Confluence of Non-Left-Linear TRSs via Relative Termination

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Confluence

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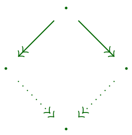
Definition

A relation $\to$ is confluent, if ${}^{*}\!\leftarrow \cdot \to^{*} \subseteq \to^{*} \cdot {}^{*}\!\leftarrow$



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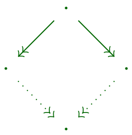
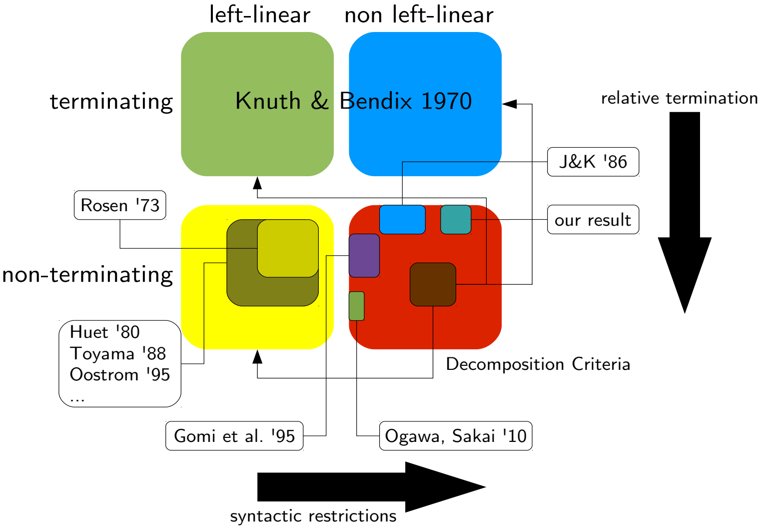


- often write $\downarrow$ ("joinable") instead of $\to^* \cdot {}^*\!\!\leftarrow$

Confluence Criteria



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overlap $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\sigma$ (of $\mathcal{R}_1$ on $\mathcal{R}_2$) is triple such that

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- $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$
- σ is mgu of ℓ_1 and $\ell_2|_p$
- if $p = \varepsilon$, then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ may not be variants of the same rule

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Example

- $\mathcal{R}_1 = \{\mathsf{f}(x, \mathsf{g}(x)) \to \mathsf{a}\}$ and $\mathcal{R}_2 = \{\mathsf{g}(x)) \to \mathsf{b}\}$ have
- overlap $(\mathsf{g}(x)) \to \mathsf{b}, 2, \mathsf{f}(y, \mathsf{g}(y)) \to \mathsf{a})_{\{y \to x\}}$

Strong Non-Overlappingness

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$\mathsf{SNO}(\mathcal{R}, \mathcal{S})$ iff $\hat{\mathcal{S}}$ and $\hat{\mathcal{R}}$ do not overlap each other,

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$\{3\}$-overlap of 1 and 2 at root:

$$(\mathsf{f}(x,x) \to \mathsf{a}, \varepsilon, \mathsf{f}(y,\mathsf{g}(y)) \to \mathsf{b})_{\{x,y \to \mathsf{c}\}}$$

with $\mathcal{S}$-critical pair (a, b).

New Confluence Criterion

Theorem

Suppose $\mathsf{SNO}(\mathcal{R}, \mathcal{S})$, *termination of* $\mathcal{R}/\mathcal{S}$, *and confluence of* $\mathcal{S}$.

$$\mathcal{R} \cup \mathcal{S} \textit{ confluent} \iff \mathsf{CP}_{\mathcal{S}}(\mathcal{R}) \subseteq {\downarrow_{\mathcal{R} \cup \mathcal{S}}}$$

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- confluence of $\mathcal{S}$ by orthogonality
- $\mathsf{CP}_{\mathcal{S}}(\mathcal{R}) = \varnothing$

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$\mathcal{S}$-critical pairs of $\mathcal{R}$:

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$$\mathsf{CP}_{\mathcal{S}}(\mathcal{R}) = \{(\mathsf{eq}(s, t, u), \mathsf{T}) \mid s, t, u \in \mathcal{T}(\mathcal{F}, \mathcal{V}),\ t \leftrightarrow_{\mathcal{S}}^{*} u\} \subseteq \downarrow_{\mathcal{R} \cup \mathcal{S}}$$

Equational Unifiability

Definition

- $\mathcal{E} = \{s_1 \approx t_1, \ldots, s_n \approx t_n\}$ is an $\mathcal{S}$-unification problem.
- $\mathcal{S}$-unifier of $\mathcal{E}$ is a substitution σ such that $\mathcal{E}\sigma \subseteq \leftrightarrow_{\mathcal{S}}^*$
- σ is more general than σ' on X if there exists τ with $x\sigma' \leftrightarrow_{\mathcal{S}}^* x\sigma\tau$ for all $x \in X$.

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Definition

Let $\mathcal{U}$ be set of $\mathcal{S}$-unifiers of $\mathcal{E}$

- $\mathcal{U}$ is complete if for every $\mathcal{S}$-unifier there exists more general one in $\mathcal{U}$
- if all unifiers in $\mathcal{U}$ are minimal, $\mathcal{U}$ is minimal complete
- σ is an $\mathcal{S}$-mgu of $\mathcal{E}$, if $\{\sigma\}$ is minimal complete set of unifiers for $\mathcal{E}$

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A term t is strongly $\mathcal{S}$-stable if for every position $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$ there are no term u and substitution σ such that $t|_p\sigma \to_{\mathcal{S}}^* \cdot \xrightarrow{\varepsilon}_{\mathcal{S}} u$.

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Lemma

Suppose $\mathsf{SNO}(\mathcal{R},\mathcal{S})$*. Then for all* $\ell \to r \in \mathcal{R}$*, the lhs* ℓ *is strongly* $\mathcal{S}$*-stable.*

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- if $\mathsf{SNO}(\mathcal{R}, \mathcal{S})$, every lhs of $\mathcal{R}$ is strongly $\mathcal{S}$-stable

Example

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- Hence $\mathsf{CP}_{\mathcal{S}}(\mathcal{R}) \subseteq {\downarrow_{\mathcal{R}\cup\mathcal{S}}}$

Experiments

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	ACP	ACP*	CSI	CSI*	Saigawa	Saigawa*
YES	12	**19**	7	**15**	0	**10**
NO	30	**4**	3	**3**	0	**2**
MAYBE	17	9	17	9	32	20
timeout (60 sec)	0	0	5	5	0	0

- 32 non-left-linear, non-terminating examples from Cops
- TTT2 for relative termination
- Timeout 60 sec. on Core Duo L7500 with 1.6 GHz

Example

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- no mgu exists for 2 and 6 and we can not exclude an $\mathcal{S}$-unifier

Comparison and Limitations

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Example

$$1\colon\ x + x \to x \qquad 2\colon\ x + y \to y + x \qquad 3\colon\ (x+y)+z \to x+(y+z)$$

- no partition with $\mathsf{SNO}(\mathcal{R}, \mathcal{S})$
- criterion by Jouannaud and Kirchner can handle it

Conclusion

New confluence criterion based on

- relative termination and
- joinability of $\mathcal{S}$-critical pairs

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Future work:

- relative termination used in criteria
  - by [Geser'90]
  - by [Hirokawa and Middeldorp,'10]
- strong-non-overlappingness used in criteria by [Gomi et al.,'96]
- $\mathcal{S}$-critical pairs in criteria by [Jouannaud and Kirchner,'86]

How to unify?