SMT-Based Techniques in Automated Termination Analysis

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Termination analysis, classically

Turing 1949

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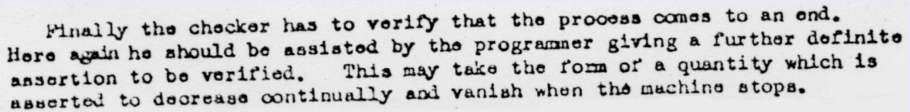
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Example (Termination can be simple)

while x $>$ 0:
    x $=$ x $-$ 1

Termination analysis, in the era of automation

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In practice:

- Encode only a proof **step** at a time
  $\rightarrow$ try to prove only **part** of the program terminating
- **Repeat** until the whole program is proved terminating

The rest of this talk

Termination proving in two parallel worlds

1. Term Rewrite Systems (TRSs)
2. Imperative Programs

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Syntactic approach for reasoning in equational first-order logic

Core functional programming language without many restrictions (and features) of "real" FP:

- first-order (usually)
- no fixed evaluation strategy
- no fixed order of rules to apply (Haskell: top to bottom)
- untyped
- no pre-defined data structures (integers, arrays, ...)

Example (Division)

$$\mathcal{R} = \begin{cases} \mathsf{minus}(x, \mathsf{0}) & \to & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\ \mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \to & \mathsf{0} \\ \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \end{cases}$$

Term rewriting: Evaluate terms by applying rules from $\mathcal{R}$

$$\mathsf{minus}(\mathsf{s}(\mathsf{s}(\mathsf{0})), \mathsf{s}(\mathsf{0})) \to_{\mathcal{R}} \mathsf{minus}(\mathsf{s}(\mathsf{0}), \mathsf{0}) \to_{\mathcal{R}} \mathsf{s}(\mathsf{0})$$

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Termination: No infinite evaluation sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \ldots$

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Show termination using Dependency Pairs

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- Find $\succ$ automatically and efficiently

Polynomial interpretations

Get $\succ$ via polynomial interpretations $[\,\cdot\,]$ over $\mathbb{N}$ [Lankford '75]
$\rightarrow$ ranking functions for rewriting

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$$\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \ \succsim\ \mathsf{minus}(x, y)$$

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$$\forall x, y. \quad x + 1 \;=\; [\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))] \;\geq\; [\mathsf{minus}(x, y)] \;=\; x$$

Use $[\,\cdot\,]$ with

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Extend to terms:

- $[x] = x$
- $[\,f(t_1, \dots, t_n)] = [\,f]([t_1], \dots, [t_n])$

$\succ$ boils down to $>$ over $\mathbb{N}$

Example (Constraints for Division)

$$\mathcal{R} \;=\; \left\{ \begin{array}{rcl} \mathsf{minus}(x, \mathsf{0}) & \succsim & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \succsim & \mathsf{minus}(x, y) \\ \mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \succsim & \mathsf{0} \\ \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \succsim & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \end{array} \right.$$

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- Abstraction: $[\mathsf{0}]$ and $[\mathsf{s}]$

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Automation

Task: Solve $\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \succsim \mathsf{minus}(x, y)$

Automation

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Task: Show satisfiability of non-linear constraints over $\mathbb{N}$

$\curvearrowright$ Prove termination of given term rewrite system

Extensions

- Polynomials with negative coefficients and max-operator
  [Hirokawa, Middeldorp, *IC '07*; Fuhs et al, *SAT '07, RTA '08*]
  - can model behavior of functions more closely:
    [pred]$(x_1) = \max(x_1 - 1, 0)$
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6. Enjoy!

Further extensions

- Constrained term rewriting [Fuhs et al, *RTA '09*; Kop, Nishida, *FroCoS '13*; Rocha, Meseguer, Muñoz, *WRLA '14*; ...]
  - term rewriting with predefined operations from SMT theories, e.g. integer arithmetic, ...
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- Complexity analysis
  [Hirokawa, Moser, *IJCAR '08*; Noschinski, Emmes, Giesl, *JAR '13*; ...]
  Can re-use termination machinery to infer and prove statements like "runtime complexity of this TRS is in $\mathcal{O}(n^3)$"

SMT solvers *from* termination analysis

Annual SMT-COMP, division QF_NIA (Quantifier-Free Non-linear Integer Arithmetic)

Year	Winner
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(disclaimer: Z3 participated only *hors concours* in the last years)

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Example (Imperative program)

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\begin{array}{l}
\textbf{if } x \geq 0\text{:} \\
\quad \textbf{while } x \neq 0\text{:} \\
\quad\quad x = x - 1
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Does this program terminate?

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```
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More: [Podelski, Rybalchenko, *VMCAI '04*, Alias et al, *SAS '10*]

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Nowadays all SMT-based!

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- Beyond sequential programs on integers:
  - structs/classes [Berdine et al, *CAV '06*; Otto et al, *RTA '10*; ...]
  - arrays (pointer arithmetic) [Ströder et al, *IJCAR '14*, ...]
  - multi-threaded programs [Cook et al, *PLDI '07*, ...]
  - IEEE floating-point numbers [Maurica, Mesnard, Payet, *SAC '16*]
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