The Generalized Subterm Criterion in $\mathsf{T_TT_2}$[*]

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Overview

- The Subterm Criterion
- A Generalization
- Implementation
- Experiments

The Subterm Criterion

Example - Addition

(recursive) function

$$\mathsf{add}(0, y) \to y$$

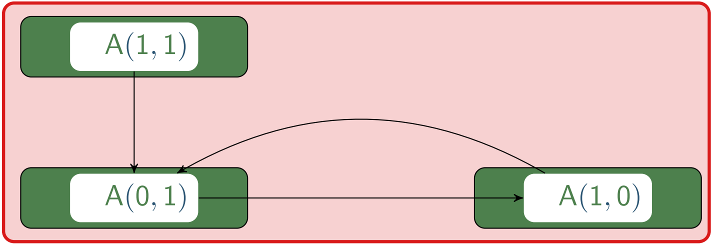
$$\mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y)) \qquad \mathsf{A}(\mathsf{s}(x), y) \to \mathsf{A}(x, y)$$

$$\mathsf{add}(x, \mathsf{s}(y)) \to \mathsf{add}(\mathsf{s}(x), y)) \qquad \mathsf{A}(x, \mathsf{s}(y)) \to \mathsf{A}(\mathsf{s}(x), y)$$

$$\mathsf{add}(\mathsf{s}(x), y) \to \mathsf{add}(x, \mathsf{s}(y)) \qquad \mathsf{A}(\mathsf{s}(x), y) \to \mathsf{A}(x, \mathsf{s}(y))$$

write n for natural number $\mathsf{s}^n(0) = \underbrace{\mathsf{s}(\mathsf{s}(\cdots \mathsf{s}}_{n \text{ times}}(0) \cdots))$

minimal infinite chain



How to avoid Infinite Minimal Chains?

Problem

- input: dependency pair problem $p = (\mathcal{P}, \mathcal{R})$ (2 TRSs)
- question: does p exclude infinite minimal chains?
- output: Yes (finite), No (infinite), Maybe

Subterm Criterion

Definition

- simple projection $\pi : \mathcal{F} \to \mathbb{N}$
- $\pi(f(t_1, \ldots, t_n)) = t_i$ if $\pi(f) = i$
- $s \trianglerighteq^{\pi} t$ abbreviates $\pi(s) \trianglerighteq \pi(t)$

Theorem (Hirokawa & Middeldorp)
Let π be a simple projection such that $\mathcal{P} \subseteq \trianglerighteq^{\pi}$.
Then $(\mathcal{P}, \mathcal{R})$ is finite iff $(\mathcal{P} \setminus \triangleright^{\pi}, \mathcal{R})$ is. □

Example - Addition

$\mathcal{P}$:

$$\mathsf{A}(\mathsf{s}(x), y) \to \mathsf{A}(x, y)$$

$\pi(\mathsf{A}) = 1$

$$\pi(\mathsf{A}(\mathsf{s}(x), y)) = \mathsf{s}(x) \rhd x = \pi(\mathsf{A}(x, y))$$

Examples

- `AG01/#3.56`

$$\mathsf{G}(\mathsf{c}(x, \mathsf{s}(y))) \to \mathsf{G}(\mathsf{c}(\mathsf{s}(x), y))$$

  but $\mathsf{c}(x, \mathsf{s}(y)) \not\trianglerighteq \mathsf{c}(\mathsf{s}(x), y)$

- `AG01/#3.1`

$$\mathsf{Q}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{Q}(\mathsf{minus}(x, y), \mathsf{s}(y))$$

  but neither $\mathsf{s}(x) \trianglerighteq \mathsf{minus}(x, y)$ nor $\mathsf{s}(y) \triangleright \mathsf{s}(y)$

- `AG01/#3.29`

$$\mathsf{F}(\mathsf{s}(x), y, y) \to \mathsf{F}(y, x, \mathsf{s}(x))$$

  but $\mathsf{s}(x) \not\trianglerighteq y$ and $y \not\trianglerighteq x$ and $y \not\trianglerighteq \mathsf{s}(x)$

- `AG01/#3.26`

$$\mathsf{F}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{F}(\mathsf{f}(x))$$

  but $\mathsf{s}(\mathsf{s}(x)) \not\trianglerighteq \mathsf{f}(x)$

A Generalization

A Generalized Subterm Criterion

Definition

multisets of natural numbers

- multiprojection $\pi : \mathcal{F} \to \mathcal{M}(\mathbb{N})$
- $$\pi(t) = \begin{cases} \pi(t_{i_1}) + \cdots + \pi(t_{i_k}) & \text{if } t = f(t_1, \ldots, t_n) \\ & \text{and } \pi(f) = \{i_1, \ldots, i_k\} \neq \varnothing, \\ \{t\} & \text{otherwise} \end{cases}$$
- $s \rhd^\pi t$ abbreviates $\pi(s) \rhd_{\mathsf{mul}} \pi(t)$, and $s \unrhd^\pi t$ iff $s \rhd^\pi t$ or $\pi(s) = \pi(t)$
- recall: $M \succ_{\mathsf{mul}} N$ iff $M = X + Z$, $N = Y + Z$, $\forall y \in Y.\, \exists x \in X.\, x \succ y$ for some $X \neq \varnothing$, Y, and Z

Theorem (Yamada, Sternagel, Thiemann & Kusakari)
Let π be a multiprojection such that $\mathcal{P} \subseteq \unrhd^\pi$ and $f(\ldots) \unrhd^\pi r$ for all $f(\ldots) \to r \in \mathcal{R}$ with $\pi(f) \neq \varnothing$.
Then $(\mathcal{P}, \mathcal{R})$ is finite iff $(\mathcal{P} \setminus \rhd^\pi, \mathcal{R})$ is. □

Examples (cont'd)

- `AG01/#3.56`
  $$\mathsf{G}(\mathsf{c}(x, \mathsf{s}(y))) \to \mathsf{G}(\mathsf{c}(\mathsf{s}(x), y))$$
  take $\pi(\mathsf{G}) = \{1\}$, $\pi(\mathsf{c}) = \{2\}$, obtain $\{\mathsf{s}(y)\} \rhd_{\mathsf{mul}} \{y\}$
- `AG01/#3.1`
  $$\mathsf{Q}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{Q}(\mathsf{minus}(x, y), \mathsf{s}(y))$$
  take $\pi(\mathsf{Q}) = \{1\}$, $\pi(\mathsf{minus}) = \{1\}$, obtain $\{\mathsf{s}(x)\} \rhd_{\mathsf{mul}} \{x\}$
- `AG01/#3.29`
  $$\mathsf{F}(\mathsf{s}(x), y, y) \to \mathsf{F}(y, x, \mathsf{s}(x))$$
  take $\pi(\mathsf{F}) = \{1, 2, 2\}$, $\pi(\mathsf{s}) = \{1, 1, 1\}$, obtain $\{x, x, x, y, y\} \rhd_{\mathsf{mul}} \{y, x, x\}$
- `AG01/#3.26`
  $$\mathsf{F}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{F}(\mathsf{f}(x))$$
  take $\pi(\mathsf{F}) = \pi(\mathsf{f}) = \pi(\mathsf{s}) = \{1, 1\}$, obtain $\{x, x, x, x, x, x, x, x\} \rhd_{\mathsf{mul}} \{x, x, x, x\}$

Implementation

Approach - SMT Encoding

Issues

- how to encode multiprojections, and thereby multisets $\pi(t)$
- how to encode $\rhd_{\mathsf{mul}}$-comparison between encoded multisets

Observation

- $\pi(t)$ contains only subterms of t, written $\mathcal{S}\mathsf{ub}(t)$
- may restrict to finite domain $\mathcal{S}\mathsf{ub}(s) \cup \mathcal{S}\mathsf{ub}(t)$ when comparing multiset $\pi(s)$ to $\pi(t)$

Comparing Multisets over Finite Domains

Lemma: Let D be a finite set, and M and N multisets over elements from D. Then, for well-founded and transitive $\succ$, $M \succ_{\mathsf{mul}} N$ is equivalent to

$$(\forall d \in D.\, \mathsf{upper}(d) \longrightarrow M(d) \geq N(d)) \text{ and } M \neq N \qquad (\star)$$

where $M(x)$ denotes the multiplicity of x in M and $\mathsf{upper}(x)$ is defined by $\forall d \in D.\, d \succ x \longrightarrow M(d) = N(d)$. □

Encoding Multiprojections

We encode the multiplicity of a term t in the multiset $\pi(s)$ by $\mathsf{M}_s(t) = \mathsf{MUL}(1, s, t)$.
$\mathsf{MUL}(w, s, t)$ is defined as follows

$$\begin{cases} \left(\bigwedge_{1 \leq i \leq n} \neg \mathsf{p}_f^i \right) ?\, w : 0 & \text{if } s = t = f(t_1, \ldots, t_n) \\ w & \text{if } s = t \text{ and } t \text{ is a variable} \\ \sum_{1 \leq i \leq n} (\mathsf{p}_f^i ?\, \mathsf{MUL}(w \cdot \mathsf{w}_f^i, s_i, t) : 0) & \text{if } t \lhd s = f(s_1, \ldots, s_n) \\ 0 & \text{otherwise} \end{cases}$$

where $b ?\, t : e$ denotes *if* b *then* t *else* e and the intended meaning of variables is that $\mathsf{p}_f^i = \top$ precisely when π *projects to the* i*-th argument of* f, in which case w_f^i gives the *weight of* i *in* $\pi(f)$.

Encoding GSC

$$(\forall s \to t \in \mathcal{P}.\, \mathsf{GEQ}(s,t)) \land (\exists s \to t \in \mathcal{P}.\, \mathsf{NEQ}(s,t)) \land$$
$$(\forall s \to t \in \mathcal{R}.\, \mathsf{RT}(s) \longrightarrow \mathsf{GEQ}(s,t)) \land (\forall f \in \mathcal{F}(\mathcal{P},\mathcal{R}).\, \mathsf{SAN}(f))$$

where

$$\mathsf{GEQ}(s,t) \text{ iff } \forall u \in \mathcal{S}\mathsf{ub}(s,t).\, \mathsf{UPPER}(u) \longrightarrow \mathsf{M}_s(u) \geq \mathsf{M}_t(u)$$
$$\mathsf{UPPER}(u) \text{ iff } \forall v \in \mathcal{S}\mathsf{ub}(s,t).\, v \rhd u \longrightarrow \mathsf{M}_s(v) = \mathsf{M}_t(v)$$
$$\mathsf{NEQ}(s,t) \text{ iff } \neg(\forall u \in \mathcal{S}\mathsf{ub}(s,t).\, \mathsf{M}_s(u) = \mathsf{M}_t(u))$$
$$\mathsf{RT}(f(s_1,\ldots,s_n)) \text{ iff } \bigvee_{1 \leq i \leq n} \mathsf{p}_f^i$$
$$\mathsf{SAN}(f) \text{ iff } \bigwedge_{1 \leq i \leq \mathsf{arity}(f)} \left(\mathsf{p}_f^i \longrightarrow \mathsf{w}_f^i > 0\right)$$

Experiments

Experiments using $\mathsf{T_TT_2}$

projections	Yes #	Yes (sec)	Maybe #	Maybe (sec)	Timeout #	Timeout (sec)	Total (sec)
simple	237	27.0	1213	230.5	48	240.0	497.6
recursive	273	32.5	1175	244.6	50	250.0	527.1
multi	332	58.6	1099	432.0	67	335.0	825.6
all	333	42.8	1112	309.5	53	265.0	617.3

Benchmark 1498 standard TRSs of TPDB 10.3

Timeout 5 seconds

Strategy DPs, DG decomposition, repeatedly apply SC

✓ +13 "Yes"s as part of $\mathsf{T_TT_2}$'s *competition strategy*

✗ no additional "Yes" with respect to other tools