Revisiting Matrix Interpretations for Polynomial Derivational Complexity of Term Rewriting

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Theorem (Moser, Schnabl and Waldmann 2008)

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- extend to $\mathbb{R}$ and $\mathbb{Q}$

Outline

- Introduction
- Preliminaries
- Establishing Derivational Complexity
- Polynomial Growth of Matrix Products
- Experimental Results

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- order $(x_1, \ldots, x_n)^T > (y_1, \ldots, y_n)^T \;:\Longleftrightarrow\; x_1 >_\delta y_1 \;\wedge\; \bigwedge_{i=2}^n \; x_i \geqslant y_i$
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- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_0 = \{x \in \mathbb{R} \mid x \geqslant 0\}$
- relation $>_\delta$ on $\mathbb{R}$ for every $\delta \in \mathbb{R}^+$: $x >_\delta y$ if $x - y \geqslant \delta$

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Observation

- matrix interpretation $\mathcal{M}$ maps

$$t = t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to \dots$$

to decreasing sequence of vectors of non-negative numbers

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if $\mathcal{M}$ is compatible with TRS $\mathcal{R}$, then $\mathrm{dc}_{\mathcal{R}}$ is asymptotically bounded by $\mathrm{growth}_{\mathcal{M}}$

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- general solution = linear combination of

$$\lambda_i^k, \quad k\,\lambda_i^k, \quad k^2\,\lambda_i^k, \quad \ldots \quad , \quad k^{m_{\lambda_i}-1}\,\lambda_i^k$$

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holds for matrix interpretations over $\mathbb{N}$, $\mathbb{Q}$ *and* $\mathbb{R}$

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$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathsf{c}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x}$$

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Outline

Experimental Results

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MSW08	46	158	177	203
NZM10	88	200	209	212

Theorem (NZM10)

If $\mathcal{R}$ admits a matrix interpretation $\mathcal{M}$ of dimension n, such that $\rho(A) \leqslant 1$ for the component-wise maximum A of all matrices in $\mathcal{M}$, then $dc_{\mathcal{R}}(k) = \mathcal{O}(k^{d+1})$

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- allows non-triangular matrices without losing polynomial boundedness

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- replace the rough overestimate $M_1 \cdot M_2 \cdot \ldots \cdot M_k \leqslant A^k$ by a more concise analysis using Joint Spectral Radius Theory

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- unify both approaches via Joint Spectral Radius Theory

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