A Proof Order for Decreasing Diagrams

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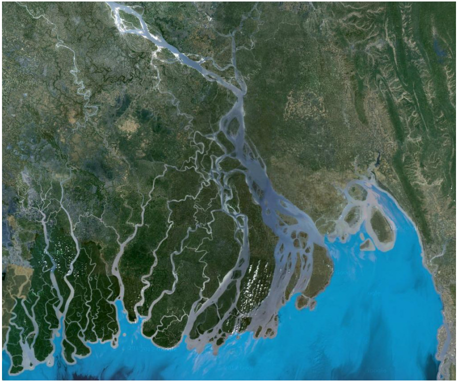
About



[Google Maps]

- abstract rewriting
- decreasing diagrams without diagrams

Contents

- Decreasing Diagrams
- Proof Order
- Confluence (modulo)

Notation

Fix

- $\mathcal{L}$ set of labels
- $\succ$ well-founded order on $\mathcal{L}$

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- $\underset{S}{\to} = \bigcup_{\alpha \in S} (\underset{\alpha}{\to})$

Decreasing Diagrams

Theorem (van Oostrom 2008)

Let $(\underset{\alpha}{\rightarrow})_{\alpha \in \mathcal{L}}$ *be a family of ARSs satisfying*

$$\underset{\alpha}{\leftarrow} \cdot \underset{\gamma}{\rightarrow} \subseteq \underset{\curlyvee\alpha}{\overset{*}{\longleftrightarrow}} \cdot \underset{\gamma}{\overset{=}{\rightarrow}} \cdot \underset{\curlyvee\alpha,\gamma}{\overset{*}{\longleftrightarrow}} \cdot \underset{\alpha}{\overset{=}{\leftarrow}} \cdot \underset{\curlyvee\gamma}{\overset{*}{\longleftrightarrow}}$$

for all $\alpha, \gamma \in \mathcal{L}$*. Then* $\underset{\mathcal{L}}{\rightarrow}$ *is confluent.*

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  - (rectangular) diagram pasting

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$\Rightarrow$ devise proof order compatible with decreasing diagrams

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Further Notation

Abstract proof steps

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- multisets $\{\!\{ \dots \}\!\}$

Proof String Orders

Definition

Proof strings are strings over $\overleftrightarrow{\mathcal{L}}$. Concatenation is $\cdot$ and ϵ is the empty string. Inversion is defined by $\epsilon^{-1} = \epsilon$ and $(\overleftrightarrow{\alpha} \cdot P)^{-1} = P^{-1} \cdot (\overleftrightarrow{\alpha})^{-1}$.

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An order on proof strings $>$ is a *proof string order* if

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- proof strings are *abstract* abstract rewrite proofs
- proof string orders are *abstract* proof orders (Bachmair, Dershowitz 1994)

Measuring Proof Strings

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We define $[\cdot]^m$ measuring proof strings by

$$
\begin{aligned}
[P]^m = {} & \{\!|(\overleftarrow{\alpha}, [R]^m) \mid P = Q \cdot \overleftarrow{\alpha} \cdot R|\!\} \\
& \cup \{\!|(\overline{\alpha}, \varnothing) \mid P = Q \cdot \overline{\alpha} \cdot R|\!\} \\
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Example

$$[\epsilon]^m$$

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Comparing Proof Strings

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We define $\succ_\oplus$ inductively by $s \succ_\oplus \{\!\{(\overleftrightarrow{\gamma_1}, t_1), \ldots, (\overleftrightarrow{\gamma_m}, t_m)\}\!\}$ iff

- $s \succ_\oplus t_i$ for $1 \leq i \leq m$
- $s \gg_{\mathsf{mul}} t$ where $\gg = \succ \times_{\mathsf{lex}} \succ_\oplus$

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- it is redundant for $\succ_\bullet$

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$\succ_\bullet$ *is a proof string order.*

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$$\begin{aligned}[\overleftarrow{\alpha} \cdot P]^m = \{\!\!\{(\overleftarrow{\alpha}, [P]^m)\}\!\!\} \\ \cup \{\!\!\{(\overleftarrow{\gamma}, [P']^m) \mid (\overleftarrow{\gamma}, [P']) \in [P]^m\}\!\!\} \\ \cup \{\!\!\{(\overline{\gamma}, \varnothing) \mid (\overline{\gamma}, \varnothing) \in [P]^m\}\!\!\} \\ \cup \{\!\!\{(\overrightarrow{\gamma}, [\overleftarrow{\alpha} \cdot P']^m) \mid (\overrightarrow{\gamma}, [P']^m) \in [P]^m\}\!\!\}\end{aligned}$$

(similar for $\overline{\alpha}$, $\overrightarrow{\alpha}$)

- comparison $[P]^m \succ_\oplus [R]^m$ carries over to $[\overleftarrow{\alpha} \cdot P]^m \succ_\oplus [\overleftarrow{\alpha} \cdot R]^m$.
- (strict) monotonicity of $\cdot$ wrt. $\succ_\bullet$ follows

Progress

- Decreasing Diagrams
- Proof Order
- Confluence (modulo)

Confluence by Decreasing Diagrams

Lemma

1. $\overleftarrow{\alpha} \succ_\bullet Q$ *for* $Q \in (\overleftrightarrow{\curlyvee \alpha})^*$
2. $\overleftarrow{\alpha} \cdot \overrightarrow{\gamma} \succ_\bullet Q$ *for* $Q \in (\overleftrightarrow{\curlyvee \alpha})^* \cdot (\overrightarrow{\gamma})^? \cdot (\overleftrightarrow{\curlyvee \alpha, \gamma})^* \cdot (\overleftarrow{\alpha})^? \cdot (\overleftrightarrow{\curlyvee \gamma})^*$

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Corollary

Let $(\underset{\alpha}{\to})_{\alpha\in\mathcal{L}}$ *be a family of ARSs satisfying*

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for all $\alpha, \gamma \in \mathcal{L}$. *Then* $\underset{\mathcal{L}}{\to}$ *is confluent.*

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- analysis of normal forms
- if $\underset{\alpha}{\rightarrow} \subseteq \overset{*}{\underset{\curlyvee\alpha}{\longleftrightarrow}}$ then local peaks involving $\underset{\alpha}{\rightarrow}$ can be excluded

Confluence Modulo

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Confluence Modulo

Lemma

1. $\overleftarrow{\alpha} \succ_\bullet Q$ *if* $Q \in (\overleftrightarrow{\curlyvee\alpha})^*$
2. $\overleftarrow{\alpha} \cdot \overrightarrow{\gamma} \succ_\bullet Q$ *if* $Q \in (\overleftrightarrow{\curlyvee\alpha})^* \cdot (\overrightarrow{\gamma})^? \cdot (\overleftrightarrow{\curlyvee\alpha,\gamma})^* \cdot (\overleftarrow{\alpha})^? \cdot (\overleftrightarrow{\curlyvee\gamma})^*$
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Example

If $\alpha \succ \eta$ and $\gamma \succ \eta$ then $\overline{\alpha} \cdot \overrightarrow{\gamma} \succ_\bullet \overrightarrow{\eta} \cdot \overrightarrow{\gamma} \cdot \overline{\alpha}$:

$$[\overline{\alpha} \cdot \overrightarrow{\gamma}]^m = \{\!\{(\overline{\alpha}, \varnothing), (\overrightarrow{\gamma}, \{\!\{(\overline{\alpha}, \varnothing)\}\!\})\}\!\}$$
$$[\overrightarrow{\eta} \cdot \overrightarrow{\gamma} \cdot \overline{\alpha}]^m = \{\!\{(\overline{\alpha}, \varnothing), (\overrightarrow{\eta}, \varnothing), (\overrightarrow{\gamma}, \{\!\{(\overrightarrow{\eta}, \varnothing)\}\!\})\}\!\}$$
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$$[\overline{\alpha} \cdot \overrightarrow{\gamma}]^m \succ_\oplus [\overrightarrow{\eta} \cdot \overrightarrow{\gamma} \cdot \overline{\alpha}]^m$$

Main Result

Theorem

Let $(\underset{\alpha}{\rightarrow})_{\alpha\in\mathcal{L}}$ and $(\underset{\alpha}{\vdash\!\dashv})_{\alpha\in\mathcal{L}}$ be families of ARSs, where $\underset{\alpha}{\vdash\!\dashv}$ is symmetric. Define $\underset{\alpha}{\Leftrightarrow} = \underset{\alpha}{\leftrightarrow} \cup \underset{\alpha}{\vdash\!\dashv}$. If

$$\underset{\alpha}{\leftarrow} \cdot \underset{\gamma}{\rightarrow} \subseteq \overset{*}{\underset{\curlyvee\alpha}{\Longleftrightarrow}} \cdot \overset{=}{\underset{\gamma}{\longrightarrow}} \cdot \overset{*}{\underset{\curlyvee\alpha,\gamma}{\Longleftrightarrow}} \cdot \overset{=}{\underset{\alpha}{\longleftarrow}} \cdot \overset{*}{\underset{\curlyvee\gamma}{\Longleftrightarrow}}$$

and

$$\underset{\alpha}{\vdash\!\dashv} \cdot \underset{\gamma}{\rightarrow} \subseteq \left(\overset{*}{\underset{\curlyvee\alpha\cap\curlyvee\gamma}{\Longleftrightarrow}} \cdot \overset{=}{\underset{\gamma}{\longrightarrow}} \cdot \overset{*}{\underset{\curlyvee\gamma}{\Longleftrightarrow}} \cdot \underset{\alpha}{\vdash\!\dashv} \cdot \overset{*}{\underset{\curlyvee\gamma}{\Longleftrightarrow}} \right) \cup \left(\overset{*}{\underset{\curlyvee\alpha}{\Longleftrightarrow}} \cdot \overset{=}{\underset{\gamma}{\longrightarrow}} \cdot \overset{*}{\underset{\curlyvee\alpha,\gamma}{\Longleftrightarrow}} \right)$$

for all $\alpha, \gamma \in \mathcal{L}$, then $\underset{\mathcal{L}}{\rightarrow}$ is Church-Rosser modulo $\overset{*}{\underset{\mathcal{L}}{\vdash\!\dashv}}$, i.e.,

$$(\leftrightarrow \cup \vdash\!\dashv)^* \subseteq \overset{*}{\rightarrow} \cdot \overset{*}{\vdash\!\dashv} \cdot \overset{*}{\leftarrow}$$

Corollary (Ohlebusch 1998)

Let $(\underset{\alpha}{\to})_{\alpha \in \mathcal{L}}$ *be a family of ARSs and* $\vdash\!\dashv$ *be a symmetric relation.*

Then $\underset{\mathcal{L}}{\to}$ *is Church-Rosser modulo* $\overset{*}{\vdash\!\dashv}$*, if for all* $\alpha, \gamma \in \mathcal{L}$

$$\underset{\alpha}{\leftarrow} \cdot \underset{\gamma}{\to} \subseteq \xrightarrow[\curlyvee\alpha]{*} \cdot \xrightarrow[\gamma]{=} \cdot \xrightarrow[\curlyvee\alpha,\gamma]{*} \cdot \overset{*}{\vdash\!\dashv} \cdot \xleftarrow[\curlyvee\alpha,\gamma]{*} \cdot \xleftarrow[\alpha]{=} \cdot \xleftarrow[\curlyvee\gamma]{*}$$

and

$$\vdash\!\dashv \cdot \underset{\alpha}{\to} \subseteq \xrightarrow[\curlyvee\alpha]{*} \cdot \overset{*}{\vdash\!\dashv} \cdot \xleftarrow[\curlyvee\alpha]{*} \cdot \xleftarrow[\alpha]{=}.$$

Corollary (Ohlebusch 1998)

Let $(\underset{\alpha}{\to})_{\alpha\in\mathcal{L}}$ *be a family of ARSs and* $\vdash\!\dashv$ *be a symmetric relation.*

Then $\underset{\mathcal{L}}{\to}$ *is Church-Rosser modulo* $\overset{*}{\vdash\!\dashv}$*, if for all* $\alpha, \gamma \in \mathcal{L}$

$$\underset{\alpha}{\leftarrow} \cdot \underset{\gamma}{\to} \subseteq \xrightarrow[\curlyvee\alpha]{*} \cdot \xrightarrow[\gamma]{=} \cdot \xrightarrow[\curlyvee\alpha,\gamma]{*} \cdot \overset{*}{\vdash\!\dashv} \cdot \xleftarrow[\curlyvee\alpha,\gamma]{*} \cdot \xleftarrow[\alpha]{=} \cdot \xleftarrow[\curlyvee\gamma]{*}$$

and

$$\vdash\!\dashv \cdot \underset{\alpha}{\to} \subseteq \xrightarrow[\curlyvee\alpha]{*} \cdot \overset{*}{\vdash\!\dashv} \cdot \xleftarrow[\curlyvee\alpha]{*} \cdot \xleftarrow[\alpha]{=}.$$

Corollary (Aoto, Toyama 2011)

Let $(\underset{\alpha}{\to})_{\alpha\in\mathcal{L}}$ *and symmetric* $(\underset{\alpha}{\vdash\!\dashv})_{\alpha\in\mathcal{L}}$ *be given, and* $\underset{\alpha}{\Leftrightarrow} = \underset{\alpha}{\leftrightarrow} \cup \underset{\alpha}{\vdash\!\dashv}$.

Then $\underset{\mathcal{L}}{\to}$ *is Church-Rosser modulo* $\overset{*}{\underset{\mathcal{L}}{\vdash\!\dashv}}$*, if for all* $\alpha, \gamma \in \mathcal{L}$*,*

$$\underset{\alpha}{\leftarrow} \cdot \underset{\gamma}{\to} \subseteq \underset{\curlyvee\alpha,\gamma}{\Longleftrightarrow} \quad \textit{and} \quad \underset{\alpha}{\vdash\!\dashv} \cdot \underset{\gamma}{\to} \subseteq \underset{\curlyvee\alpha,\gamma}{\Longleftrightarrow}.$$

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Future

- commutation modulo (cf. van Oostrom 2008)

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- commutation modulo (cf. van Oostrom 2008)
- concrete criteria for TRSs

Summary

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Thank you!

Bonus

Example

Let the linear TRS $\mathcal{R}$ be given by

$$\begin{array}{llll} \mathsf{f}(\mathsf{u}(\mathsf{O}),\mathsf{u}(y)) \to \mathsf{A} & \mathsf{O} \to \mathsf{u}(\mathsf{O}) & \mathsf{u}(x) \to x & \mathsf{f}(x,y) \to \mathsf{f}(x,\mathsf{u}(y)) \\ \mathsf{f}(\mathsf{v}(x),\mathsf{v}(\mathsf{O})) \to \mathsf{B} & \mathsf{O} \to \mathsf{v}(\mathsf{O}) & \mathsf{v}(x) \to x & \mathsf{f}(x,y) \to \mathsf{f}(\mathsf{v}(x),y) \end{array}$$

$\mathcal{R}$ is not confluent:

$$\mathsf{A} \xleftarrow[\mathcal{R}]{*} \mathsf{f}(\mathsf{O},\mathsf{O}) \xrightarrow[\mathcal{R}]{*} \mathsf{B}$$

$\mathcal{R}$ has deeply joinable critical peaks:

- $\xleftarrow{+} \mathsf{f}(\mathsf{u}(\mathsf{O}),\mathsf{u}(y)) \xrightarrow{+}$ reach A
- $\xleftarrow{+} \mathsf{f}(\mathsf{v}(x),\mathsf{v}(\mathsf{O})) \xrightarrow{+}$ reach B
- $\xleftarrow{+} \mathsf{f}(x,y) \xrightarrow{+}$ reach $\mathsf{f}(x,y)$
- $\xleftarrow{+} \mathsf{O} \xrightarrow{+}$ reach O