Proving Equality of Simply Typed Term Rewriting Systems

Yuki Chiba Takahito Aoto

Japan Advanced Institute of Science and Technology (JAIST)

chiba@jaist.ac.jp

August 7, 2010

Program Transformation by Templates

Chiba et al. 2005, 2006, 2008, 2010.

- Pattern matching
- Correct templates
  - Equivalent Transformation of TRSs (Toyama 1991)
  - Inductionless induction
- RAPT
  - Automated Verification of the correctness of transformations

Higher-order Program Transformation

Based on 1st-order term rewriting

$\Rightarrow$ Difficult to apply higher-order programs (e.g. `map`, `filter`, `foldr` $\cdots$)

Higher-order Program Transformation

Based on 1st-order term rewriting

$\Rightarrow$ Difficult to apply higher-order programs (e.g. `map`, `filter`, `foldr` $\cdots$)

```
fun sum xs = foldr (op +) 0 xs
```

Higher-order Program Transformation

Based on 1st-order term rewriting

⇒ Difficult to apply higher-order programs (e.g. `map`, `filter`, `foldr` $\cdots$)

```
fun sum xs = foldr (op +) 0 xs
```

```
fun sum xs = sum1 xs 0
and sum1 [] y = y
  | sum1 (x::xs) y = sum1 xs (y+x);
```

Simply Typed Term Rewriting System

(Yamada 2001)

$$
\begin{aligned}
(\mathsf{sum}\ xs) &\rightarrow (\mathsf{foldr}\ +\ 0\ xs)\\
(\mathsf{foldr}\ f\ e\ [\,])&\rightarrow e\\
(\mathsf{foldr}\ f\ e\ (:\ x\ xs))&\rightarrow (f\ x\ (\mathsf{foldr}\ f\ e\ xs))\\
(+\ 0\ y)&\rightarrow y\\
(+\ (\mathsf{s}\ x)\ y)&\rightarrow (\mathsf{s}\ (+\ x\ y))
\end{aligned}
$$

$\mathscr{F}^{\mathsf{Nat}} = \{0\}$, $\mathscr{F}^{\mathsf{Nat}\rightarrow\mathsf{Nat}} = \{\mathsf{s}\}$, $\mathscr{F}^{\mathsf{Nat}\times\mathsf{Nat}\rightarrow\mathsf{Nat}} = \{+\}$, $\mathscr{F}^{\mathsf{List}} = \{[\,]\}$, $\mathscr{F}^{\mathsf{Nat}\times\mathsf{List}\rightarrow\mathsf{List}} = \{:\}$, $\mathscr{F}^{(\mathsf{Nat}\times\mathsf{Nat}\rightarrow\mathsf{Nat})\times\mathsf{Nat}\times\mathsf{List}\rightarrow\mathsf{List}} = \{\mathsf{foldr}\}$, $\mathscr{F}^{\mathsf{List}\rightarrow\mathsf{Nat}} = \{\mathsf{sum}\}$

Purpose

Extending the framework to STTRS

Purpose

Extending the framework to STTRS

1. Equivalent transformation
2. Transformation templates
3. Pattern matching
4. Correct templates

Contents

The equality of STTRSs

$\mathcal{R} \simeq_{\mathscr{G}} \mathcal{R}' \overset{\text{def}}{\Leftrightarrow} \forall s \in \mathrm{T}(\mathscr{G}).\forall t \in \mathrm{T}(\mathscr{C}).\ s \xrightarrow{*}_{\mathcal{R}} t \text{ iff } s \xrightarrow{*}_{\mathcal{R}'} t$

The equality of STTRSs

$$\mathcal{R} \simeq_{\mathscr{G}} \mathcal{R}' \overset{\text{def}}{\Leftrightarrow} \forall s \in \mathrm{T}(\mathscr{G}).\forall t \in \mathrm{T}(\mathscr{C}).\ s \xrightarrow{*}_{\mathcal{R}} t \text{ iff } s \xrightarrow{*}_{\mathcal{R}'} t$$

$$\left\{\begin{array}{ll}(\mathsf{sum}\ xs) & \to (\mathsf{foldr}\ +\ 0\ xs) \\ (\mathsf{foldr}\ f\ e\ [\,]) & \to e \\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \to y \\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y))\end{array}\right\}$$

$$\left\{\begin{array}{ll}(\mathsf{sum}\ xs) & \to (\mathsf{sum1}\ xs) \\ (\mathsf{sum1}\ [\,]\ z]) & \to z \\ (\mathsf{sum1}\ (:\ x\ xs)\ z) & \to (\mathsf{sum1}\ xs\ (+\ z\ s)) \\ (+\ 0\ y) & \to y \\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y))\end{array}\right\}$$

The equality of STTRSs

$$\mathcal{R} \simeq_{\mathscr{G}} \mathcal{R}' \overset{\text{def}}{\Leftrightarrow} \forall s \in \mathrm{T}(\mathscr{G}).\forall t \in \mathrm{T}(\mathscr{C}).\ s \xrightarrow{*}_{\mathcal{R}} t \text{ iff } s \xrightarrow{*}_{\mathcal{R}'} t$$

$$\left\{ \begin{array}{ll} (\mathsf{sum}\ xs) & \rightarrow (\mathsf{foldr}\ +\ 0\ xs) \\ (\mathsf{foldr}\ f\ e\ [\,]) & \rightarrow e \\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \rightarrow (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \rightarrow y \\ (+\ (\mathsf{s}\ x)\ y) & \rightarrow (\mathsf{s}\ (+\ x\ y)) \end{array} \right\}$$

$$\simeq_{\{\mathsf{sum},+,[\,],:,\mathsf{s},0\}}$$

$$\left\{ \begin{array}{ll} (\mathsf{sum}\ xs) & \rightarrow (\mathsf{sum1}\ xs) \\ (\mathsf{sum1}\ [\,]\ z]) & \rightarrow z \\ (\mathsf{sum1}\ (:\ x\ xs)\ z) & \rightarrow (\mathsf{sum1}\ xs\ (+\ z\ s)) \\ (+\ 0\ y) & \rightarrow y \\ (+\ (\mathsf{s}\ x)\ y) & \rightarrow (\mathsf{s}\ (+\ x\ y)) \end{array} \right\}$$

Equivalent Transformation

$\mathcal{R}_0$: left-linear CS over $\mathscr{F}_0$, $\mathcal{E}$: set of equations over $\mathscr{F}_0$

- Introduction

  $$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{(f\ x_1\ \ldots\ x_n) \to r\}\ (= \mathcal{R}_{k+1})$$

  $$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$$

  - $(f\ x_1\ \ldots\ x_n)$ is linear
  - $f \notin \mathscr{F}_k$,
  - $r \in T(\mathscr{F}_k, \mathscr{V})$, and
  - $x_1, \ldots, x_n$ are basic

- Addition

  $$\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}\ (= \mathcal{R}_{k+1})$$

  - $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$

- Elimination

  $$\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}\ (= \mathcal{R}_{k+1})$$

Example

$$\left\{\begin{array}{ll} (\mathsf{sum}\ xs) & \rightarrow (\mathsf{foldr}\ +\ 0\ xs) \\ (\mathsf{foldr}\ f\ e\ [\,]) & \rightarrow e \\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \rightarrow (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \rightarrow y \\ (+\ (\mathsf{s}\ x)\ y) & \rightarrow (\mathsf{s}\ (+\ x\ y)) \end{array}\right\}$$

$$\left\{\begin{array}{ll} (+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z) \\ (+\ x\ 0) & \approx x \end{array}\right\}$$

Example

$$\left\{\begin{array}{ll}(\mathsf{sum}\ xs) & \to (\mathsf{foldr}\ +\ 0\ xs) \\ (\mathsf{foldr}\ f\ e\ [\,]) & \to e \\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \to y \\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y))\end{array}\right\}$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{sum}}$

$$\left\{\begin{array}{ll}(+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z) \\ (+\ x\ 0) & \approx x\end{array}\right\}$$

Example

$$\left\{\begin{array}{ll} (\textsf{sum}\ xs) & \to (\textsf{foldr}\ +\ 0\ xs) \\ (\textsf{foldr}\ f\ e\ [\,]) & \to e \\ (\textsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\textsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \to y \\ (+\ (\textsf{s}\ x)\ y) & \to (\textsf{s}\ (+\ x\ y)) \end{array}\right\}$$

- $\mathcal{R}_0 = \mathcal{R}_{\textsf{sum}}$
- $\mathcal{R}_0 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{(\textsf{sum1}\ xs\ y) \to (+\ y\ (\textsf{sum}\ xs))\} = \mathcal{R}_1$

$$\left\{\begin{array}{ll} (+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z) \\ (+\ x\ 0) & \approx x \end{array}\right\}$$

Example

$$\left\{\begin{array}{ll}(\mathsf{sum}\ xs) & \rightarrow (\mathsf{foldr}\ +\ \mathsf{0}\ xs)\\ (\mathsf{foldr}\ f\ e\ [\,]) & \rightarrow e\\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \rightarrow (f\ x\ (\mathsf{foldr}\ f\ e\ xs))\\ (+\ \mathsf{0}\ y) & \rightarrow y\\ (+\ (\mathsf{s}\ x)\ y) & \rightarrow (\mathsf{s}\ (+\ x\ y))\end{array}\right\}$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{sum}}$
- $\mathcal{R}_0 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{(\mathsf{sum1}\ xs\ y) \rightarrow (+\ y\ (\mathsf{sum}\ xs))\} = \mathcal{R}_1$
- $\mathcal{R}_1 \underset{A}{\Rightarrow} \mathcal{R}_1 \cup \{(\mathsf{sum1}\ [\,]\ y) \rightarrow y\} = \mathcal{R}_2$

$$\left\{\begin{array}{ll}(+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z)\\ (+\ x\ 0) & \approx x\end{array}\right\}$$

Example

$$\left\{\begin{array}{ll} (\mathsf{sum}\ xs) & \to (\mathsf{foldr}\ +\ 0\ xs) \\ (\mathsf{foldr}\ f\ e\ [\,]) & \to e \\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \to y \\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y)) \end{array}\right\}$$

$$\left\{\begin{array}{ll} (+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z) \\ (+\ x\ 0) & \approx x \end{array}\right\}$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{sum}}$
- $\mathcal{R}_0 \underset{I}{\Rrightarrow} \mathcal{R}_0 \cup \{(\mathsf{sum1}\ xs\ y) \to (+\ y\ (\mathsf{sum}\ xs))\} = \mathcal{R}_1$
- $\mathcal{R}_1 \underset{A}{\Rrightarrow} \mathcal{R}_1 \cup \{(\mathsf{sum1}\ [\,]\ y) \to y\} = \mathcal{R}_2$
- $\mathcal{R}_2 \underset{A}{\Rrightarrow} \mathcal{R}_2 \cup \{(\mathsf{sum1}\ (:\ x\ xs)\ y) \to (\mathsf{sum1}\ xs\ (+\ y\ x))\}$ $= \mathcal{R}_3$

$$\begin{array}{lll} (\mathsf{sum1}\ (:\ x\ xs)\ y) & \to_{\mathcal{R}_2} & (+\ y\ (\mathsf{sum}\ (:\ x\ xs))) \\ & \to_{\mathcal{R}_2} & (+\ y\ (\mathsf{foldr}\ +\ 0\ (:\ x\ xs))) \\ & \to_{\mathcal{R}_2} & (+\ y\ (+\ x\ (\mathsf{foldr}\ +\ 0\ xs))) \\ & \approx & (+\ (+\ y\ x)\ (\mathsf{foldr}\ +\ 0\ xs)) \\ & \leftarrow_{\mathcal{R}_2} & (\mathsf{sum1}\ xs\ (+\ y\ x)) \end{array}$$

Example

$$\left\{\begin{array}{ll} (\mathsf{sum}\ xs) & \to (\mathsf{foldr}\ +\ 0\ xs) \\ (\mathsf{foldr}\ f\ e\ [\,]) & \to e \\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \to y \\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y)) \end{array}\right\}$$

$$\left\{\begin{array}{ll} (+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z) \\ (+\ x\ 0) & \approx x \end{array}\right\}$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{sum}}$
- $\mathcal{R}_0 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{(\mathsf{sum1}\ xs\ y) \to (+\ y\ (\mathsf{sum}\ xs))\} = \mathcal{R}_1$
- $\mathcal{R}_1 \underset{A}{\Rightarrow} \mathcal{R}_1 \cup \{(\mathsf{sum1}\ [\,]\ y) \to y\} = \mathcal{R}_2$
- $\mathcal{R}_2 \underset{A}{\Rightarrow} \mathcal{R}_2 \cup \{(\mathsf{sum1}\ (:\ x\ xs)\ y) \to (\mathsf{sum1}\ xs\ (+\ y\ x))\} = \mathcal{R}_3$

$$\begin{array}{rl} (\mathsf{sum1}\ (:\ x\ xs)\ y) & \to_{\mathcal{R}_2} (+\ y\ (\mathsf{sum}\ (:\ x\ xs))) \\ & \to_{\mathcal{R}_2} (+\ y\ (\mathsf{foldr}\ +\ 0\ (:\ x\ xs))) \\ & \to_{\mathcal{R}_2} (+\ y\ (+\ x\ (\mathsf{foldr}\ +\ 0\ xs))) \\ & \approx \quad (+\ (+\ y\ x)\ (\mathsf{foldr}\ +\ 0\ xs)) \\ & \leftarrow_{\mathcal{R}_2} (\mathsf{sum1}\ xs\ (+\ y\ x)) \end{array}$$

- $\mathcal{R}_3 \underset{A}{\Rightarrow} \mathcal{R}_3 \cup \{(\mathsf{sum}\ xs) \to (\mathsf{sum1}\ xs\ 0)\} = \mathcal{R}_4$

$$\begin{array}{rl} (\mathsf{sum}\ xs) & \leftarrow_{\mathcal{R}_3} (+\ 0\ (\mathsf{sum}\ xs)) \\ & \leftarrow_{\mathcal{R}_3} (\mathsf{sum1}\ xs\ 0) \end{array}$$

Example

$$\left\{\begin{array}{ll}(\mathsf{sum}\ xs) & \to (\mathsf{foldr}\ +\ 0\ xs)\\ (\mathsf{foldr}\ f\ e\ [\,]) & \to e\\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs))\\ (+\ 0\ y) & \to y\\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y))\end{array}\right\}$$

$$\left\{\begin{array}{ll}(+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z)\\ (+\ x\ 0) & \approx x\end{array}\right\}$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{sum}}$
- $\mathcal{R}_0 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{(\mathsf{sum1}\ xs\ y) \to (+\ y\ (\mathsf{sum}\ xs))\} = \mathcal{R}_1$
- $\mathcal{R}_1 \underset{A}{\Rightarrow} \mathcal{R}_1 \cup \{(\mathsf{sum1}\ [\,]\ y) \to y\} = \mathcal{R}_2$
- $\mathcal{R}_2 \underset{A}{\Rightarrow} \mathcal{R}_2 \cup \{(\mathsf{sum1}\ (:\ x\ xs)\ y) \to (\mathsf{sum1}\ xs\ (+\ y\ x))\} = \mathcal{R}_3$

$$\begin{array}{rll}(\mathsf{sum1}\ (:\ x\ xs)\ y) & \to_{\mathcal{R}_2} & (+\ y\ (\mathsf{sum}\ (:\ x\ xs)))\\ & \to_{\mathcal{R}_2} & (+\ y\ (\mathsf{foldr}\ +\ 0\ (:\ x\ xs)))\\ & \to_{\mathcal{R}_2} & (+\ y\ (+\ x\ (\mathsf{foldr}\ +\ 0\ xs)))\\ & \approx & (+\ (+\ y\ x)\ (\mathsf{foldr}\ +\ 0\ xs))\\ & \leftarrow_{\mathcal{R}_2} & (\mathsf{sum1}\ xs\ (+\ y\ x))\end{array}$$

- $\mathcal{R}_3 \underset{A}{\Rightarrow} \mathcal{R}_3 \cup \{(\mathsf{sum}\ xs) \to (\mathsf{sum1}\ xs\ 0)\} = \mathcal{R}_4$

$$\begin{array}{rll}(\mathsf{sum}\ xs) & \leftarrow_{\mathcal{R}_3} & (+\ 0\ (\mathsf{sum}\ xs))\\ & \leftarrow_{\mathcal{R}_3} & (\mathsf{sum1}\ xs\ 0)\end{array}$$

- $\mathcal{R}_4 \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

Example

$$\left\{\begin{array}{ll} (\mathsf{sum}\ xs) & \to (\mathsf{foldr} + 0\ xs) \\ (\mathsf{foldr}\ f\ e\ [\,]) & \to e \\ (\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\ (+\ 0\ y) & \to y \\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y)) \end{array}\right\}$$

$$\left\{\begin{array}{ll} (+\ x\ (+\ y\ z)) & \approx (+\ (+\ x\ y)\ z) \\ (+\ x\ 0) & \approx x \end{array}\right\}$$

$$\left\{\begin{array}{ll} (\mathsf{sum}\ xs) & \to (\mathsf{sum1}\ xs) \\ (\mathsf{sum1}\ [\,]\ z]) & \to z \\ (\mathsf{sum1}\ (:\ x\ xs)\ z) & \to (\mathsf{sum1}\ xs\ (+\ z\ s)) \\ (+\ 0\ y) & \to y \\ (+\ (\mathsf{s}\ x)\ y) & \to (\mathsf{s}\ (+\ x\ y)) \end{array}\right.$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{sum}}$
- $\mathcal{R}_0 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{(\mathsf{sum1}\ xs\ y) \to (+\ y\ (\mathsf{sum}\ xs))\} = \mathcal{R}_1$
- $\mathcal{R}_1 \underset{A}{\Rightarrow} \mathcal{R}_1 \cup \{(\mathsf{sum1}\ [\,]\ y) \to y\} = \mathcal{R}_2$
- $\mathcal{R}_2 \underset{A}{\Rightarrow} \mathcal{R}_2 \cup \{(\mathsf{sum1}\ (:\ x\ xs)\ y) \to (\mathsf{sum1}\ xs\ (+\ y\ x))\}$ $= \mathcal{R}_3$

$$\begin{array}{rl} (\mathsf{sum1}\ (:\ x\ xs)\ y) & \to_{\mathcal{R}_2} (+\ y\ (\mathsf{sum}\ (:\ x\ xs))) \\ & \to_{\mathcal{R}_2} (+\ y\ (\mathsf{foldr} + 0\ (:\ x\ xs))) \\ & \to_{\mathcal{R}_2} (+\ y\ (+\ x\ (\mathsf{foldr} + 0\ xs))) \\ & \approx \quad (+\ (+\ y\ x)\ (\mathsf{foldr} + 0\ xs)) \\ & \leftarrow_{\mathcal{R}_2} (\mathsf{sum1}\ xs\ (+\ y\ x)) \end{array}$$

- $\mathcal{R}_3 \underset{A}{\Rightarrow} \mathcal{R}_3 \cup \{(\mathsf{sum}\ xs) \to (\mathsf{sum1}\ xs\ 0)\} = \mathcal{R}_4$

$$\begin{array}{rl} (\mathsf{sum}\ xs) & \leftarrow_{\mathcal{R}_3} (+\ 0\ (\mathsf{sum}\ xs)) \\ & \leftarrow_{\mathcal{R}_3} (\mathsf{sum1}\ xs\ 0) \end{array}$$

- $\mathcal{R}_4 \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

Verifying Equality of TRSs (1st-order)

Theorem

- $\mathcal{R}$ is a left-linear CS over $\mathscr{G}$
- $\mathcal{R}'$ is a TRS over $\mathscr{G}'$
- $\mathcal{E}$ is a set of equations over $\mathscr{G}$
- $\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \cdot \overset{*}{\underset{A}{\Rightarrow}} \cdot \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$ under $\mathcal{E}$ $\quad\Longrightarrow\quad$ $\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $CR(\mathcal{R})$, $SC(\mathcal{R}, \mathscr{G})$
- $SC(\mathcal{R}', \mathscr{G}')$

Conjectures

$$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$$

1. $\mathrm{CR}(\mathcal{R})$ implies $\mathrm{CR}(\mathcal{R}_I)$
2. Let $\mathcal{R}$ and $\mathcal{R}_I$ be STTRSs over $\mathscr{G}$ and $\mathscr{F}$. $\forall s \in \mathrm{T}(\mathscr{F}).\exists t \in \mathrm{T}(\mathscr{G})$ s.t. $s \overset{*}{\to}_{\mathcal{R}_I} t$

Proof

$$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$$

1. $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$
2. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$
3. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$
4. $\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$

$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

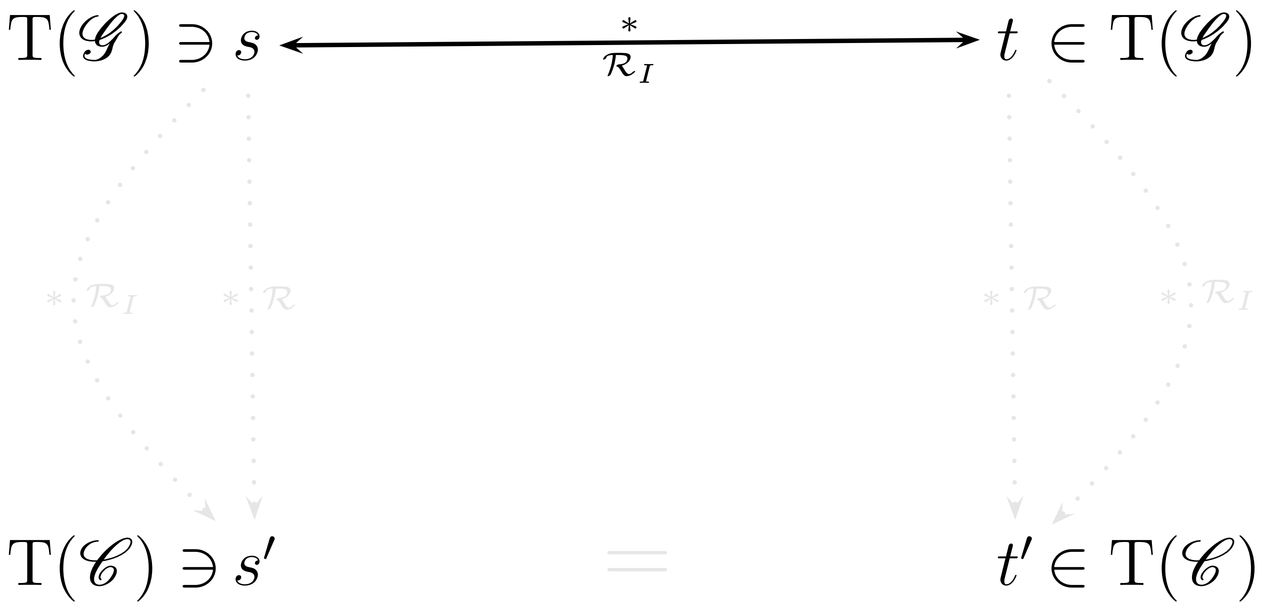
$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Introduction

$$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{(f\ x_1\ \ldots\ x_n) \to r\}\ (= \mathcal{R}_{k+1})$$

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$$

- $(f\ x_1\ \ldots\ x_n)$ is linear
- $f \notin \mathscr{F}_k$,
- $r \in T(\mathscr{F}_k, \mathscr{V})$, and
- $x_1, \ldots, x_n$ are basic



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$
- $\mathcal{R} \subseteq \mathcal{R}_I$
- $\mathrm{CR}(\mathcal{R}_I)$ and $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

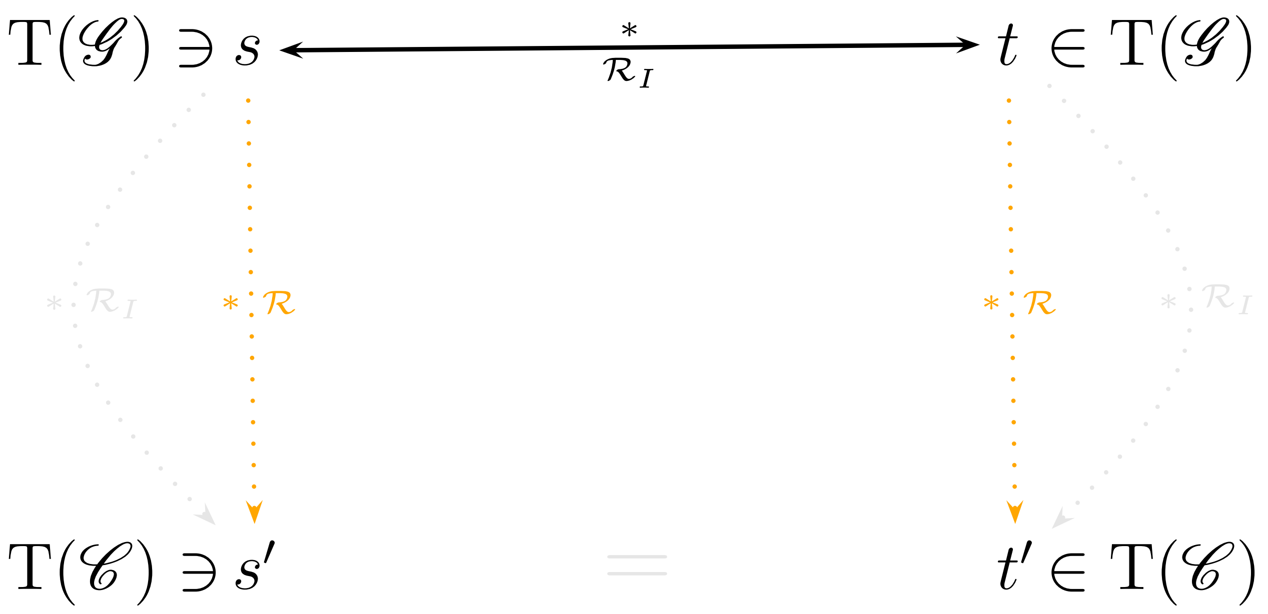
$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Introduction

$$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{(f\ x_1\ \ldots\ x_n) \to r\}\ (= \mathcal{R}_{k+1})$$

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$$

- $(f\ x_1\ \ldots\ x_n)$ is linear
- $f \notin \mathscr{F}_k$,
- $r \in T(\mathscr{F}_k, \mathscr{V})$, and
- $x_1, \ldots, x_n$ are basic



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$
- $\mathcal{R} \subseteq \mathcal{R}_I$
- $\mathrm{CR}(\mathcal{R}_I)$ and $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

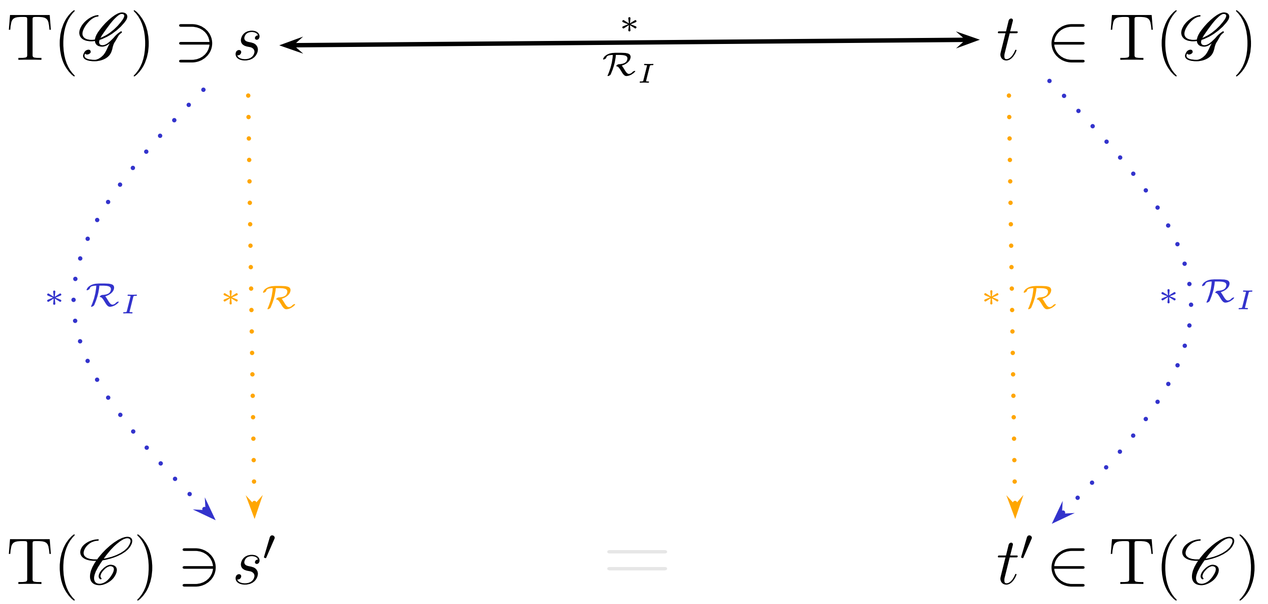
$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Introduction

$$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{(f\ x_1\ \ldots\ x_n) \to r\}\ (= \mathcal{R}_{k+1})$$

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$$

  - $(f\ x_1\ \ldots\ x_n)$ is linear
  - $f \notin \mathscr{F}_k$,
  - $r \in T(\mathscr{F}_k, \mathscr{V})$, and
  - $x_1, \ldots, x_n$ are basic



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$
- $\mathcal{R} \subseteq \mathcal{R}_I$
- $\mathrm{CR}(\mathcal{R}_I)$ and $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

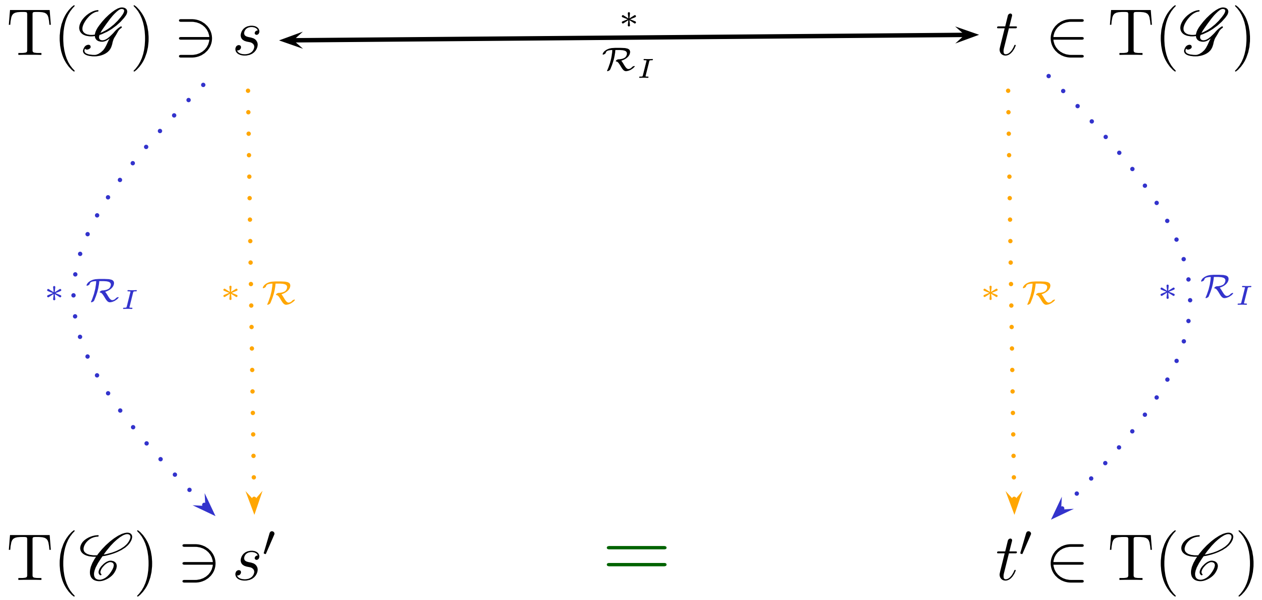
$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Introduction

$$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{(f\ x_1\ \ldots\ x_n) \to r\}\ (= \mathcal{R}_{k+1})$$
$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$$

  - $(f\ x_1\ \ldots\ x_n)$ is linear
  - $f \notin \mathscr{F}_k$,
  - $r \in T(\mathscr{F}_k, \mathscr{V})$, and
  - $x_1, \ldots, x_n$ are basic



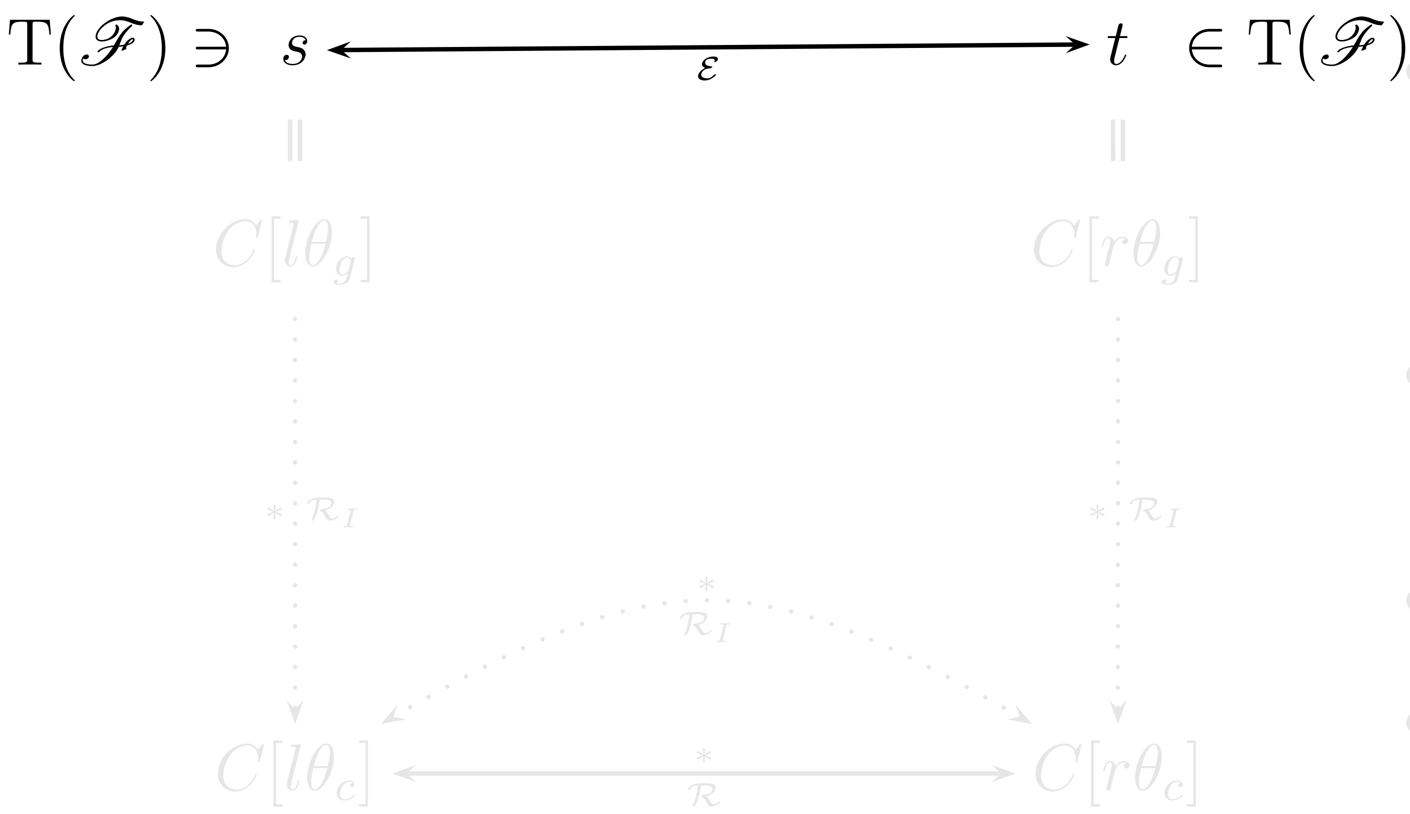
- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$
- $\mathcal{R} \subseteq \mathcal{R}_I$
- $\mathrm{CR}(\mathcal{R}_I)$ and $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$ – $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}}\, r$



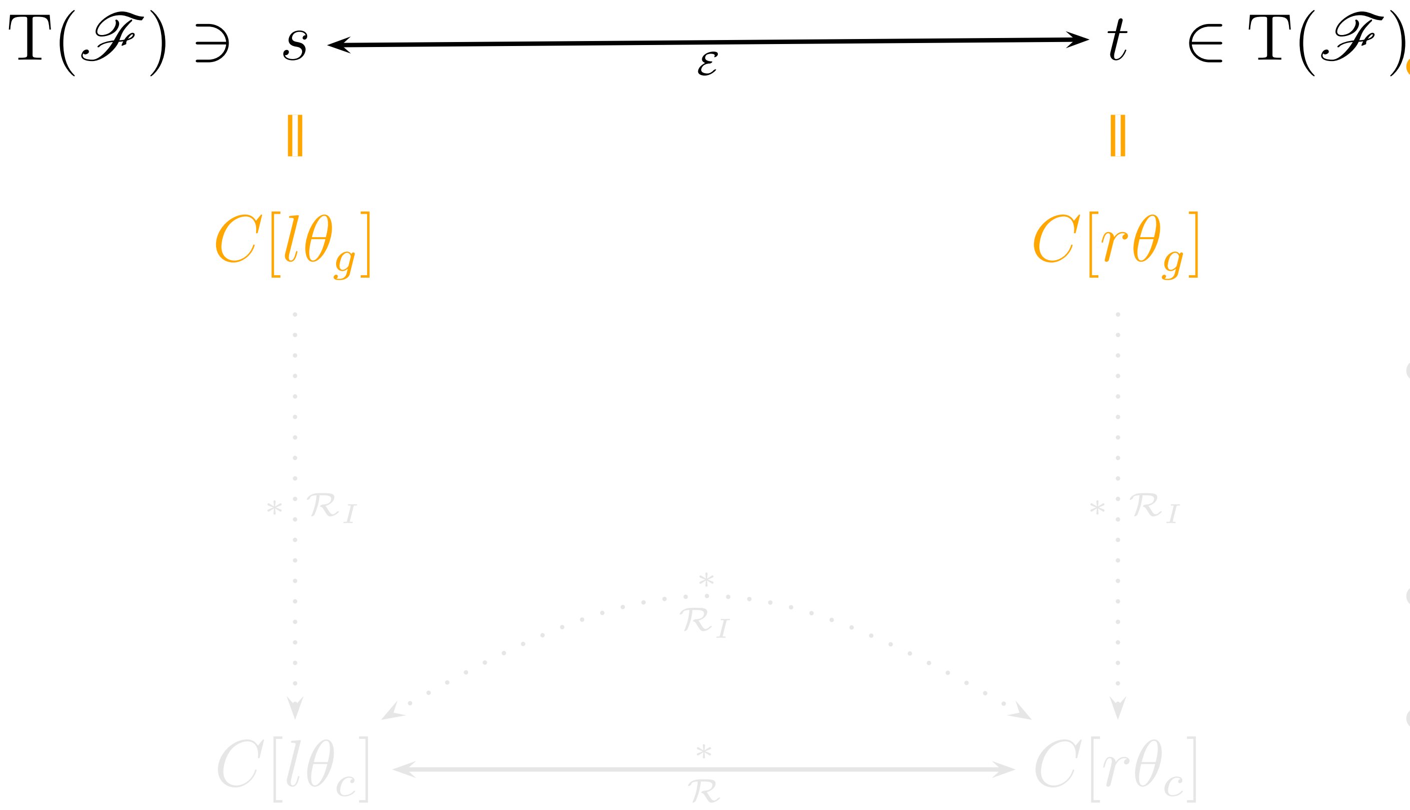
- The definition of reduction relation where
  - $\theta_g : \mathscr{V} \to \mathrm{T}(\mathscr{F})$
- $\mathrm{SC}(\mathcal{R}_I, \mathscr{F})$ where
  - $\theta_c : \mathscr{V} \to \mathrm{T}(\mathscr{C})$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$
  - $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



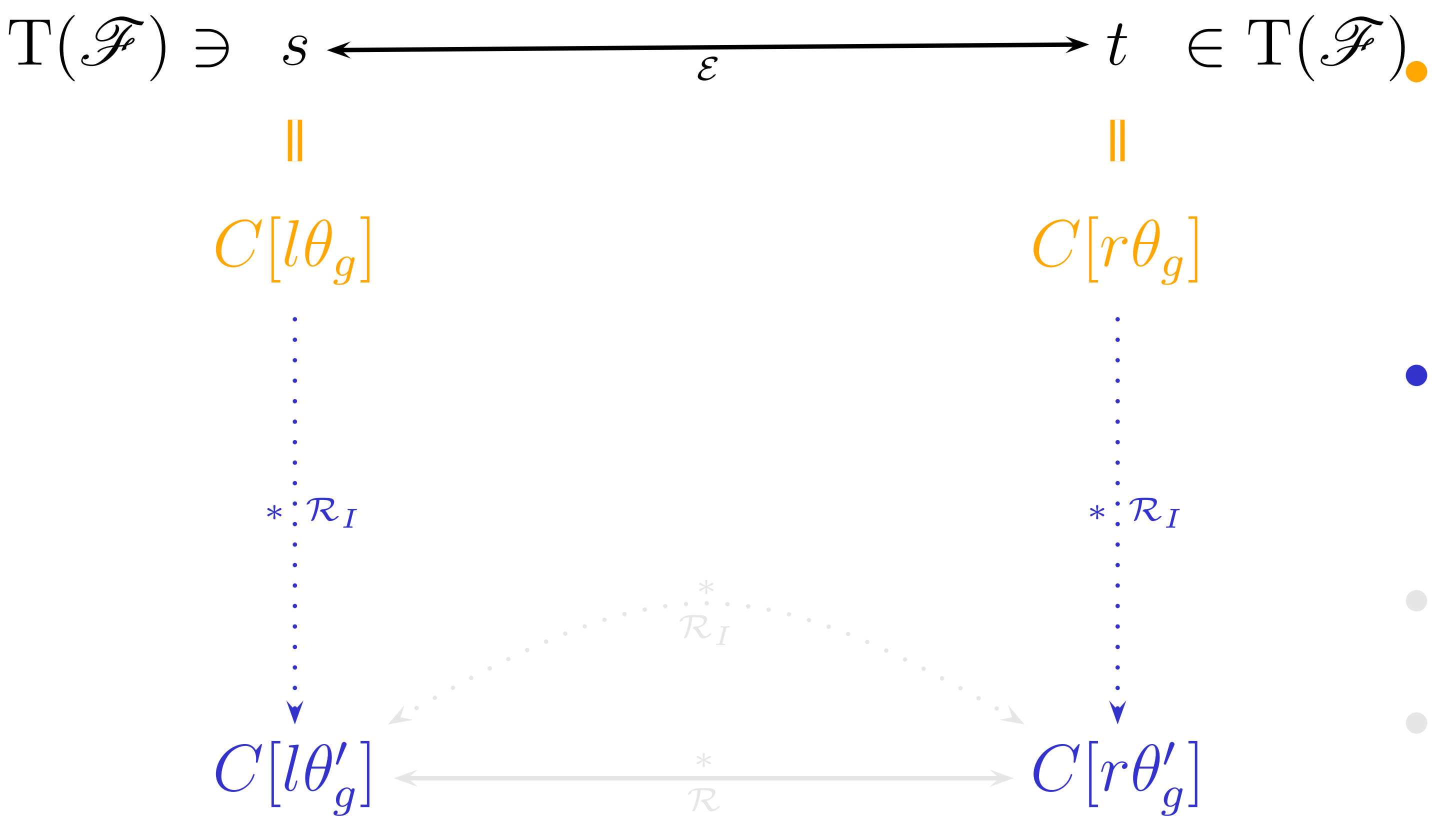
- The definition of reduction relation where
  - $\theta_g : \mathscr{V} \to \mathrm{T}(\mathscr{F})$
- $\mathrm{SC}(\mathcal{R}_I, \mathscr{F})$ where
  - $\theta_c : \mathscr{V} \to \mathrm{T}(\mathscr{C})$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$ – $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



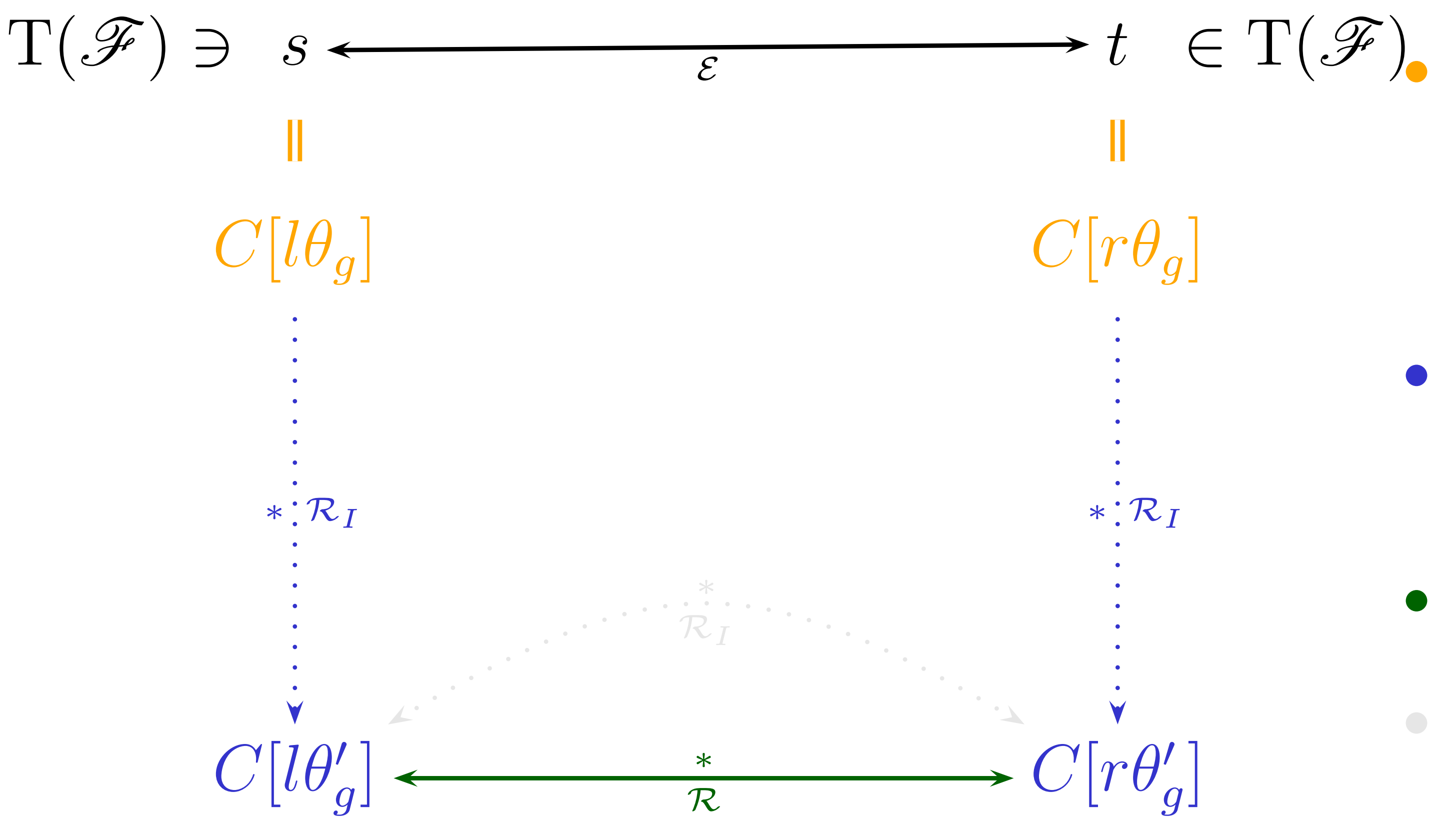
- The definition of reduction relation where
  - $\theta_g : \mathscr{V} \to \mathrm{T}(\mathscr{F})$
- Conjecture 2 where
  - $\theta'_g : \mathscr{V} \to \mathrm{T}(\mathscr{G})$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$ – $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



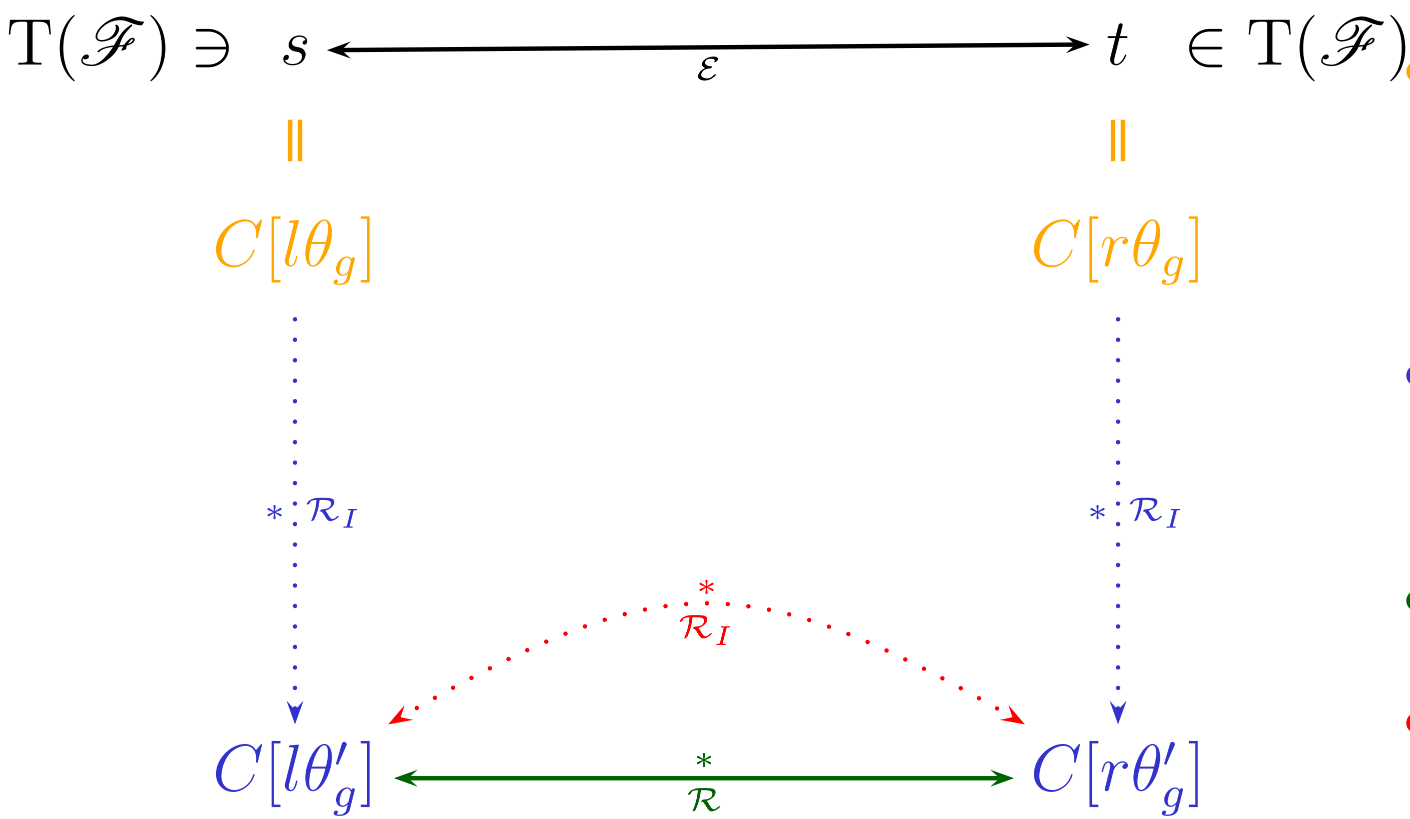
- The definition of reduction relation where
  - $\theta_g : \mathscr{V} \to \mathrm{T}(\mathscr{F})$
- Conjecture 2 where
  - $\theta'_g : \mathscr{V} \to \mathrm{T}(\mathscr{G})$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}\ (= \mathcal{R}_{k+1})$ – $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



- The definition of reduction relation where
  - $\theta_g : \mathscr{V} \to \mathrm{T}(\mathscr{F})$
- Conjecture 2 where
  - $\theta'_g : \mathscr{V} \to \mathrm{T}(\mathscr{G})$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Elimination
  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$

$$\mathrm{T}(\mathscr{G}') \ni s \overset{*}{\underset{\mathcal{R}_I}{\longleftrightarrow}} t \in \mathrm{T}(\mathscr{G}')$$

$\mathcal{R}_I$ $*$ $*$ $\mathcal{R}'$ $*$ $\mathcal{R}'$ $\mathcal{R}_I$ $*$

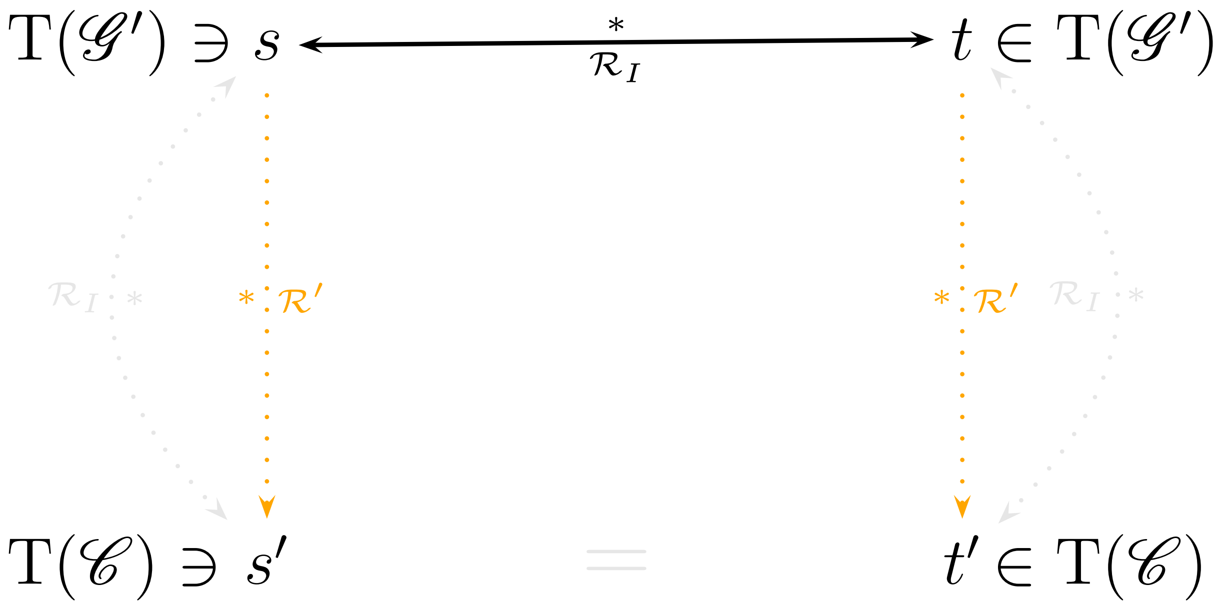
$$\mathrm{T}(\mathscr{C}) \ni s' \quad = \quad t' \in \mathrm{T}(\mathscr{C})$$

- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftrightarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}_A} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$
- $\mathrm{CR}(\mathcal{R}_I)$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Elimination
  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$



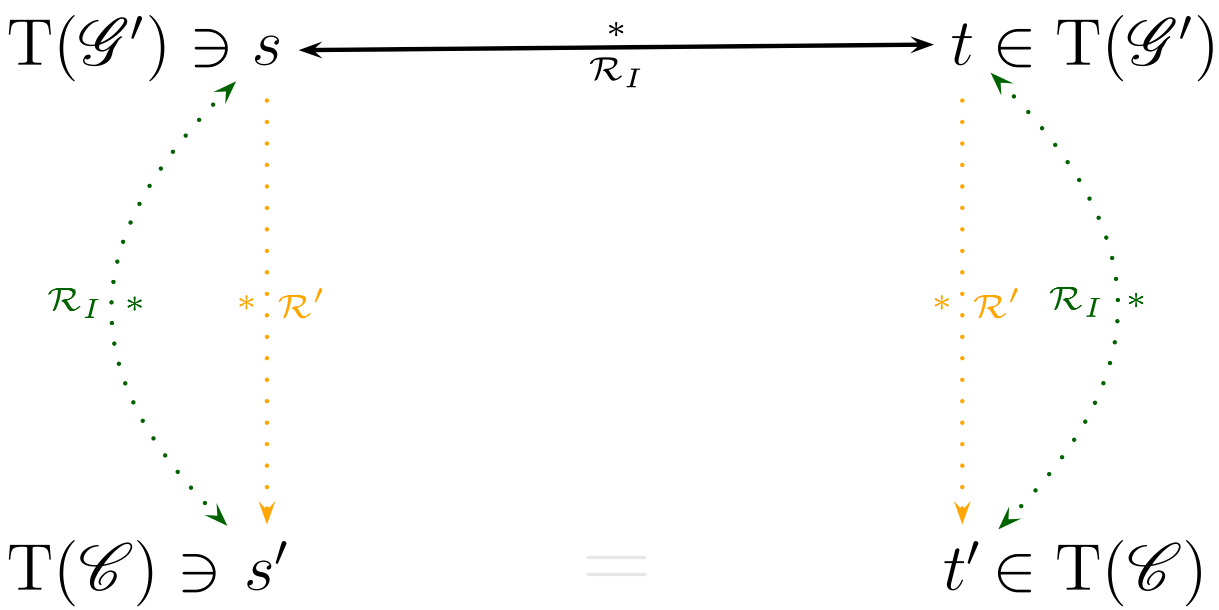
- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftrightarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}_A} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$
- $\mathrm{CR}(\mathcal{R}_I)$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Elimination

  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$



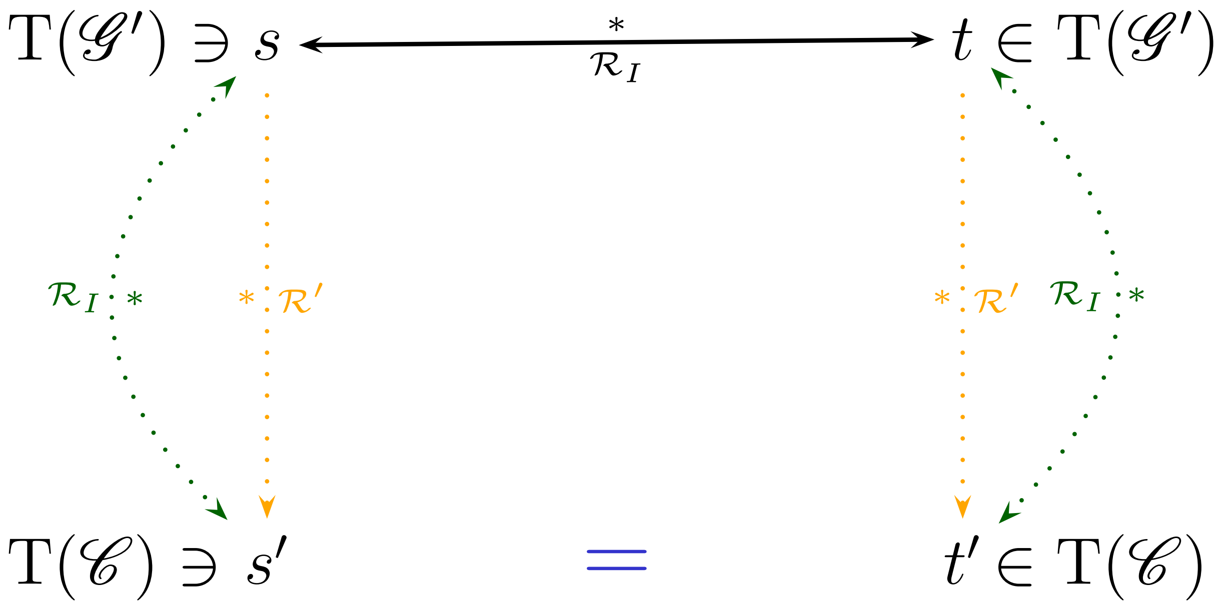
- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftrightarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}_A} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$
- $\mathrm{CR}(\mathcal{R}_I)$,

  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

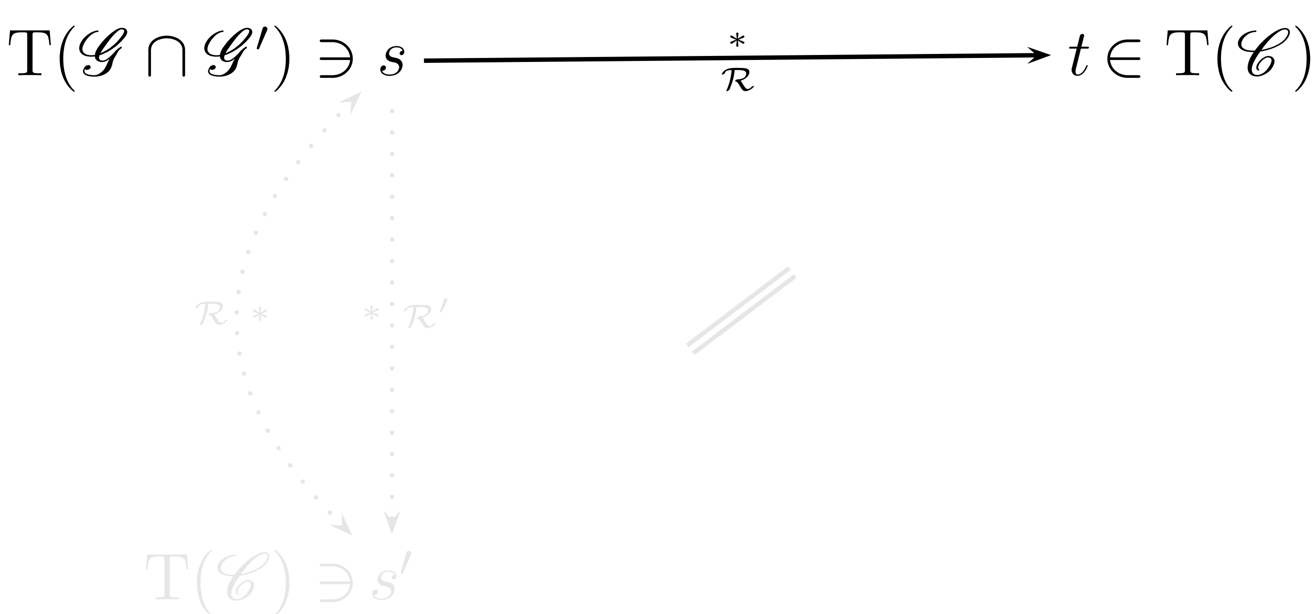
- Elimination
  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$



- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftrightarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}_A} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$
- $\mathrm{CR}(\mathcal{R}_I)$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ ($\subseteq$)

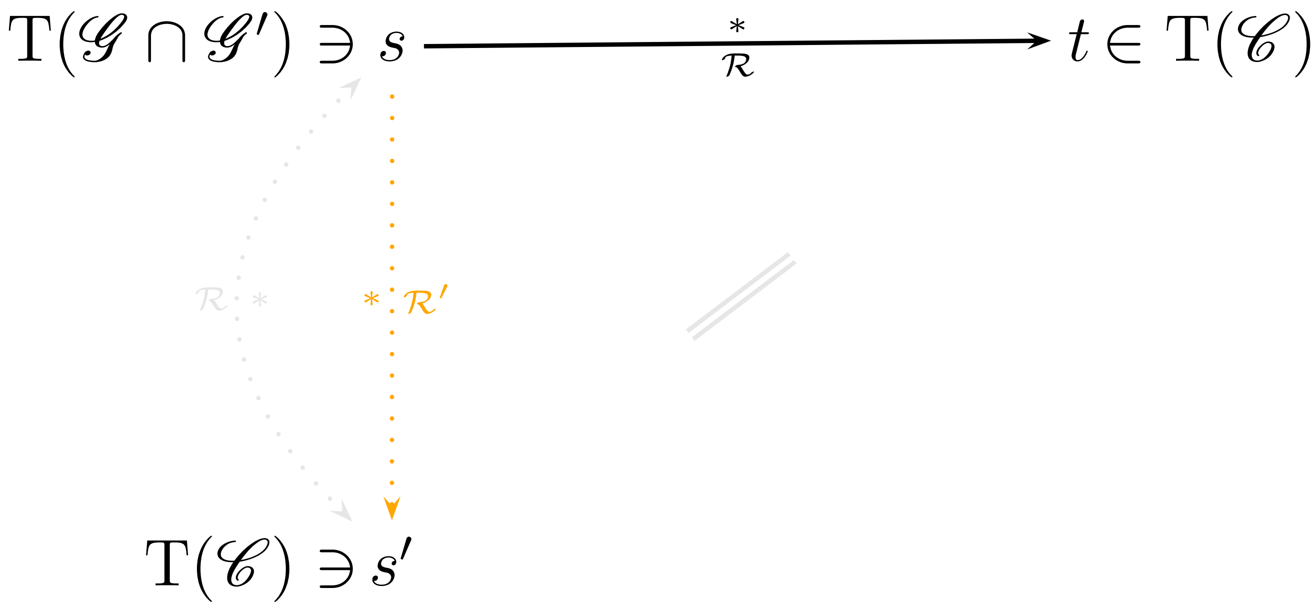
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$ ($\subseteq$)

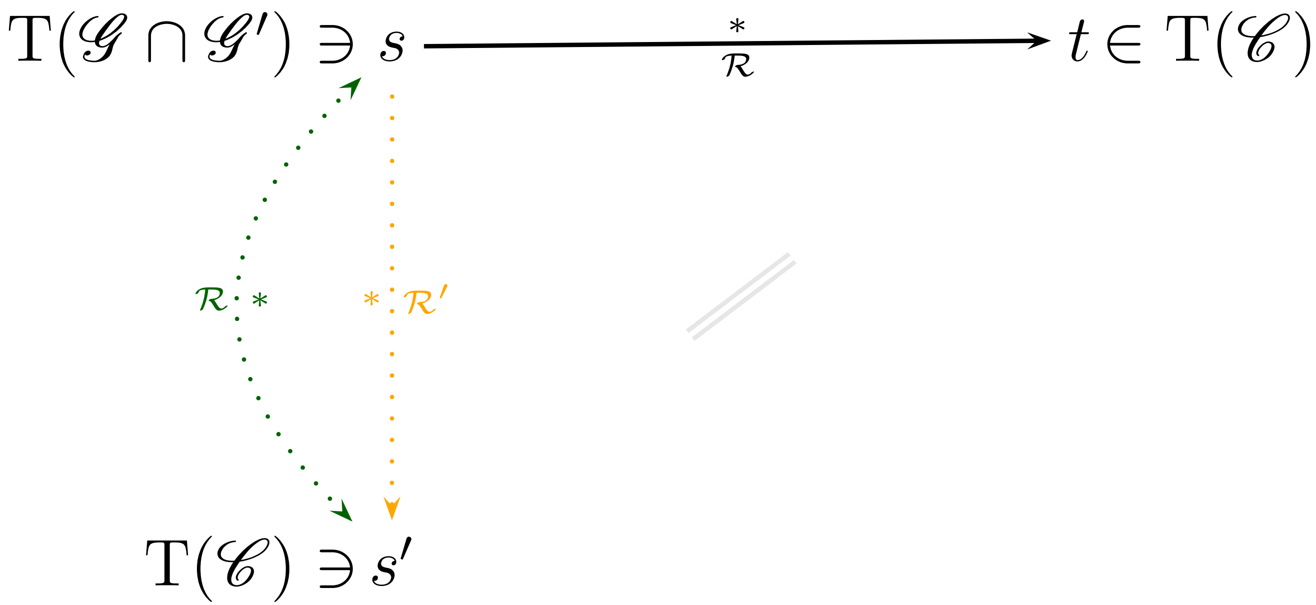
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ $(\subseteq)$

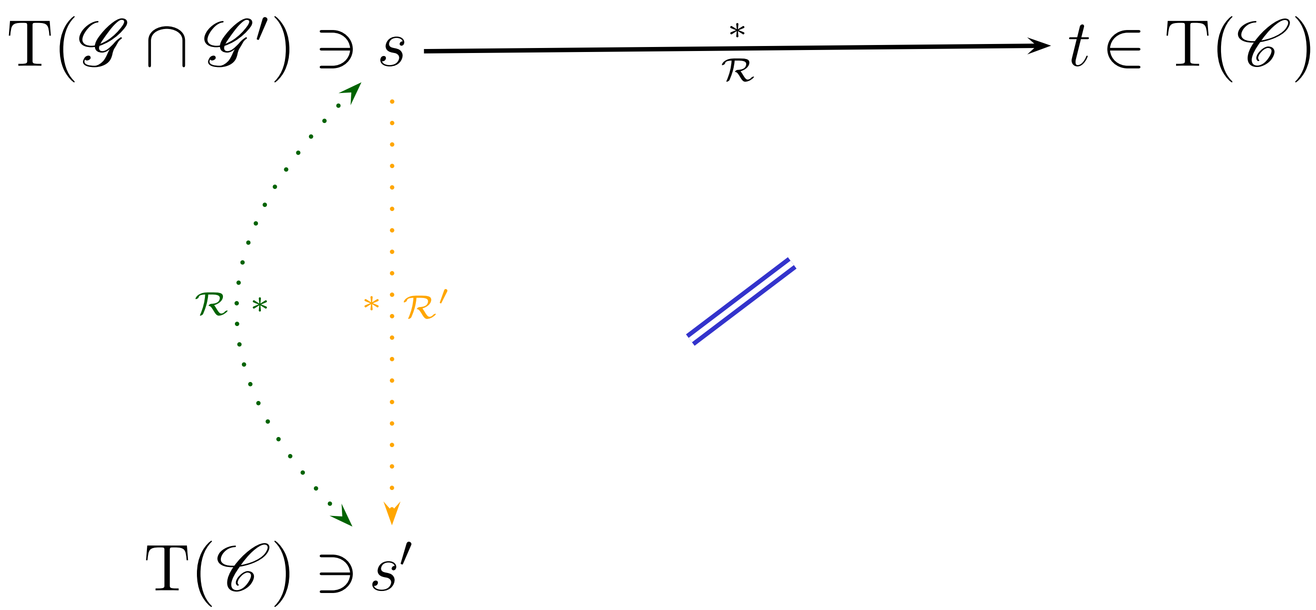
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ ($\subseteq$)

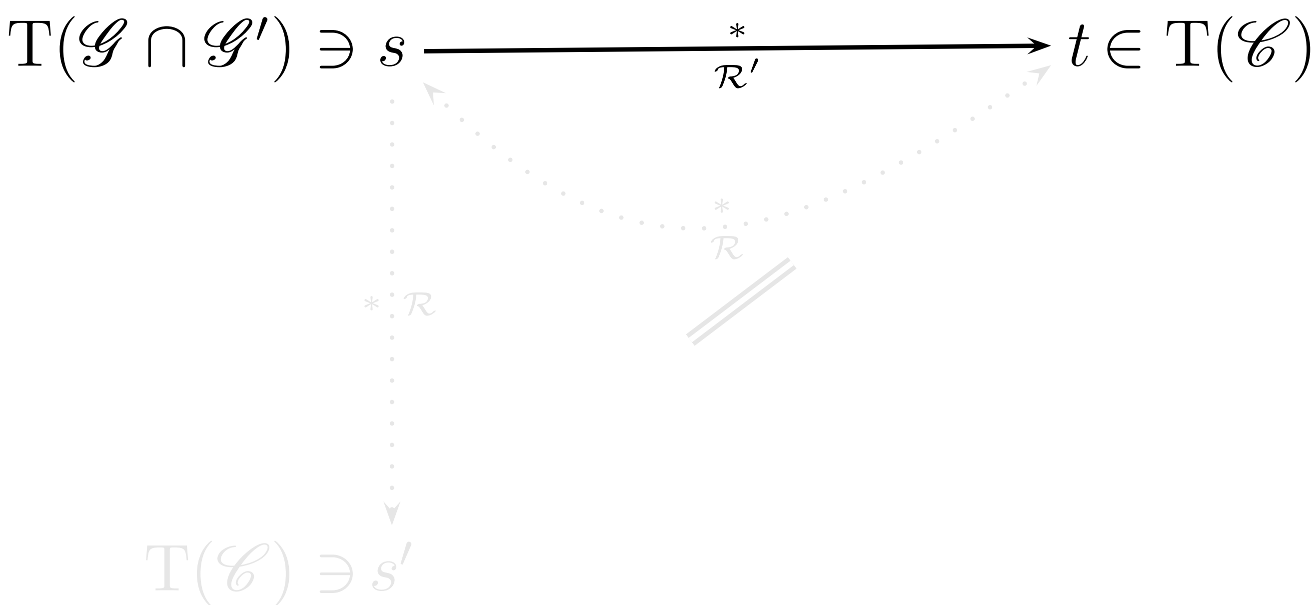
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$ ($\supseteq$)

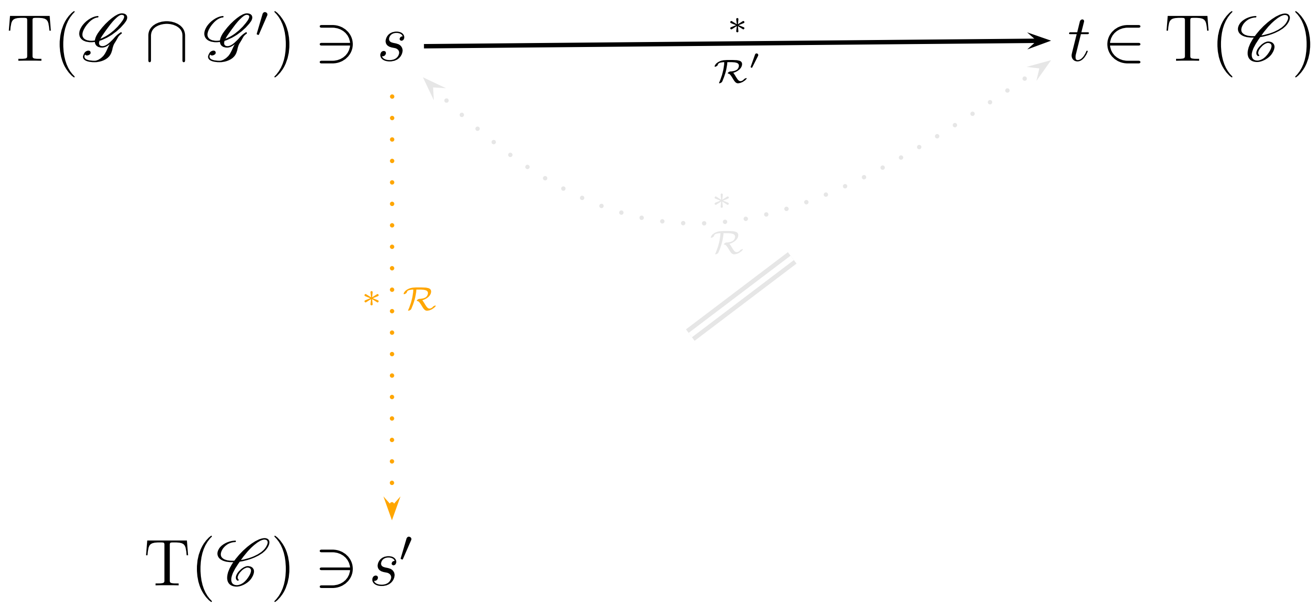
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ $(\supseteq)$

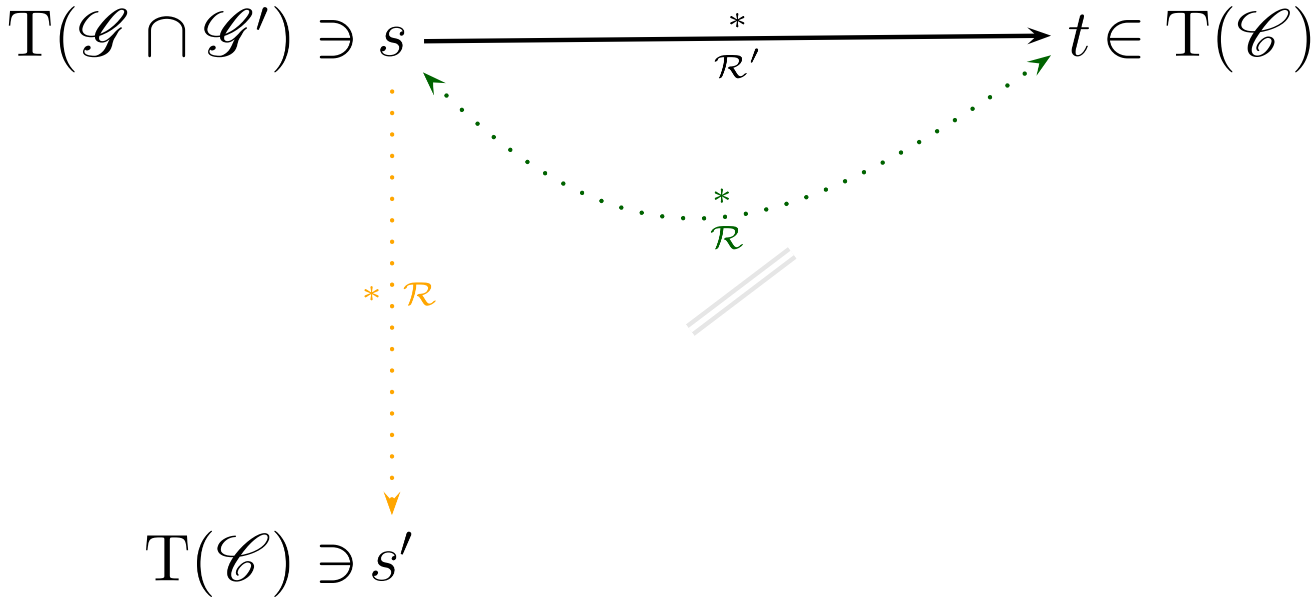
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$ ($\supseteq$)

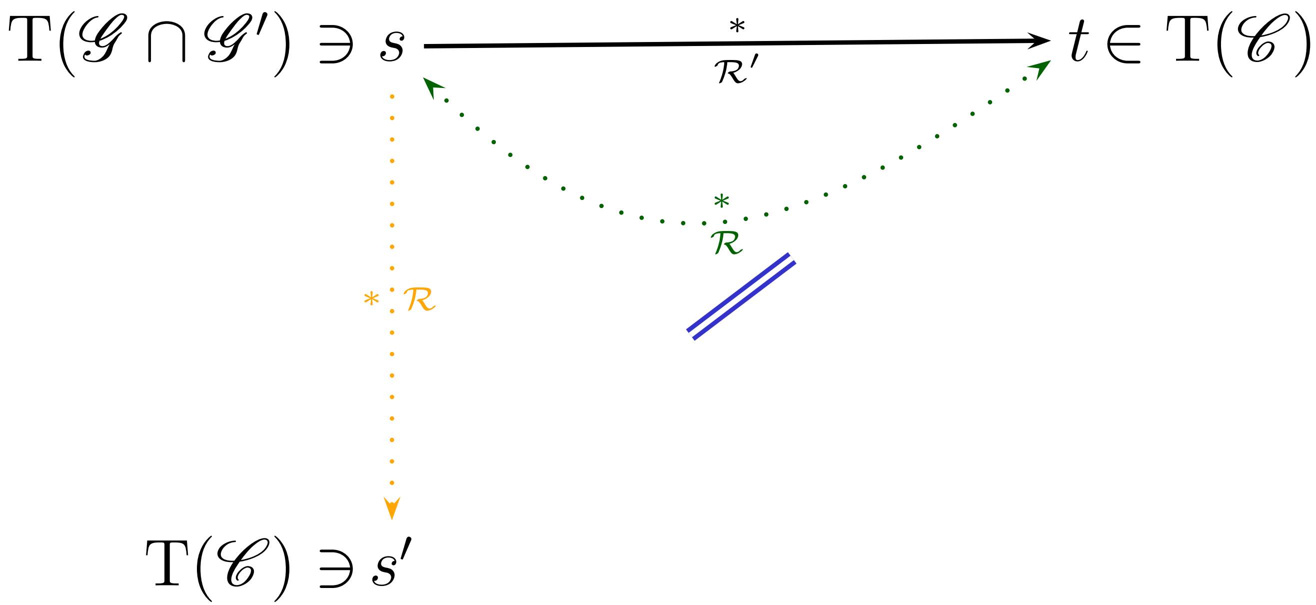
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ $(\supseteq)$

From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

Conjectures

$$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$$

1. $\mathrm{CR}(\mathcal{R})$ implies $\mathrm{CR}(\mathcal{R}_I)$
2. Let $\mathcal{R}$ and $\mathcal{R}_I$ be STTRSs over $\mathscr{G}$ and $\mathscr{F}$. $\forall s \in \mathrm{T}(\mathscr{F}).\exists t \in \mathrm{T}(\mathscr{G})$ s.t. $s \xrightarrow{*}_{\mathcal{R}_I} t$

Higher-order sufficient completeness

(Aoto, Yamada and Toyama, 2004)

Definition

An STTRS $\mathcal{R}$ is said to be *higher-order sufficiently complete* $\mathrm{HSC}(R)$ if for any basic simply typed term $s \in \mathrm{T}^{\mathrm{b}}(\mathscr{F}, \mathscr{V}^{\mathrm{h}})$ there exists $t \in \mathrm{T}^{\mathrm{b}}(\mathscr{F}, \mathscr{V}^{\mathrm{h}})$ such that $s \xrightarrow{*}_{\mathcal{R}} t$ and that either t has an expanded variable occurrence or t is a ground constructor term.

$$
\begin{array}{ll}
(\mathsf{foldr}\ f\ e\ [\,]) & \to e \\
(\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\
(+\ \mathsf{0}\ y) & \to y \\
(+\ \mathsf{s}(x)\ y) & \to (\mathsf{s}\ (+\ x\ y))
\end{array}
$$

$(\mathsf{foldr}\ f\ 0\ [1,2,3]) \to_{\mathcal{R}} (f\ 1\ (\mathsf{foldr}\ f\ 0\ [2,3]))$,

Higher-order sufficient completeness

(Aoto, Yamada and Toyama, 2004)

Definition

An STTRS $\mathcal{R}$ is said to be *higher-order sufficiently complete* $\mathrm{HSC}(R)$ if for any basic simply typed term $s \in \mathrm{T}^{\mathrm{b}}(\mathscr{F}, \mathscr{V}^{\mathrm{h}})$ there exists $t \in \mathrm{T}^{\mathrm{b}}(\mathscr{F}, \mathscr{V}^{\mathrm{h}})$ such that $s \xrightarrow{*}_{\mathcal{R}} t$ and that either t has an expanded variable occurrence or t is a ground constructor term.

$$
\begin{array}{ll}
(\mathsf{foldr}\ f\ e\ [\,]) & \to e \\
(\mathsf{foldr}\ f\ e\ (:\ x\ xs)) & \to (f\ x\ (\mathsf{foldr}\ f\ e\ xs)) \\
(+\ \mathsf{0}\ y) & \to y \\
(+\ \mathsf{s}(x)\ y) & \to (\mathsf{s}\ (+\ x\ y))
\end{array}
$$

$(\mathsf{foldr}\ f\ 0\ [1,2,3]) \to_{\mathcal{R}} (f\ 1\ (\mathsf{foldr}\ f\ 0\ [2,3]))$,
$(\mathsf{foldr}\ +\ 0\ [1,2,3]) \to_{\mathcal{R}} (+\ 1\ (\mathsf{foldr}\ +\ 0\ [2,3]))$

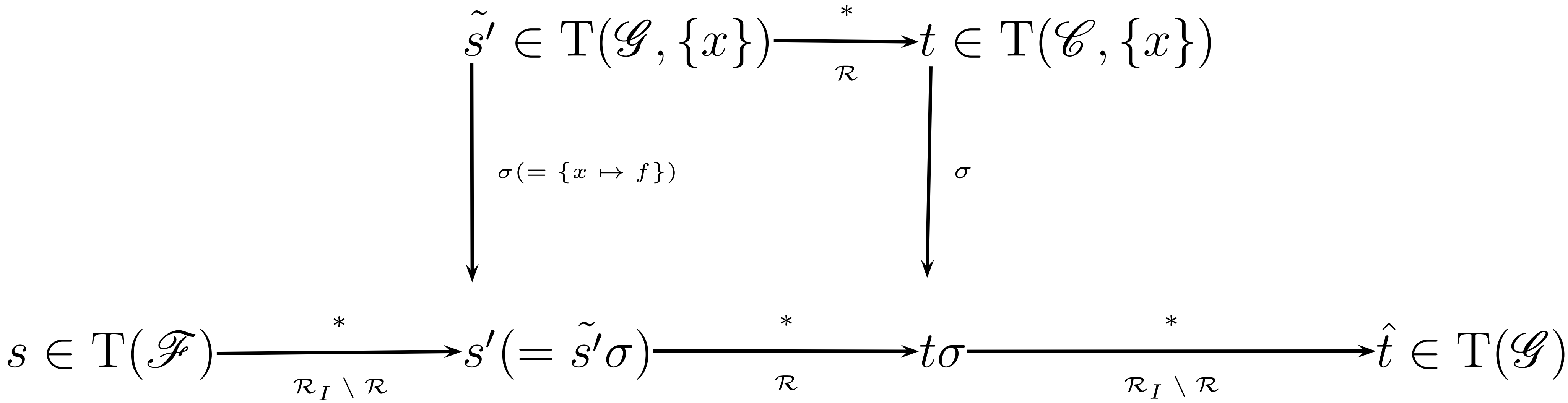
Conjectures

$$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$$

Let $\mathcal{R}$ and $\mathcal{R}_I$ be STTRSs over $\mathscr{G}$ and $\mathscr{F}$. $\forall s \in \mathrm{T}(\mathscr{F}).\exists t \in \mathrm{T}(\mathscr{G})$ s.t. $s \xrightarrow{*}_{\mathcal{R}_I} t$

Proof (sketch)

$\mathrm{HSC}(\mathcal{R})$ and $\mathrm{WN}(\mathcal{R})$



Conclusion

- Equivalent transformation for STTRSs
- Verification of the equality of STTRSs

Conclusion

- Transformation templates
- Examples