Full Substitution Strategy

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joint work with

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Full Substitution

Definition

for left-linear TRS

- set of redexes are simultaneous if no redex overlaps with other redexes

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- $s \multimap_A t$ is complete development step of simultaneous set A for s
- $s \twoheadrightarrow t$ if $s \multimap_A t$ for some maximal A full substitution

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- $s \mathrel{\bullet\!\!\!\rightarrow} t$ if $s \mathrel{\multimap\!\!\rightarrow}_A t$ for some maximal A full substitution

DEFINITION simultaneous step

$\mathrel{\multimap\!\!\rightarrow}$ is inductively defined as

- $x \mathrel{\multimap\!\!\rightarrow} x$ if x is variable
- $f(s_1, \ldots, s_n) \mathrel{\multimap\!\!\rightarrow} f(t_1, \ldots, t_n)$ if $s_i \mathrel{\multimap\!\!\rightarrow} t_i$ for all i
- $\ell\sigma \mathrel{\multimap\!\!\rightarrow} r\tau$ if $\ell \rightarrow r \in \mathcal{R}$ and $x\sigma \mathrel{\multimap\!\!\rightarrow} x\tau$ for all variable x

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LEMMA

$s \multimap\!\!\!\rightarrow t \iff s \multimap\!\!\!\rightarrow_A t$ for some A for left-linear TRS

Definition

$\multimap\!\!\!\!\!\!\!\bullet\!\!\rightarrow$ is normalising for $\mathcal{R}$ if every maximal sequence from weakly normalising term ends with normal form of $\mathcal{R}$

Normalizing Strategies

TRS

$$\mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y) \qquad \mathsf{nat} \to \mathsf{0} : \mathsf{inc}(\mathsf{nat})$$

$$\mathsf{hd}(x : y) \to x$$

$$\mathsf{tl}(x : y) \to y$$

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☞ full-substitution strategy $\multimap\!\!\!\to$ is normalising for orthogonal TRSs

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This Talk

decreasingly confluent TRS

Hirokawa & Middeldorp, IJCAR 2010

$$\mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y)$$
$$\mathsf{hd}(x : y) \to x$$
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$$\mathsf{nat} \to 0 : \mathsf{inc}(\mathsf{nat})$$
$$\mathsf{inc}(\mathsf{tl}(\mathsf{nat})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nat}))$$

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AIM to show

full-substitution strategy is normalising for decreasingly confluent TRS

$$\mathsf{hd}(\mathsf{inc}(\mathsf{tl}(\mathsf{nat}))) \multimap\!\!\to^* \mathsf{s}(\mathsf{s}(0))$$

Rest of Talk

prove

1. $\longrightarrow\!\!\!\!\!\!\bullet\;\;$ is normalising for orthogonal TRSs known

Rest of Talk

prove

1. $\multimap\!\!\!\!\bullet\!\!\to$ is normalising for orthogonal TRSs — known

2. $\multimap\!\!\!\!\bullet\!\!\to$ is normalising for decreasingly confluent TRSs — new

Critical Pairs

DEFINITION

$(l_1 \to r_1, p, l_2 \to r_2)_\mu$ is overlap of TRS $\mathcal{R}$ if

- $l_1 \to r_1, l_2 \to r_2 \in \mathcal{R}$ without common variables by renaming
- $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(l_2)$
- $\mu = \mathrm{mgu}(l_2|_p, l_1)$, and
- $l_1 \to r_1$ and $l_2 \to r_2$ are not variants if $p = \epsilon$

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$$(l_2\mu)[l_1\mu]_p = l_2\mu$$

$$(l_2\mu)[r_1\mu]_p \quad \leftarrow \rtimes \rightarrow \quad r_2\mu$$

critical pair induced from overlap

Orthogonal TRS

THEOREM orthogonality Rosen, 1973

$\mathsf{LL} \quad \& \quad \leftarrow\!\rtimes\!\rightarrow = \varnothing \quad \Longrightarrow \quad \mathsf{CR}$

PROOF

Orthogonal TRS

Theorem orthogonality Rosen, 1973

$\mathsf{LL} \quad \& \quad \leftarrow \rtimes \rightarrow = \varnothing \quad \Longrightarrow \quad \mathsf{CR}$

Proof

$\leftarrow\!\circ\!- \cdot -\!\bullet\!\rightarrow \subseteq -\!\circ\!\rightarrow$ triangle commutation $\square$

Normalising for Orthogonal TRS

THEOREM

full substitution strategy is normalising for orthogonal TRSs

PROOF

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by $\leftarrow\!\circ\!- \cdot -\!\bullet\!\rightarrow \subseteq -\!\circ\!\rightarrow$

Normalising for Orthogonal TRS

Theorem

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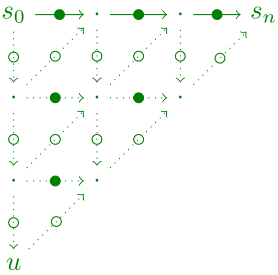

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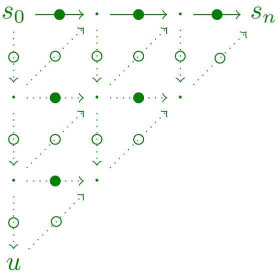

if $t \in \mathsf{NF}$ then $s_n = u = t \in \mathsf{NF}$ □

Decreasingly Confluent TRS

DEFINITION critical pair steps

$$\mathsf{CPS}(\mathcal{R}) = \left\{ \begin{array}{c} l_2\mu \rightarrow l_2\mu[r_1\mu]_p \\ l_2\mu \rightarrow r_2\mu \end{array} \middle| \; (l_1 \rightarrow r_1, p, l_2 \rightarrow r_2)_\mu \text{ is overlap of } \mathcal{R} \right\}$$

$$(l_2\mu)[l_1\mu]_p = l_2\mu$$

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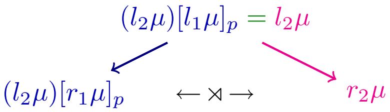

DEFINITION relative rewriting — Geser, 1990

$$\to_{\mathcal{R}/\mathcal{S}} \; = \; \to^*_{\mathcal{S}} \cdot \to_{\mathcal{R}} \cdot \to^*_{\mathcal{S}}$$

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Definition relative rewriting — Geser, 1990

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Theorem — Hirokawa & Middeldorp, 2010

$$\mathsf{LL} \quad \& \quad \leftarrow\!\rtimes\!\rightarrow \subseteq \downarrow \quad \& \quad \mathsf{CPS}(\mathcal{R})/\mathcal{R} : \mathsf{SN} \quad \Longrightarrow \quad \mathsf{CR}$$

Example

consider TRS $\mathcal{R}$

$$\begin{aligned} \mathsf{inc}(x : y) &\to \mathsf{s}(x) : \mathsf{inc}(y) & \mathsf{nat} &\to \mathsf{0} : \mathsf{inc}(\mathsf{nat}) \\ \mathsf{hd}(x : y) &\to x & \mathsf{inc}(\mathsf{tl}(\mathsf{nat})) &\to \mathsf{tl}(\mathsf{inc}(\mathsf{nat})) \\ \mathsf{tl}(x : y) &\to y \end{aligned}$$

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$\mathsf{CPS}(\mathcal{R})$ consists of

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hence, $\mathcal{R}$ is decreasingly confluent ☞ demo?

Normalizing for Decreasingly Confluent TRS

Theorem

full substitution strategy is normalising for decreasingly confluent TRSs

Proof

Normalizing for Decreasingly Confluent TRS

THEOREM

full substitution strategy is normalising for decreasingly confluent TRSs

PROOF

$t \leftarrow\!\circ\!- s \;-\!\bullet\!\rightarrow u \quad\Longrightarrow\quad t -\!\circ\!\rightarrow u$ or $t < s > u$

for $> \;=\; \rightarrow_{\mathsf{CPS}(\mathcal{R})/\mathcal{R}}$

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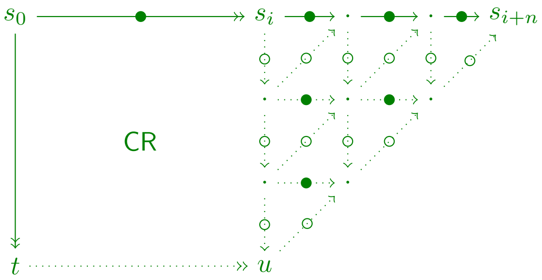

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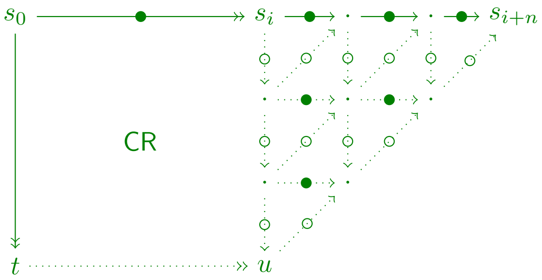

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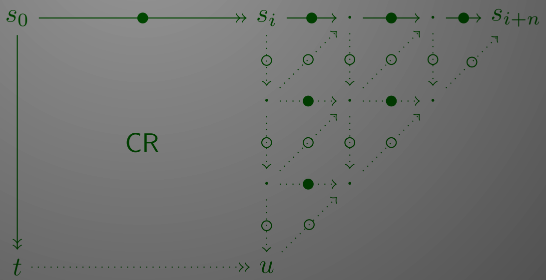

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Conclusion

for decreasingly confluent TRSs

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CURRENT WORK

- parallel outermost strategy is normalising?