Exponential Path Order EPO*

Martin Avanzini[1] Naohi Eguchi[2]

[1]Computational Logic
Faculty of Computer Science, University of Innsbruck, Austria

[2]School of Information Science
Japan Advanced Institute of Science and Technology, Japan

Outline

$\mathcal{B}$	polynomial time

Stephen Bellantoni and Stephen Cook
A new Recursion-Theoretic Characterization of the Polytime Functions.
CC, pages 97–110, 1992

Outline

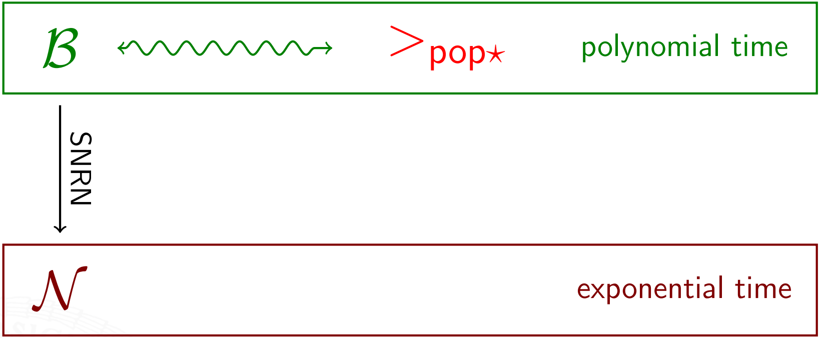
$\mathcal{B}$ polynomial time

↓ SNRN

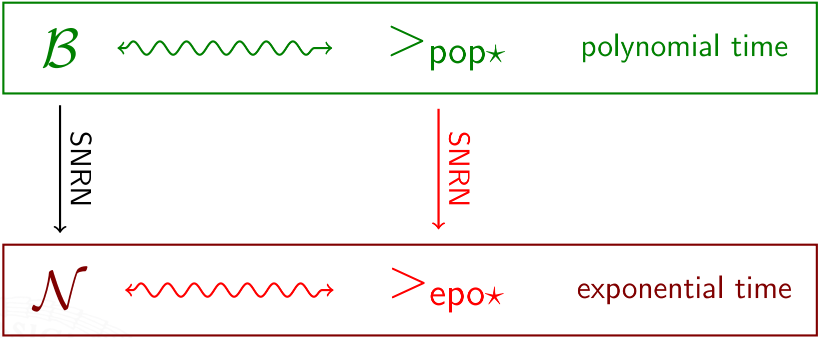
$\mathcal{N}$ exponential time

Toshiyasu Arai and Naohi Eguchi
A new Function Algebra of EXPTIME Functions by Safe Nested Recursion.
TCL, pages 130–146, 2008

Outline



Martin Avanzini and Georg Moser
Complexity Analysis by Rewriting.
FLOPS '09, pages 130–146, 2008



$\mathcal{B}$
$>_{\mathsf{pop}\star}$
polynomial time
SNRN
SNRN
$\mathcal{N}$
$>_{\mathsf{epo}\star}$
exponential time

this talk

Outline

$\mathcal{B}$ $\leftsquigarrow\!\rightsquigarrow$ $>_{\mathsf{pop}\star}$ polynomial time

Geprüfte Qualität AMA GÜTESIEGEL AUSTRIA

$\downarrow$ SNRN $\qquad \downarrow$ SNRN

$\mathcal{N}$ $\leftsquigarrow\!\rightsquigarrow$ $>_{\mathsf{epo}\star}$ exponential time

MADE IN JAPAN

MADE IN JAPAN

Geprüfte Qualität AMA GÜTESIEGEL AUSTRIA

this talk

The class $\mathcal{B}$

Recursion Theoretic Characterisation of FP

Alternative Characterisation of FP

Safe and Normal Argument Positions

Idea

break strength of recursion scheme by separation of arguments positions

$$\mathsf{f}(\underbrace{x_1, \dots, x_k}_{\text{normal}}; \underbrace{y_1, \dots, y_l}_{\text{safe}})$$

Alternative Characterisation of FP

Defining Functions by Recursion

Definition (Safe Recursion on Notation)

Suppose $g \in \mathcal{B}^{k,l}$ and $h_0, h_1 \in \mathcal{B}^{k+1,l+1}$. Then $f \in \mathcal{B}^{k+1,l}$ where

$$f(\epsilon, \overline{x} ; \overline{y}) = g(\overline{x} ; \overline{y})$$
$$f(zi, \overline{x} ; \overline{y}) = h_i(z, \overline{x} ; \overline{y}, f(z, \overline{x} ; \overline{y})) \qquad (i \in \{0, 1\})$$

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- . . . on Notation recursion on binary representation

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- Safe ... no recursion on recursively computed result
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$$\text{where } h_i(\epsilon, \overline{x} ; \overline{y}) = j_i(\overline{x} ; \overline{y})$$
$$h_i(zi, \overline{x} ; \overline{y}) = k_{i,j}(z, \overline{x} ; \overline{y}, h_i(z, \overline{x} ; \overline{y}))$$

- Safe … no recursion on recursively computed result
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Defining Functions by Recursion

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$$f(zi, \overline{x} ; \overline{y}) = h_i(z, \overline{x} ; \overline{y}, f(z, \overline{x} ; \overline{y})) \qquad (i \in \{0,1\})$$

Definition (Safe Composition)

Suppose $h \in \mathcal{B}^{k',l'}$, $\overline{r} \in \mathcal{B}^{k,0}$ and $\overline{s} \in \mathcal{B}^{k,l}$. Then $g \in \mathcal{B}^{k,l}$ where

$$g(\overline{x} ; \overline{y}) = h(\overline{r(\overline{x};)}; \overline{s(\overline{x} ; \overline{y})})$$

Alternative Characterisation of FP

Bellantoni and Cook, 1992

Definition

Class $\mathcal{B}$ is smallest class

1. containing certain initial functions
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Theorem (S. Bellantoni and S. Cook,1992)

$$\mathcal{B} = \mathsf{FP}$$

The class $\mathcal{N}$

Recursion Theoretic Characterisation of FEXP

Alternative Characterisation of FEXP

The class $\mathcal{N}$

$$\mathcal{N} \quad \approx \quad \mathcal{B} + \text{safe nested recursion on notation}$$

Alternative Characterisation of FEXP

Safe Nested Recursion on Notation

1. nesting of recursive function calls

$$f(\epsilon\,;\, y) = g(\,;\, y)$$
$$f(xi\,;\, y) = r_i(x\,;\, y, \qquad f(x\,;\, y) \quad)$$

Alternative Characterisation of FEXP

Safe Nested Recursion on Notation

1. nesting of recursive function calls

$$f(\epsilon\,;\,y) = g(\,;\,y)$$
$$f(xi\,;\,y) = r_i(x\,;\,y, f(x\,;\,s_i(x\,;\,y, f(x\,;\,y))))$$

Alternative Characterisation of FEXP

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2. recursion on multiple parameters

$$f(\epsilon,\epsilon\,;\,\overline{z}) = g(\,;\,\overline{z})$$
$$f(xi,\epsilon\,;\,\overline{z}) = r_{i,\epsilon}(x,\epsilon\,;\,\overline{z}, f(x,y\,;\,s_{i,\epsilon}(x,\epsilon\,;\,f(x,\epsilon\,;\,\overline{z}))))$$
$$f(\epsilon,yi\,;\,\overline{z}) = r_{\epsilon,j}(\epsilon,y\,;\,\overline{z}, f(x,y\,;\,s_{\epsilon,j}(\epsilon,y\,;\,f(\epsilon,y\,;\,\overline{z}))))$$
$$f(xi,yj\,;\,\overline{z}) = r_{i,j}(x,y\,;\,\overline{z}, f(xi,y\,;\,s_{i,j}(x,\epsilon\,;\,f(x,yj\,;\,\overline{z}))))$$

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$$f(\epsilon, yi\,;\,\overline{z}) = r_{\epsilon,j}(\epsilon, y\,;\,\overline{z}, f(x, y\,;\,s_{\epsilon,j}(\epsilon, y\,;\,f(\epsilon, y\,;\,\overline{z}))))$$
$$f(xi, yj\,;\,\overline{z}) = r_{i,j}(x, y\,;\,\overline{z}, f(xi, y\,;\,s_{i,j}(x, \epsilon\,;\,f(x, yj\,;\,\overline{z}))))$$

3. lexicographic decreasing recursive parameters

$$(xi, yj) >^1_{\mathsf{lex}} (xi, y) \qquad (xi, yj) >^1_{\mathsf{lex}} (x, yj)$$

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Suppose $g \in \mathcal{N}^{0,l}$ and $r_w, \overline{s_w} \in \mathcal{N}^{k,l+1}$. Then $f \in \mathcal{N}^{k+1,l+l'}$ where

$$f(\overline{\epsilon}\,;\,\overline{y},\overline{z}) = g(\,;\,\overline{y},\overline{z})$$

$$f(\overline{u}\,;\,\overline{y},\overline{z}) = r_w(\overline{v_1}\,;\,\overline{y}, f(\overline{v_1}\,;\,\overline{y}, \overline{s_w(\overline{v_2}\,;\,\overline{y}, f(\overline{v_2}\,;\,\overline{y},\overline{z}))})) \;\forall w. \in \{0,1,\epsilon\}^k$$

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- w is sequence of last bits (or ϵ) of $\overline{u}$

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  - $u >^1 v$ if and only if $u = vi$ for some i,

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  - $u >^1 v$ if and only if $u = vi$ for some i,
  - $(u_1, \dots, u_n) >^1_{\text{lex}} (v_1, \dots, v_n)$ if for some $1 \leqslant k \leqslant n$
    - $u_j = v_j$ for all $1 \leqslant j < k$, and
    - $u_k >^1 v_k$, and
    - for each $k < j \leqslant n$, $u_i \geqslant^1 v_j$ for some $1 \leqslant i \leqslant n$

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- arbitrary level of nesting allowed

Alternative Characterisation of FEXP

Safe Nested Recursion on Notation

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Note

"Ackermann-like" functions cannot be defined

$$\mathrm{A}(\varepsilon\,;\,n) = n1$$
$$\mathrm{A}(mi\,;\,\varepsilon) = \mathrm{A}(m\,;\,1)$$
$$\mathrm{A}(mi\,;\,nj) = \mathrm{A}(m\,;\,\mathrm{A}(mi\,;\,n))$$

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$$\mathrm{A}(mi\,;\,\varepsilon) = \mathrm{A}(m\,;\,1)$$
$$\mathrm{A}(mi\,;\,nj) = \mathrm{A}(m\,;\,\mathrm{A}(mi\,;\,n)) \qquad (mi) \not>^{1}_{\mathrm{lex}} (mi)$$

Alternative Characterisation of FEXP

Eguchi, 2009

Definition

Class $\mathcal{N}$ is smallest class

1. containing the initial functions of $\mathcal{B}$
2. closed under safe nested recursion on notation
3. closed under weak safe composition

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Definition (Weak Safe Composition)

$$g(\overline{x}\,;\,\overline{y}) = h(x_{i_1},\ldots,x_{i_k}\,;\,\overline{s(\overline{x};\overline{y})}) \qquad \{x_{i_1},\ldots,x_{i_k}\} \subseteq \{\overline{x}\}$$

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Theorem (T. Arai and N. Eguchi,2008)

$$\mathcal{N} = \text{FEXP}$$

The order $>_{\mathsf{epo}\star}$

A Path Order based on $\mathcal{N}$

Polynomial Path Order $>_{\mathsf{pop}\star}$

- restriction of multiset path orders $>_{\mathsf{pop}\star} \subseteq >_{\mathsf{mpo}}$

Polynomial Path Order $>_{pop\star}$

- restriction of multiset path orders $>_{pop\star} \subseteq >_{mpo}$
- induced by precedence *and* set of safe argument positions per symbol

Polynomial Path Order $>_{\mathrm{pop}\star}$

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```
> tct -a irc -p -s "pop*" times.trs
  YES(?,POLY)

  'Polynomial Path Orders'
  ------------------------
  Answer:            YES(?,POLY)
  Input Problem:     innermost runtime-complexity with respect to
    Rules:
      {  *(x, s(y)) -> +(x, *(x, y))
       , *(x, 0()) -> 0()
       , +(s(x), y) -> s(+(x, y))
       , +(0(), y) -> y}
  Details:
    Rules in Predicative Notation:
     {  *(x, s(; y);) -> +(x; *(x, y;))
      , *(x, 0();) -> 0()
      , +(s(; x); y) -> s(; +(x; y))
      , +(0(); y) -> y}
    Precedence:
     * > +
    Safe Argument Positions:
     safe(+) = {2}, safe(*) = {}
```

Polynomial Path Order $>_{\mathsf{pop}\star}$

- restriction of multiset path orders $\quad >_{\mathsf{pop}\star} \subseteq >_{\mathsf{mpo}}$
- induced by precedence *and* set of safe argument positions per symbol

$\mathcal{R} \subseteq >_{\mathsf{pop}\star} \quad \rightsquigarrow$ "$\mathcal{R}$ obeys safe recursion and safe composition"

Polynomial Path Order $>_{\mathsf{pop}\star}$

- restriction of multiset path orders $\qquad >_{\mathsf{pop}\star} \subseteq >_{\mathsf{mpo}}$
- induced by precedence *and* set of safe argument positions per symbol

$\mathcal{R} \subseteq >_{\mathsf{pop}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe recursion and safe composition"

$\rightsquigarrow$ innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ polynomial

Polynomial Path Order $>_{\mathsf{pop}\star}$

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$\mathcal{R} \subseteq >_{\mathsf{pop}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe recursion and safe composition"

$\rightsquigarrow$ innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ polynomial

$$\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}(n) = \max\{\, \mathsf{dl}(t, \xrightarrow{\mathsf{i}}_{\mathcal{R}}) \mid |t| \leqslant n \text{ and arguments from } \mathcal{T}(\mathcal{C}, \mathcal{V}) \,\}$$

where $\mathsf{dl}(t, \to) = \max\{\ell \mid \exists (t_1, \ldots, t_\ell).\ t \to t_1 \to \cdots \to t_\ell\}$

Polynomial Path Order $>_{\mathsf{pop}\star}$

- restriction of multiset path orders $>_{\mathsf{pop}\star} \subseteq >_{\mathsf{mpo}}$
- induced by precedence *and* set of safe argument positions per symbol

$\mathcal{R} \subseteq >_{\mathsf{pop}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe recursion and safe composition"

$\rightsquigarrow$ innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ polynomial

$\rightsquigarrow$ $\mathcal{R}$ defines only functions from $\mathcal{B} = \mathsf{FP}$

Exponential Path Order $>_{\mathsf{epo}\star}$

- restriction of lexicographic path orders $\quad >_{\mathsf{epo}\star} \subseteq >_{\mathsf{lpo}}$
- induced by precedence *and* set of safe argument positions per symbol

$\mathcal{R} \subseteq >_{\mathsf{epo}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe nested recursion and safe composition"

$\rightsquigarrow$ innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ exponential

$\rightsquigarrow$ $\mathcal{R}$ defines only functions from $\mathcal{N} = \mathsf{FEXP}$

Exponential Path Order $>_{\text{epo}\star}$

The Order $>_{\text{epo}\star}$

Let $s = f(s_1, \ldots, s_l \, ; \, s_{l+1}, \ldots, s_m)$, $\succ$ be a precedence with $\forall c \in \mathcal{C}$ minimal

① $$\frac{s_i \geqslant_{\text{epo}\star} t}{f(s_1, \ldots, s_l \, ; \, s_{l+1}, \ldots, s_m) >_{\text{epo}\star} t}$$ for some $1 \leqslant i \leqslant m$

Exponential Path Order $>_{\mathsf{epo}\star}$

The Order $>_{\mathsf{epo}\star}$

Let $s = f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m)$, $\succ$ be a precedence with $\forall c \in \mathcal{C}$ minimal

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② $$\frac{s \sqsupset_{\mathsf{epo}\star} t_1 \quad \cdots \quad s \sqsupset_{\mathsf{epo}\star} t_k \qquad s >_{\mathsf{epo}\star} t_{k+1} \quad \cdots \quad s >_{\mathsf{epo}\star} t_n}{f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) >_{\mathsf{epo}\star} g(t_1, \ldots, t_k \,;\, t_{k+1}, \ldots, t_n)} \quad \begin{array}{l} f \succ g, \\ f \in \mathcal{D} \end{array}$$

Exponential Path Order $>_{\mathsf{epo}\star}$

The Order $>_{\mathsf{epo}\star}$

Let $s = f(s_1, \ldots, s_l \mathbin{;} s_{l+1}, \ldots, s_m)$, $\succ$ be a precedence with $\forall c \in \mathcal{C}$ minimal

① $$\frac{s_i \geqslant_{\mathsf{epo}\star} t}{f(s_1, \ldots, s_l \mathbin{;} s_{l+1}, \ldots, s_m) >_{\mathsf{epo}\star} t} \text{ for some } 1 \leqslant i \leqslant m$$

② $$\frac{s \sqsupset_{\mathsf{epo}\star} t_1 \quad \cdots \quad s \sqsupset_{\mathsf{epo}\star} t_k \qquad s >_{\mathsf{epo}\star} t_{k+1} \quad \cdots \quad s >_{\mathsf{epo}\star} t_n}{f(s_1, \ldots, s_l \mathbin{;} s_{l+1}, \ldots, s_m) >_{\mathsf{epo}\star} g(t_1, \ldots, t_k \mathbin{;} t_{k+1}, \ldots, t_n)} \quad \begin{array}{l} f \succ g, \\ f \in \mathcal{D} \end{array}$$

$\sqsupset_{\mathsf{epo}\star} \subseteq >_{\mathsf{epo}\star}$

Exponential Path Order $>_{\mathrm{epo}\star}$

The Order $>_{\mathrm{epo}\star}$

Let $s = f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m)$, $\succ$ be a precedence with $\forall c \in \mathcal{C}$ minimal

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$\mathrm{e}^{\mathrm{e}}(x\,;\,) >^{?}_{\text{epo}\star} \mathrm{e}(\mathrm{e}(x\,;\,)\,;\,)$ $\qquad$ $\mathrm{e}^{\mathrm{e}} \succ \mathrm{e} \quad \mathrm{e}^{\mathrm{e}}, \mathrm{e} \in \mathcal{D}$

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$\mathsf{e}^{\mathsf{e}}(x\,;\,) \sqsupset^{?}_{\mathsf{epo}\star} \quad \mathsf{e}(x\,;\,)$ $\qquad\qquad$ $\mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$

Exponential Path Order $>_{\text{epo}\star}$

Auxiliary Order $\sqsupset_{\text{epo}\star}$

① $$\frac{s_i \sqsupseteq_{\text{epo}\star} t}{c(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) \sqsupset_{\text{epo}\star} t}$$ $c \in \mathcal{C}$ and some $1 \leqslant i \leqslant m$

$\mathsf{e}^{\mathsf{e}}(x\,;\,) \sqsupset^{?}_{\text{epo}\star} \quad \mathsf{e}(x\,;\,)$ $\qquad \mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$

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$\mathsf{e}^{\mathsf{e}}(x\,;\,) \sqsupset^{?}_{\mathsf{epo}\star} \mathsf{e}(x\,;\,)$ $\qquad$ $\mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$

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$$x \quad \sqsupseteq^{?}_{\mathsf{epo}\star} \quad \mathsf{e}(x\,;\,) \qquad\qquad \mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$$

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Auxiliary Order $\sqsupset_{\mathsf{epo}\star}$

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$x \quad \sqsupseteq^{?}_{\mathsf{epo}\star} \quad \mathsf{e}(x\,;\,) \qquad\qquad \mathsf{e^e} \succ \mathsf{e} \quad \mathsf{e^e}, \mathsf{e} \in \mathcal{D}$ ↯

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$\mathsf{e}^{\mathsf{e}}(x\,;\,) \not>_{\mathsf{epo}\star} \mathsf{e}(\mathsf{e}(x\,;\,)\,;\,)$ $\qquad \mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$ ☺

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$$f(\overline{x} \,;\, \overline{y}) >_{\mathsf{epo}\star} g(x_{i_1}, \ldots, x_{i_k} \,;\, \overline{s(\overline{x} \,;\, \overline{y})}) \qquad f \succ g, \overline{s} \quad f \in \mathcal{D}$$

☺

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Recall

Safe Nested Recursion on Notation

$$f(\overline{u} \,;\, \overline{z}) \quad = \quad r_w(\overline{v_1} \,;\, \overline{z}, f(\overline{v_1} \,;\, s_w(\overline{v_2} \,;\, \overline{z}, f(\overline{v_2} \,;\, \overline{z})))) \qquad \overline{u} >^1_{\mathsf{lex}} \overline{v_i}$$

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③ $$\frac{(s_1, \dots, s_l)\ (\sqsupset_{\text{epo}\star})_{\text{lex}}\ (t_1, \dots, t_l) \qquad s >_{\text{epo}\star} t_{l+1} \quad \cdots \quad s >_{\text{epo}\star} t_m}{f(s_1, \dots, s_l \,;\, s_{l+1}, \dots, s_m) >_{\text{epo}\star} f(t_1, \dots, t_l \,;\, t_{l+1}, \dots, t_m)} \quad f \in \mathcal{D}$$

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Recall Safe Nested Recursion on Notation

$$f(\overline{u} \,;\, \overline{z}) >_{\text{epo}\star} r_w(\overline{v_1} \,;\, \overline{z}, f(\overline{v_1} \,;\, s_w(\overline{v_2} \,;\, \overline{z}, f(\overline{v_2} \,;\, \overline{z})))) \quad f \succ r_w, s_w \quad f \in \mathcal{D}$$

Exponential Path Order $>_{\mathsf{epo}\star}$

Soundness and Completeness

Theorem

Every function from FEXP *is computed by some TRS compatible with an instance* $>_{\mathsf{epo}\star}$.

Exponential Path Order $>_{\mathsf{epo}\star}$

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Conjecture

Suppose $\mathcal{R} \subseteq >_{\mathsf{epo}\star}$ for constructor TRS $\mathcal{R}$. Then the innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ is bounded by an exponential.

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Every function from FEXP *is computed by some TRS compatible with an instance* $>_{\mathsf{epo}\star}$.

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Suppose $\mathcal{R} \subseteq >_{\mathsf{epo}\star}$ for constructor TRS $\mathcal{R}$. Then the innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ is bounded by an exponential.

- almost finished proof on paper

Conclusion

$\mathcal{B} \rightsquigarrow >_{\mathsf{pop}\star}$ polynomial time

↓ SNRN ↓ SNRN

$\mathcal{N} \rightsquigarrow >_{\mathsf{epo}\star}$ exponential time