On Implementing Relative Complexity

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Overview

- Derivational Complexity
- Relative Rewriting
- Modular Complexity
- Experiments
- Conclusion

Derivational Complexity

Definition (Derivation Height)

$\mathsf{dh}(t, \to) = \max\{n \mid t = t_0 \to t_1 \to \cdots \to t_n\}$

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$\mathsf{dc}(m, \to) = \max\{\mathsf{dh}(t, \to) \mid |t| \leqslant m\}$

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Lemma

$>$ *rewrite relation*

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$>$ *rewrite relation and* $\mathcal{R} \subseteq >$ $\longrightarrow$ $\mathrm{dc}(m, >) \geqslant \mathrm{dc}(m, \to_{\mathcal{R}})$

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Current approaches

TMIs

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$$\mathsf{a}(x) \to \mathsf{b}(x)$$

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$[t]_{\mathbb{N}} \leqslant 2 \cdot |t|$ $\longrightarrow$ $\mathrm{dc}(m, >_{\mathbb{N}}) = \mathcal{O}(m)$ $\longrightarrow$ $\mathrm{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m)$

Example ($\mathcal{R} = \text{AG01}/\#3.7$)

$$\mathsf{half}(0) \to 0 \qquad \mathsf{log}(\mathsf{s}(0)) \to 0$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad \mathsf{log}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{log}(\mathsf{s}(\mathsf{half}(x))))$$

Example ($\mathcal{R}$ = AG01/#3.7)

$$\begin{aligned} \mathsf{half}(0) &> 0 & \mathsf{log}(\mathsf{s}(0)) &> 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &> \mathsf{s}(\mathsf{half}(x)) & \mathsf{log}(\mathsf{s}(\mathsf{s}(x))) &> \mathsf{s}(\mathsf{log}(\mathsf{s}(\mathsf{half}(x)))) \end{aligned}$$

TMI

$$\mathsf{log}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x \qquad \mathsf{s}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\mathsf{half}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 0_{\mathcal{M}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

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TMI (0.7 sec)

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$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m^2)$

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AMI

$$\mathsf{log}_{\mathcal{M}}(x) = \begin{pmatrix} 4 & 0 \\ 5 & 0 \end{pmatrix} x \qquad \mathsf{s}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & 4 \\ 6 & 0 \end{pmatrix} x$$

$$\mathsf{half}_{\mathcal{M}}(x) = \begin{pmatrix} 2 & -\infty \\ 0 & -\infty \end{pmatrix} x \qquad 0_{\mathcal{M}} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

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Implementation

Strategy

```
LINEAR    = (arctic -dim 1 -ib 4 -ob 5 -direct ||
             arctic -dim 2 -ib 3 -ob 4 -direct ||
             arctic -dim 3 -ib 2 -ob 3 -direct ||
             bounds ||
             matrix -triangle -dim 1 -ib 5 -ob 6 -direct ||
             ...)

QUADRATIC = (matrix -triangle -dim 2 -ib 4 -ob 5 -direct || ...)

CUBIC     = (matrix -triangle -dim 3 -ib 3 -ob 4 -direct || ...)

QUARTIC   = (matrix -triangle -dim 4 -ib 2 -ob 3 -direct || ...)
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Execution

run `LINEAR || QUADRATIC || CUBIC || QUARTIC`

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Execution

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and take tightest bound

Relative Termination

Definition

$\rightarrow_{\mathcal{R}/\mathcal{S}} = \rightarrow_{\mathcal{S}}^{*} \cdot \rightarrow_{\mathcal{R}} \cdot \rightarrow_{\mathcal{S}}^{*}$

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Proof idea

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$t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{S}} t_3 \rightarrow_{\mathcal{S}} t_4 \rightarrow_{\mathcal{R}} \cdots$

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Relative Termination for Complexity

Question

$$\mathsf{dc}(m, \to_{\mathcal{R} \cup \mathcal{S}}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathsf{dc}(m, \to_{\mathcal{S}}))$$

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Counterexample (Hofbauer, 2006)

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Relative Complexity

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R} \cup \mathcal{S}}) \;=\; \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathsf{dc}(m, \to_{\mathcal{S}/\mathcal{R}}))$

Proof idea

$\mathcal{R} \subseteq > \; \mathcal{S} \subseteq \geqslant$

$\mathcal{R} \subseteq \geqslant \; \mathcal{S} \subseteq >$

$$t_1 \;\to_{\mathcal{R}}\; t_2 \;\to_{\mathcal{S}}\; t_3 \;\to_{\mathcal{S}}\; t_4 \;\to_{\mathcal{R}}\; \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix}$$

Relative Complexity

Lemma

$$\mathrm{dc}(m, \to_{\mathcal{R} \cup \mathcal{S}}) \;=\; \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathrm{dc}(m, \to_{\mathcal{S}/\mathcal{R}}))$$

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$\mathcal{R} \subseteq \geqslant \;\; \mathcal{S} \subseteq >$

$$t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{S}} t_3 \to_{\mathcal{S}} t_4 \to_{\mathcal{R}} \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix}$$

Relative Complexity

Lemma

$$\mathrm{dc}(m, \to_{\mathcal{R} \cup \mathcal{S}}) \;=\; \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathrm{dc}(m, \to_{\mathcal{S}/\mathcal{R}}))$$

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$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_3 \\ t_3 \end{pmatrix}$$

Relative Complexity

Lemma

$$\mathrm{dc}(m, \to_{\mathcal{R} \cup \mathcal{S}}) = \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathrm{dc}(m, \to_{\mathcal{S}/\mathcal{R}}))$$

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$\mathcal{R} \subseteq \geqslant \mathcal{S} \subseteq >$

$$t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{S}} t_3 \to_{\mathcal{S}} t_4 \to_{\mathcal{R}} \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_3 \\ t_3 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_4 \\ t_4 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \cdots$$

Relative Complexity

Lemma

$$\mathrm{dc}(m, \to_{\mathcal{R}\cup\mathcal{S}}) \;=\; \Theta(\mathrm{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathrm{dc}(m, \to_{\mathcal{S}/\mathcal{R}}))$$

Proof idea

$\mathcal{R} \subseteq > \; \mathcal{S} \subseteq \geqslant$

$\mathcal{R} \subseteq \geqslant \; \mathcal{S} \subseteq >$

$$t_1 \;\to_{\mathcal{R}}\; t_2 \;\to_{\mathcal{S}}\; t_3 \;\to_{\mathcal{S}}\; t_4 \;\to_{\mathcal{R}}\; \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_3 \\ t_3 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_4 \\ t_4 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \;\cdots$$

Modular Complexity

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathsf{dc}(m, \to_{\mathcal{R}/\varnothing})$

Modular Complexity

Lemma

$\mathsf{dc}(m, \rightarrow_{\mathcal{R}}) = \mathsf{dc}(m, \rightarrow_{\mathcal{R}/\varnothing})$ $\quad\quad$ $\mathsf{dc}(m, \rightarrow_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

Modular Complexity

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathsf{dc}(m, \to_{\mathcal{R}/\varnothing}) \qquad \mathsf{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

Theorem

$\mathsf{dc}(m, \to_{(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{S}}) = \Theta(\mathsf{dc}(m, \to_{\mathcal{R}_1/(\mathcal{S} \cup \mathcal{R}_2)}) + \mathsf{dc}(m, \to_{\mathcal{R}_2/(\mathcal{S} \cup \mathcal{R}_1)}))$

Modular Complexity

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathsf{dc}(m, \to_{\mathcal{R}/\varnothing}) \qquad \mathsf{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

Theorem

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Definition (Complexity Pair)

$(>, \geqslant)$ with rewrite relations $>$, $\geqslant$

Modular Complexity

Lemma

$\mathrm{dc}(m, \to_{\mathcal{R}}) = \mathrm{dc}(m, \to_{\mathcal{R}/\varnothing}) \qquad \mathrm{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

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Definition (Complexity Pair)

$(>, \geqslant)$ with rewrite relations $>$, $\geqslant$ and $> \cdot \geqslant \subseteq >$ and $\geqslant \cdot > \subseteq >$

Modular Complexity

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathsf{dc}(m, \to_{\mathcal{R}/\varnothing}) \qquad \mathsf{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

Theorem

$\mathsf{dc}(m, \to_{(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{S}}) = \Theta(\mathsf{dc}(m, \to_{\mathcal{R}_1/(\mathcal{S} \cup \mathcal{R}_2)}) + \mathsf{dc}(m, \to_{\mathcal{R}_2/(\mathcal{S} \cup \mathcal{R}_1)}))$

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$(>, \geqslant)$ with rewrite relations $>$, $\geqslant$ and $> \cdot \geqslant \subseteq >$ and $\geqslant \cdot > \subseteq >$

Lemma

$(>, \geqslant)$ *complexity pair*

Modular Complexity

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathsf{dc}(m, \to_{\mathcal{R}/\varnothing})$ $\qquad$ $\mathsf{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

Theorem

$\mathsf{dc}(m, \to_{(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{S}}) = \Theta(\mathsf{dc}(m, \to_{\mathcal{R}_1/(\mathcal{S} \cup \mathcal{R}_2)}) + \mathsf{dc}(m, \to_{\mathcal{R}_2/(\mathcal{S} \cup \mathcal{R}_1)}))$

Definition (Complexity Pair)

$(>, \geqslant)$ with rewrite relations $>$, $\geqslant$ and $> \cdot \geqslant \subseteq >$ and $\geqslant \cdot > \subseteq >$

Lemma

$(>, \geqslant)$ *complexity pair,* $\mathcal{R} \subseteq >$, $\mathcal{S} \subseteq \geqslant$

Modular Complexity

Lemma

$\mathrm{dc}(m, \to_{\mathcal{R}}) = \mathrm{dc}(m, \to_{\mathcal{R}/\varnothing}) \qquad \mathrm{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

Theorem

$\mathrm{dc}(m, \to_{(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{S}}) = \Theta(\mathrm{dc}(m, \to_{\mathcal{R}_1/(\mathcal{S} \cup \mathcal{R}_2)}) + \mathrm{dc}(m, \to_{\mathcal{R}_2/(\mathcal{S} \cup \mathcal{R}_1)}))$

Definition (Complexity Pair)

$(>, \geqslant)$ with rewrite relations $>$, $\geqslant$ and $> \cdot \geqslant \subseteq >$ and $\geqslant \cdot > \subseteq >$

Lemma

$(>, \geqslant)$ *complexity pair*, $\mathcal{R} \subseteq >$, $\mathcal{S} \subseteq \geqslant \quad \longrightarrow \mathrm{dc}(m, >) \geqslant \mathrm{dc}(m, \to_{\mathcal{R}/\mathcal{S}})$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \quad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \quad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \quad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \quad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \quad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \quad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

Example

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 1_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) > \mathsf{f}(\mathsf{g}(1)) \quad \mathsf{f}(\mathsf{f}(x)) > \mathsf{f}(x) \quad \mathsf{g}(0) \geqslant \mathsf{g}(\mathsf{f}(0)) \quad \mathsf{g}(\mathsf{g}(x)) \geqslant \mathsf{g}(x)$$

Example

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 1_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \geqslant \mathsf{f}(\mathsf{g}(1)) \quad \mathsf{f}(\mathsf{f}(x)) \geqslant \mathsf{f}(x) \quad \mathsf{g}(0) > \mathsf{g}(\mathsf{f}(0)) \quad \mathsf{g}(\mathsf{g}(x)) > \mathsf{g}(x)$$

Example

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 1_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example

$$\mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \quad 1_{\mathcal{M}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 0_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \quad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \quad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \quad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

Example

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Example

$$\mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \quad 1_{\mathcal{M}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 0_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Lemma

$\mathrm{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m^2)$

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and take tightest bound

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.28)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & & \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.28)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & & \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

Root-Labeling and TMI

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.28)

$$\begin{aligned} \mathsf{half}(0) &\geqslant 0 & \mathsf{bits}(0) &\geqslant 0 \\ \mathsf{half}(\mathsf{s}(0)) &\geqslant 0 & & \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\geqslant \mathsf{s}(\mathsf{half}(x)) & \mathsf{bits}(\mathsf{s}(x)) &> \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

Root-Labeling and TMI (0.5 sec)

$\mathcal{O}(m^2)$ (too large for display)

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.28)

$$\mathsf{half}(0) \to 0 \qquad \mathsf{bits}(0) \to 0$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad \mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$$

AMI

$$\mathsf{bits}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & 0 & -\infty \\ 0 & 0 & 3 \\ -\infty & 1 & 0 \end{pmatrix} x \quad \mathsf{s}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & 0 \\ 2 & 0 & 2 \\ 2 & -\infty & 1 \end{pmatrix} x \quad \mathsf{half}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ -\infty \\ -\infty \end{pmatrix}$$

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.28)

$$\begin{aligned} \mathsf{half}(0) &\geqslant 0 & \qquad \mathsf{bits}(0) &\geqslant 0 \\ \mathsf{half}(\mathsf{s}(0)) &\geqslant 0 & & \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\geqslant \mathsf{s}(\mathsf{half}(x)) & \qquad \mathsf{bits}(\mathsf{s}(x)) &> \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

AMI (2.8 sec)

$$\mathsf{bits}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & 0 & -\infty \\ 0 & 0 & 3 \\ -\infty & 1 & 0 \end{pmatrix} x \quad \mathsf{s}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & 0 \\ 2 & 0 & 2 \\ 2 & -\infty & 1 \end{pmatrix} x \quad \mathsf{half}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ -\infty \\ -\infty \end{pmatrix}$$

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.28)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & & \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

AMI (2.8 sec)

$$\mathsf{bits}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & 0 & -\infty \\ 0 & 0 & 3 \\ -\infty & 1 & 0 \end{pmatrix} x \quad \mathsf{s}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & 0 \\ 2 & 0 & 2 \\ 2 & -\infty & 1 \end{pmatrix} x \quad \mathsf{half}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ -\infty \\ -\infty \end{pmatrix}$$

Problems

partial proofs

Implementation

Approach

run `LINEAR || QUADRATIC || CUBIC || QUARTIC` and ~~take tightest bound~~

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.28)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & & \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

AMI (2.8 sec)

$$\mathsf{bits}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & 0 & -\infty \\ 0 & 0 & 3 \\ -\infty & 1 & 0 \end{pmatrix} x \quad \mathsf{s}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & 0 \\ 2 & 0 & 2 \\ 2 & -\infty & 1 \end{pmatrix} x \quad \mathsf{half}_{\mathcal{M}}(x) = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ -\infty \\ -\infty \end{pmatrix}$$

Problems

partial proofs, incomparable proofs

The “It does not matter” Slide

Notation

$\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_4$

The "It does not matter" Slide

Notation

$\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_4$ $\quad$ $\mathcal{R}_{1,3} = \mathcal{R}_1 \cup \mathcal{R}_3$ $\quad$ $\cdots$

The “It does not matter” Slide

Approach

(1) find proof

The "It does not matter" Slide

Approach

(1) find proof

Example

$$\mathcal{R}/\varnothing$$

$$\varnothing/\mathcal{R}$$

The "It does not matter" Slide

Approach

(1) find proof

Example

$$\mathcal{R}/\varnothing$$

$$|\qquad \mathrm{dc}(m, \to_{\mathcal{R}_2/\mathcal{R}_{1,3,4}}) = \mathcal{O}(m)$$

$$\mathcal{R}_{1,3,4}/\mathcal{R}_2$$

$$\varnothing/\mathcal{R}$$

The "It does not matter" Slide

Approach

(1) find proof

Example

$$\mathcal{R}/\varnothing$$

$$\mathrm{dc}(m, \rightarrow_{\mathcal{R}_2/\mathcal{R}_{1,3,4}}) = \mathcal{O}(m)$$

$$\mathcal{R}_{1,3,4}/\mathcal{R}_2$$

$$\mathrm{dc}(m, \rightarrow_{\mathcal{R}_{1,4}/\mathcal{R}_{2,3}}) = \mathcal{O}(m^3)$$

$$\mathcal{R}_3/\mathcal{R}_{1,2,4}$$

$$\varnothing/\mathcal{R}$$

The “It does not matter” Slide

Approach

(1) find proof

Example

$$\mathcal{R}/\varnothing$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_2/\mathcal{R}_{1,3,4}}) = \mathcal{O}(m)$$

$$\mathcal{R}_{1,3,4}/\mathcal{R}_2$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_{1,4}/\mathcal{R}_{2,3}}) = \mathcal{O}(m^3)$$

$$\mathcal{R}_3/\mathcal{R}_{1,2,4}$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_3/\mathcal{R}_{1,2,4}}) = \mathcal{O}(m^2)$$

$$\varnothing/\mathcal{R}$$

The "It does not matter" Slide

Approach

(1) find proof ($\mathcal{O}(m^3)$)

Example

$$\mathcal{R}/\varnothing$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_2/\mathcal{R}_{1,3,4}}) = \mathcal{O}(m)$$

$$\mathcal{R}_{1,3,4}/\mathcal{R}_2$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_{1,4}/\mathcal{R}_{2,3}}) = \mathcal{O}(m^3)$$

$$\mathcal{R}_3/\mathcal{R}_{1,2,4}$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_3/\mathcal{R}_{1,2,4}}) = \mathcal{O}(m^2)$$

$$\varnothing/\mathcal{R}$$

The "It does not matter" Slide

Approach

(1) find proof ($\mathcal{O}(m^3)$) (2) tighten bound

Example

$$\mathcal{R}/\varnothing$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_2/\mathcal{R}_{1,3,4}}) = \mathcal{O}(m)$$

$$\mathcal{R}_{1,3,4}/\mathcal{R}_2$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_{1,4}/\mathcal{R}_{2,3}}) = \mathcal{O}(m^3)$$

$$\mathcal{R}_3/\mathcal{R}_{1,2,4}$$

$$\mathrm{dc}(m, \to_{\mathcal{R}_3/\mathcal{R}_{1,2,4}}) = \mathcal{O}(m^2)$$

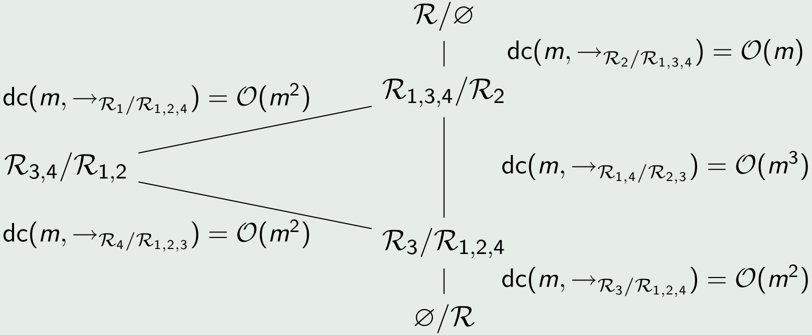
$$\varnothing/\mathcal{R}$$

The "It does not matter" Slide

Approach

(1) find proof ($\mathcal{O}(m^3)$) (2) tighten bound

Example



The "It does not matter" Slide

Approach

(1) find proof ($\mathcal{O}(m^3)$) (2) tighten bound ($\mathcal{O}(m^2)$)

Example

$\mathcal{R}/\varnothing$

$\mathrm{dc}(m, \to_{\mathcal{R}_2/\mathcal{R}_{1,3,4}}) = \mathcal{O}(m)$

$\mathrm{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_{1,2,4}}) = \mathcal{O}(m^2)$

$\mathcal{R}_{1,3,4}/\mathcal{R}_2$

$\mathcal{R}_{3,4}/\mathcal{R}_{1,2}$

$\mathrm{dc}(m, \to_{\mathcal{R}_{1,4}/\mathcal{R}_{2,3}}) = \mathcal{O}(m^3)$

$\mathrm{dc}(m, \to_{\mathcal{R}_4/\mathcal{R}_{1,2,3}}) = \mathcal{O}(m^2)$

$\mathcal{R}_3/\mathcal{R}_{1,2,4}$

$\mathrm{dc}(m, \to_{\mathcal{R}_3/\mathcal{R}_{1,2,4}}) = \mathcal{O}(m^2)$

$\varnothing/\mathcal{R}$

Does it really not matter?

Answer

for complexity pairs it does not matter

Does it really not matter?

Answer

for complexity pairs it does not matter

Reason

$\mathcal{R} = \bigcup \mathcal{R}_i$

Does it really not matter?

Answer

for complexity pairs it does not matter

Reason

$\mathcal{R} = \bigcup \mathcal{R}_i \longrightarrow$ for any i $\mathcal{R}_i \subseteq >_i$ or $\mathcal{R}_i \subseteq \geqslant_i$

Does it really not matter?

Answer

for complexity pairs it does not matter

Reason

$\mathcal{R} = \bigcup \mathcal{R}_i \longrightarrow$ for any i $\mathcal{R}_i \subseteq >_i$ or $\mathcal{R}_i \subseteq \geqslant_i$

Answer (2)

for non-complexity pairs situation changes

Does it really not matter?

Answer

for complexity pairs it does not matter

Reason

$\mathcal{R} = \bigcup \mathcal{R}_i \longrightarrow$ for any i $\mathcal{R}_i \subseteq >_i$ or $\mathcal{R}_i \subseteq \geqslant_i$

Answer (2)

for non-complexity pairs situation changes

Reason

some rules might be ignored

Does it really not matter?

Answer

for complexity pairs it does not matter

Reason

$\mathcal{R} = \bigcup \mathcal{R}_i \longrightarrow$ for any i $\mathcal{R}_i \subseteq >_i$ or $\mathcal{R}_i \subseteq \geqslant_i$

Answer (2)

for non-complexity pairs situation changes

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$\mathrm{dc}(m, \to_{\mathcal{R} \cup \mathcal{S}}) = \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathrm{dc}(m, \to_{\mathcal{S}}))$

Does it really not matter?

Answer

for complexity pairs it does not matter

Reason

$\mathcal{R} = \bigcup \mathcal{R}_i \longrightarrow$ for any i $\mathcal{R}_i \subseteq >_i$ or $\mathcal{R}_i \subseteq \geqslant_i$

Answer (2)

for non-complexity pairs situation changes

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$$\mathrm{dc}(m, \rightarrow_{\mathcal{R} \cup \mathcal{S}}) = \mathcal{O}(\mathrm{dc}(m, \rightarrow_{\mathcal{R}/\mathcal{S}}) + \mathrm{dc}(m, \rightarrow_{\mathcal{S}}))$$

if $\mathcal{S} \subseteq >_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}$ *non-duplicating*

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

direct
modular
modular*

Experiments (direct vs modular)

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	$\mathcal{O}(n)$
direct	168
modular	
modular*	

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	$\mathcal{O}(n)$
direct	168
modular	193
modular*	

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$
direct	168
modular	193
modular*	?

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
direct	168	252
modular	193	
modular*	?	

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
direct	168	252
modular	193	283
modular*	?	?

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$
direct	168	252	287
modular	193	283	
modular*	?	?	

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$
direct	168	252	287
modular	193	283	312
modular*	?	?	?

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$	time
direct	168	252	287	0.79
modular	193	283	312	
modular*	?	?	?	

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$	time
direct	168	252	287	0.79
modular	193	283	312	1.45
modular*	?	?	?	?

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$	time	timeout
direct	168	252	287	0.79	194
modular	193	283	312	1.45	
modular*	?	?	?	?	

Experiments (direct vs modular)

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	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$	time	timeout
direct	168	252	287	0.79	194
modular	193	283	312	1.45	347
modular*	?	?	?	?	?

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	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$	time	timeout
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modular*	?	?	?	?	?

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- experiments