, \ldots, \delta_n \in \mathcal{R}$, $m, n \leqslant \ell$, and

$$s \rightarrow_{\gamma_1} \cdot \cdots \cdot \rightarrow_{\gamma_m} \cdot {}_{\delta_n}\!\leftarrow \cdot \cdots \cdot {}_{\delta_1}\!\leftarrow t$$

$\sqsupseteq$ on sequences : $(a_1, \ldots, a_n) \sqsupseteq (a_{i_1}, \ldots, a_{i_m})$
whenever $1 \leqslant i_1 < \cdots < i_m \leqslant n$

$J^\ell(s, t)$ denotes set of minimal ℓ-join instances

ℓ-join constraint $\mathsf{J}^\ell_{\mathcal{R}}(s, t)$:

$$\bigvee \{ \gamma_1 \wedge \cdots \wedge \gamma_n \wedge \delta_1 \wedge \cdots \wedge \delta_m \mid (\vec{\gamma}, \vec{\delta}) \in J^\ell(s, t) \}$$

Lemma

Assume $\mathsf{SN}(\mathcal{R})$ *and* $\mathcal{R} \subseteq \mathcal{S}$. *Then* $\mathcal{R} \models \mathsf{J}^\ell_{\mathcal{S}}(s, t)$ *iff* $s \downarrow^\ell_{\mathcal{R}} t$.

WCR-constraints

Definition

local confluence constraint WCR_k^ℓ is the conjunction of

$$\neg\, l_1 \to r_1 \;\vee\; \neg\, l_2 \to r_2 \;\vee\; \mathsf{J}^\ell_{\mathcal{B}_k}((l_2\mu)[r_1\mu]_p, r_2\mu)$$

for all overlaps $(l_1 \to r_1, p, l_2 \to r_2)_\mu$ of $\mathcal{B}_k$

Lemma

Let $\mathcal{R} \subseteq \mathcal{B}_k$. If $\mathcal{R} \models \mathsf{WCR}_k^\ell$ then $\mathcal{R}$ is complete.

constraint for bounded completion BC_k^ℓ:

$$\bigwedge\{\mathsf{J}^\ell_{\mathcal{B}_k}(s,t) \mid s \approx t \in \mathcal{E}\} \;\wedge\; \mathsf{WCR}_k^\ell$$

Soundness

Theorem
$\mathcal{R}$ is a complete TRS for $\mathcal{E}$ if $\mathcal{R} \subseteq \mathcal{B}_k$ and $\mathcal{R} \models \mathsf{BC}_k^\ell$ for some k, ℓ

Example:

$$1:\ \mathsf{f}(\mathsf{g}(x)) \approx x \qquad\qquad 2:\ \mathsf{f}(x) \approx \mathsf{g}(\mathsf{g}(x))$$

- $\mathcal{A}_0 = \{1, 2\}$ and $\mathcal{B}_0 = \{\overrightarrow{1}, \overrightarrow{2}, \overleftarrow{2}\}$
- $\mathcal{A}_1 = \{1, 2, \ldots, 5\}$ and $\mathcal{B}_1 = \mathcal{B}_0 \cup \{\overrightarrow{3}, \overleftarrow{3}, \overrightarrow{4}, \overleftarrow{4}, \overleftarrow{5}\}$, where

$$3:\ \mathsf{f}(\mathsf{f}(x)) \approx \mathsf{g}(x) \quad 4:\ \mathsf{g}(\mathsf{f}(x)) \approx \mathsf{f}(\mathsf{g}(x)) \quad 5:\ x \approx \mathsf{g}(\mathsf{g}(\mathsf{g}(x)))$$

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Example:

$$1\colon\ \mathsf{f}(\mathsf{g}(x)) \approx x \qquad\qquad 2\colon\ \mathsf{f}(x) \approx \mathsf{g}(\mathsf{g}(x))$$

- WCR e.g. the overlap of $\overrightarrow{1}$ and $\overleftarrow{2}$

  $$\neg\,\overrightarrow{1} \vee \neg\,\overleftarrow{2} \vee \mathsf{J}^\ell_{\mathcal{B}_1}(\mathsf{g}(x), \mathsf{f}(\mathsf{f}(x)))$$

  For $\ell = 1$, this is

  $$\neg\,\overrightarrow{1} \vee \neg\,\overrightarrow{2} \vee \overrightarrow{3} \vee \overleftarrow{3}$$

Incompleteness

Example

Consider the ES $\mathcal{E}$

$$\mathsf{f}(\mathsf{a}, x, z) \approx \mathsf{f}(y, \mathsf{b}, z) \qquad \mathsf{f}(\mathsf{a}, \mathsf{b}, z) \approx \mathsf{b}$$

Complete TRS for $\mathcal{E}$:

$$\mathsf{f}(\mathsf{a}, x, z) \to \mathsf{b} \qquad \mathsf{f}(y, \mathsf{b}, z) \to \mathsf{b} \qquad \mathsf{f}(\mathsf{a}, \mathsf{b}, z) \to \mathsf{b}$$

Bounded completion fails for any bound k.

$$\mathcal{A}_0 = \mathcal{E} \qquad \mathcal{B}_0 = \{\mathsf{f}(\mathsf{a}, \mathsf{b}, z) \to \mathsf{b}\}$$

$\mathcal{A}_i = \mathcal{A}_0$ and $\mathcal{B}_i = \mathcal{B}_0$ for all $i \in \mathbb{N}$, but $\mathcal{B}_0$ not complete for $\mathcal{E}$

Knuth-Bendix completion

Given reduction order $>$

$$\textsc{Deduce}\quad \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \quad \text{if } s \approx t \in \mathrm{CP}(\mathcal{R})$$

$$\textsc{Orient}\quad \frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}} \quad \text{if } s > t$$

$$\textsc{Delete}\quad \frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

$$\textsc{Simplify}\quad \frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}} \quad \text{if } s \to_{\mathcal{R}} u$$

$$\textsc{Compose}\quad \frac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}} \quad \text{if } t \to_{\mathcal{R}} u$$

$$\textsc{Collapse}\quad \frac{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}} \quad \text{if } s \xrightarrow{\cdot\!>}_{\mathcal{R}} u$$

Knuth-Bendix completion

Consider run with $\langle \mathcal{E}_0, \mathcal{R}_0 \rangle = \langle \mathcal{E}, \varnothing \rangle$:

$$\langle \mathcal{E}_0, \mathcal{R}_0 \rangle \vdash \langle \mathcal{E}_1, \mathcal{R}_1 \rangle \vdash \langle \mathcal{E}_1, \mathcal{R}_2 \rangle \vdash \cdots$$

$\mathcal{R}_\omega := \bigcup_{i \geqslant 0} \bigcap_{j \geqslant i} \mathcal{R}_j$ A run is fair if $\mathrm{CP}(\mathcal{R}_\omega) \subseteq \bigcup_{i \geqslant 0} \mathcal{E}_i$

Theorem (Knuth,Bendix)

Let $\langle \mathcal{E}, \varnothing \rangle \vdash^* \langle \varnothing, \mathcal{R} \rangle$ for some $>$. Then $\mathcal{R}$ is complete for $\mathcal{E}$.

Theorem

Let $\mathcal{E}$ be an ES and $\langle \mathcal{E}, \varnothing \rangle \vdash^k \langle \emptyset, \mathcal{R} \rangle$ a fair run under $>$.

Then $\mathcal{R} \subseteq \mathcal{B}_k$ and $\mathcal{R} \models \mathrm{BC}_k^\ell$ for some $\ell \in \mathbb{N}$.

Strict subsumption

Consider ES $\mathcal{E}$

$$1\colon\ \mathsf{f}(x) \approx \mathsf{f}(\mathsf{a}) \qquad\qquad 2\colon\ \mathsf{f}(\mathsf{b}) \approx \mathsf{b}$$

$\mathcal{A}_2 = \mathcal{A}_1 \cup$

$$3\colon\ \mathsf{f}(\mathsf{a}) \approx \mathsf{b} \qquad\qquad 4\colon\ \mathsf{f}(x) \approx \mathsf{b}$$

Let $\mathcal{R} = \{\overrightarrow{4}\}$. $\mathcal{R} \models \mathsf{BC}_2^1$, since

- $\mathcal{R}$ is terminating and $\mathcal{R} \subseteq \mathcal{B}_2$
- $\leftrightarrow_{\mathcal{E}} \subseteq \downarrow_{\mathcal{R}}^1$, thus $\mathcal{R} \models \bigwedge\{\mathsf{J}_{\mathcal{B}_k}^{\ell}(s,t) \mid s \approx t \in \mathcal{E}\}$
- $\mathsf{CP}(\mathcal{R}) = \varnothing$, thus $\mathcal{R} \models \mathsf{WCR}_1^1$

Knuth-Bendix completion: assume $\mathsf{f}(\mathsf{b}) > \mathsf{b}$ holds.
Only two possible runs:

$$\langle \mathcal{E}, \varnothing \rangle \qquad\qquad \langle \mathcal{E}, \varnothing \rangle \vdash \langle \{1\}, \{\overrightarrow{2}\} \rangle$$

Classes of completeable systems

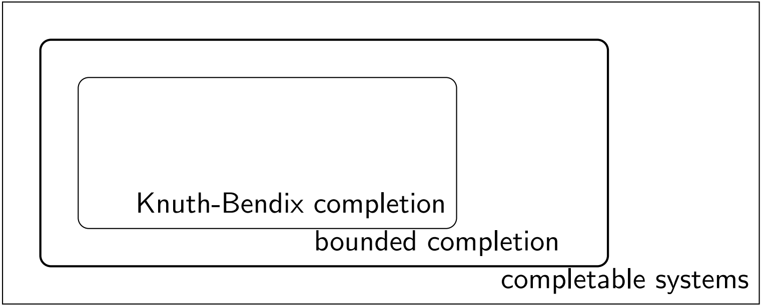

Figure: Hierarchy

Automation

$\mathcal{A}_i$ explode, i.e. for group-theory $|\mathcal{A}_2| = 550$, $|\mathcal{A}_3| =?$

heuristical optimization

- restrict $\mathcal{A}_i$ to eqs $s \approx t$ with $|s| + |t| < d$
- reduce eqs in $\mathcal{A}_i$ by a restricted version of $\mathcal{B}_i$
- reduce during computation of join-instances

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tool-demo

Experiments

- 52 examples (from mkbTT- homepage)
- 30 secs timeout
- order constraints for lpo and MI with dim 2

	lpo	matrix interpretations
bc	27	22

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Conclusion

- bc allows optimization of order constraints
- search by SAT-solvers / generalized term. provers
- candidate sets of $\mathcal{A}_i$ need refinement

Future work:

- DPLL-approach
- indexing
- unfailing completion

Thanks for your attention!