Lower Runtime Bounds for Integer Programs

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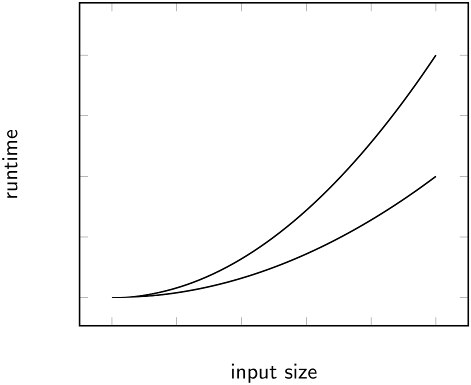
Lower Bounds?



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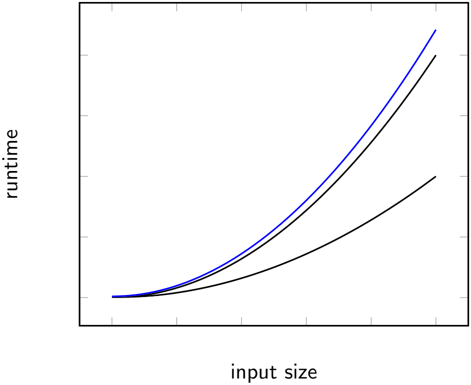

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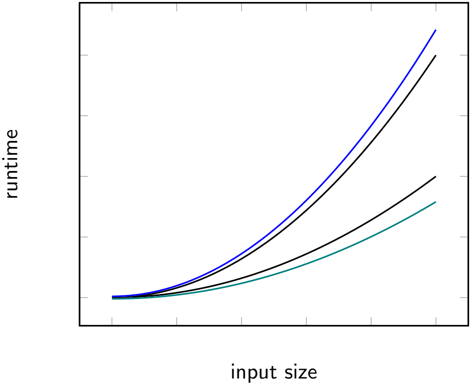

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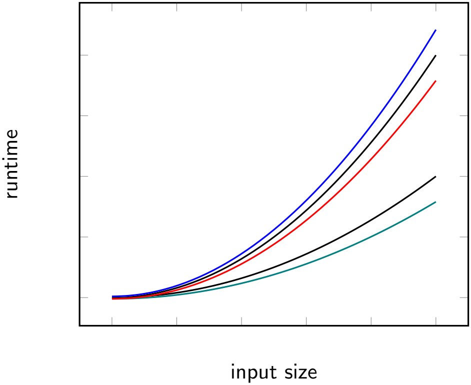

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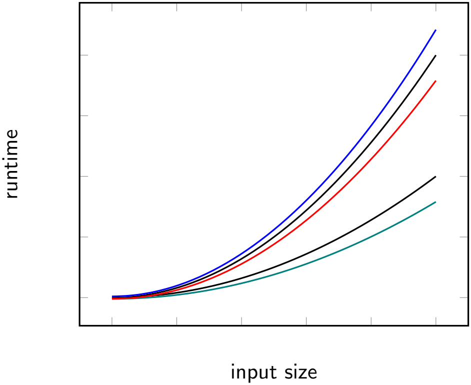

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Why?

- *tight* bounds

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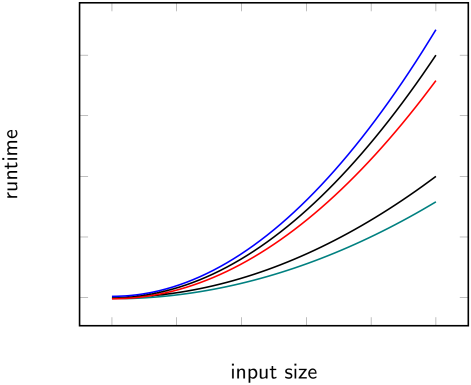

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Why?

- *tight* bounds
- identify attacks

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non-recursive integer programs:

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$z = y$
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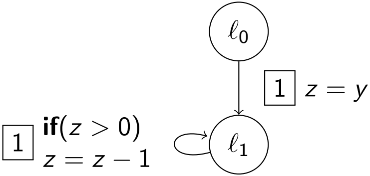

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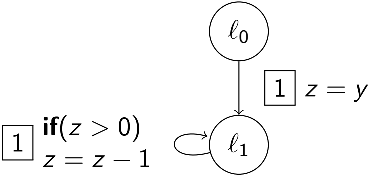

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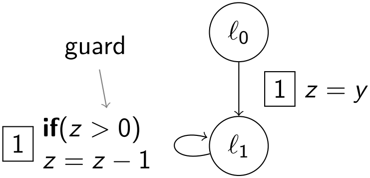

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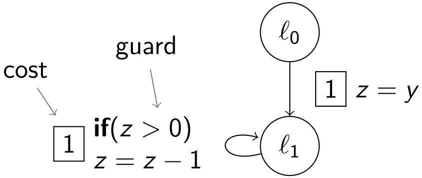

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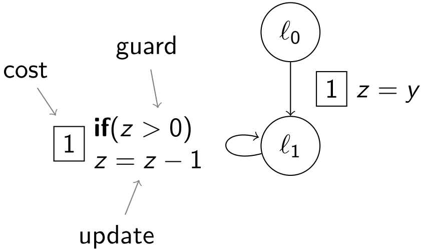

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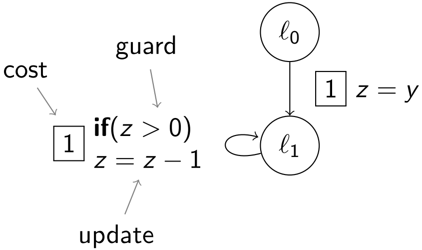

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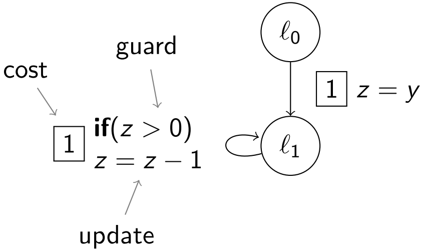

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- cost: can be specified in the input, nonnegative

The Technique

- **step 1**: underapproximating program simplification

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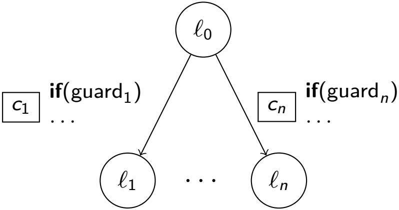

The Technique

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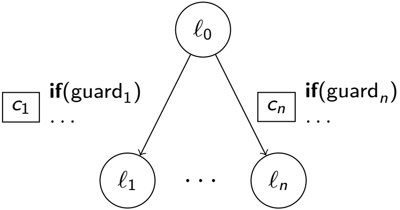


- **step 2**: infer asymptotic lower bound

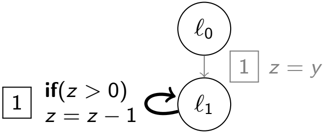
Program Simplification

Acceleration and *Chaining*

Program Simplification

Acceleration and *Chaining*

- **accelerate simple loops**



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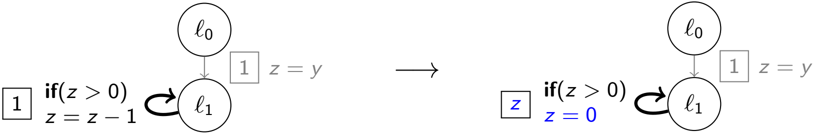

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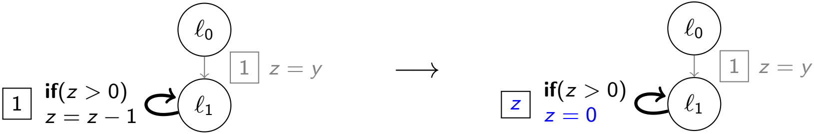

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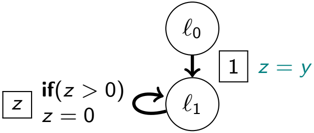

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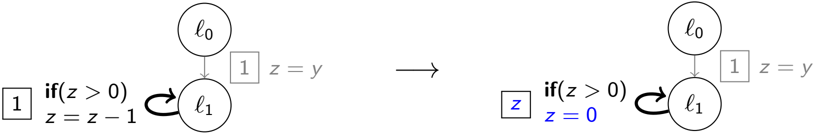

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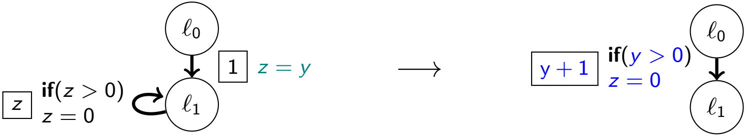

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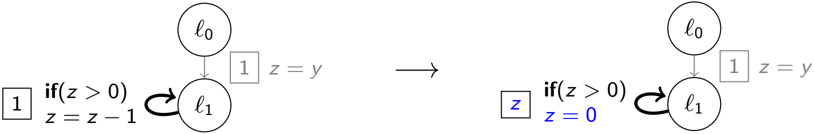

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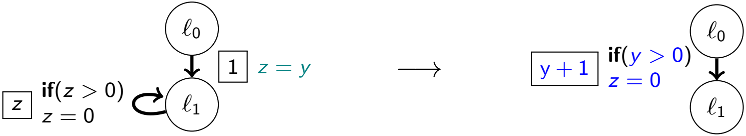
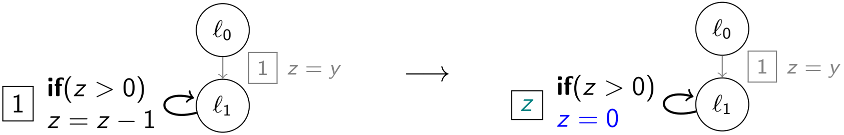


- **iterate**

Loop Acceleration: Challenges



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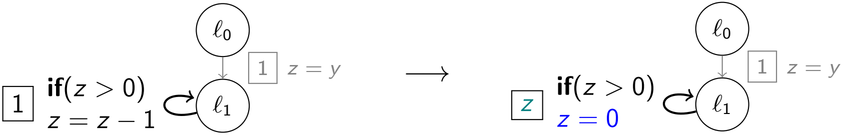

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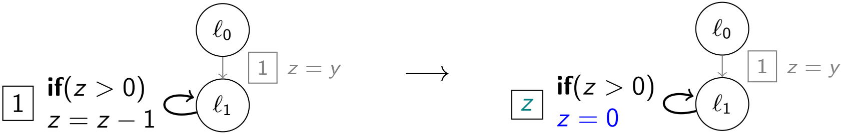

- What's the **result**?

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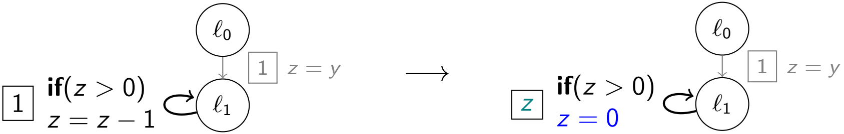

$\boxed{1}$ **if**$(z > 0)$ $z = z - 1$ ℓ_0 ℓ_1 $\boxed{1}$ $z = y$

$\longrightarrow$

$\boxed{z}$ **if**$(z > 0)$ $z = 0$ ℓ_0 ℓ_1 $\boxed{1}$ $z = y$

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- How many **repetitions**?

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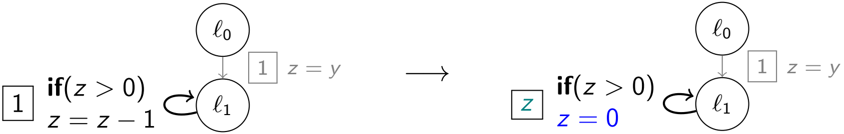

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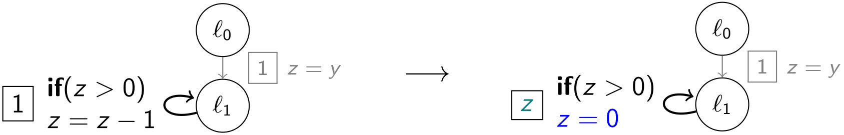

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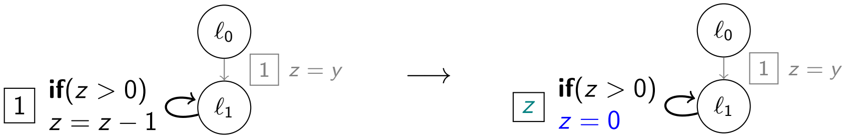

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  - $z^{(1)} = z - 1$ and $z^{(n+1)} = z^{(n)} - 1 \quad \curvearrowright \quad z^{(n)} = z - n$
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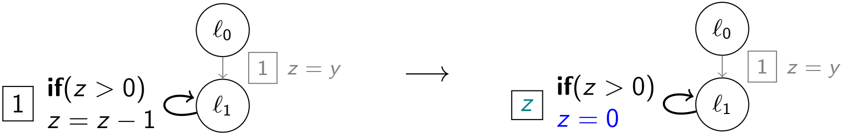

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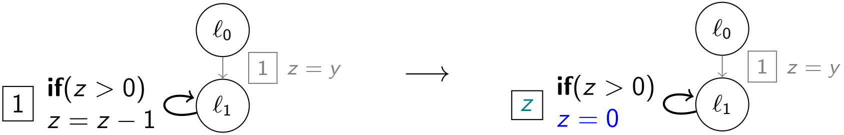

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- How many **repetitions**?
  - use *metering functions* $\quad \curvearrowright \quad n = z$

Metering Functions

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while (guard) do
    update
done
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- variation of *ranking functions*

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- b is a *metering* *function* iff

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$$\text{guard} \Rightarrow b > 0 \text{ and guard} \Rightarrow \text{update}(b) \leq b - 1$$

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Metering Functions

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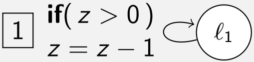

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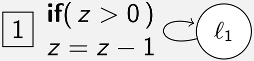

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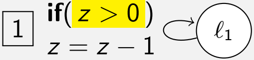

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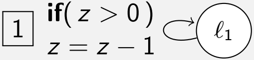

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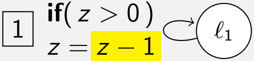

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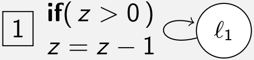

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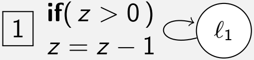

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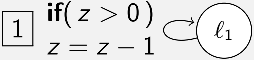

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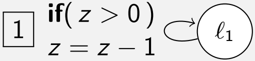

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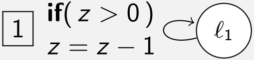

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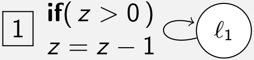

$$z \leq 0 \Rightarrow b(z) \leq 0 \text{ and } z > 0 \Rightarrow b(z-1) \geq b(z) - 1$$

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Metering Functions

b is a *metering function* iff

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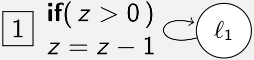

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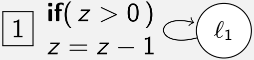

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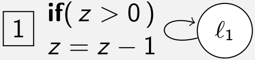

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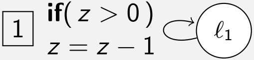

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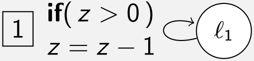

$\boxed{1}$ **if**($z > 0$)
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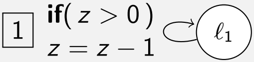

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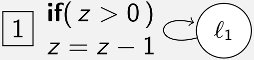

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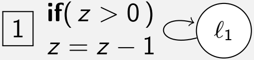

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ℓ_1

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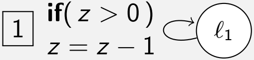

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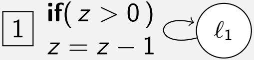

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finding them:

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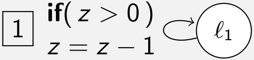

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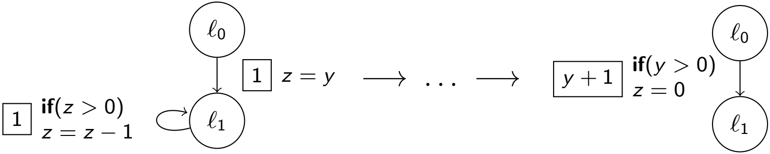
finding them: just like ranking functions

Program Simplification

Algorithm

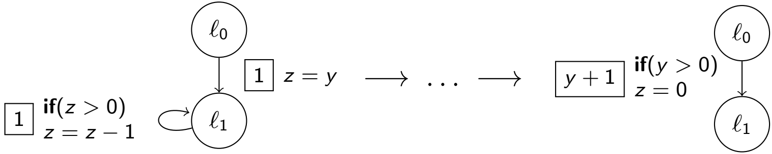
- while there is a path of length > 1



Program Simplification

Algorithm

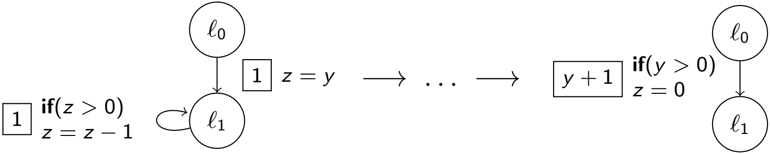
- while there is a path of length > 1
    - **accelerate** simple loops



Program Simplification

Algorithm

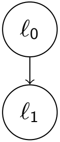
- while there is a path of length > 1
  - **accelerate** simple loops
  - **chain** subsequent transitions



Simplified Programs

$\boxed{x^2 - y}$ **if**($0 < x < 10 \land y + x < 0$)

...



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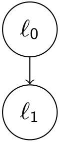
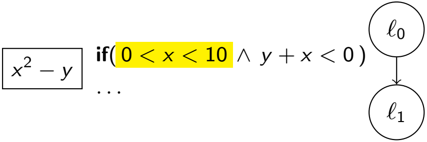
. . .



- inferring lower bound still non-trivial

Example

Simplified Programs



$\boxed{x^2 - y}$ **if**($0 < x < 10 \land y + x < 0$)

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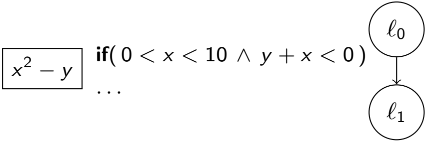
- inferring lower bound still non-trivial
- runtime depends on cost **and guard**

Example

Simplified Programs



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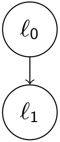

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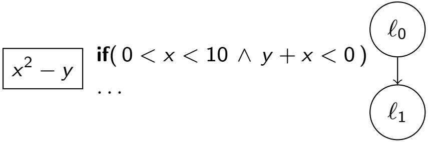
Example

Simplified Programs



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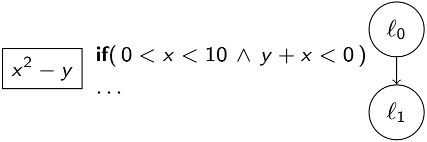
Example

- $\mathbf{v}_n = \{x/1, y/-n\}$

Simplified Programs



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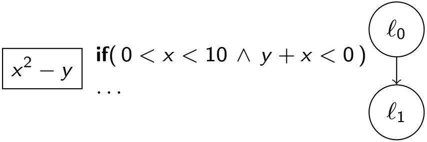
Example

- $\mathbf{v}_n = \{x/1, y/-n\}$ satisfies guard for $n \geq 2$

Simplified Programs



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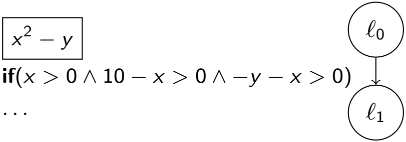
Example

- $\mathbf{v}_n = \{x/1, y/-n\}$ satisfies guard for $n \geq 2$
- $\mathbf{v}_n(x^2 - y) = 1 + n \implies \Omega(n)$

Example



$\boxed{x^2 - y}$

if$(x > 0 \land 10 - x > 0 \land -y - x > 0)$

$\ldots$

goal: infer $\mathbf{v}_n = \{x/1, y/-n\}$

Example

$\boxed{x^2 - y}$

if$(x > 0 \land 10 - x > 0 \land -y - x > 0)$

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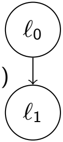

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$$\{x^{+!}, (10 - x)^{+!}, (-y - x)^{+}\}$$

Example

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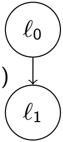

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- a^{+} : $\mathbf{v}_n(a)$ "increases with n" $\quad$ a^{-} : $\mathbf{v}_n(a)$ "decreases with n"
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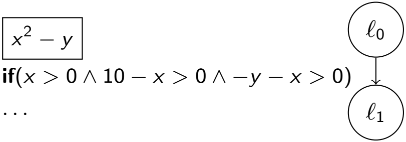

$\boxed{x^2 - y}$

$\textbf{if}(x > 0 \land 10 - x > 0 \land -y - x > 0)$

$\ldots$

goal: infer $\mathbf{v}_n = \{x/1, y/-n\}$

observe : $(a-b)^{+}$ if a^{+} and $b^{+!}$

$$\{x^{+!}, (10-x)^{+!}, (-y-x)^{+}\}$$

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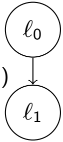

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Example

$\boxed{x^2 - y}$

if$(x > 0 \wedge 10 - x > 0 \wedge -y - x > 0)$

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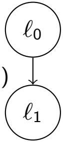

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observe : $(a-b)^+$ if a^+ and $b^{+!}$
$(-a)^+$ if a^-

$$\{x^{+!}, (10-x)^{+!}, (-y-x)^+\} \quad \rightsquigarrow \quad \{x^{+!}, (10-x)^{+!}, (-y)^+\}$$

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Example

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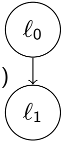

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$$\rightsquigarrow \quad \{x^{+!}, (10 - x)^{+!}, y^{-}\}$$

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Example

$\boxed{x^2 - y}$

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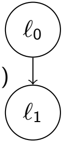

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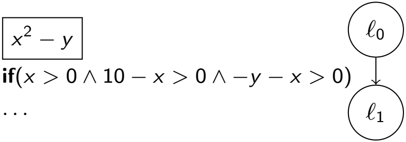

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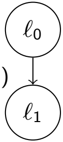

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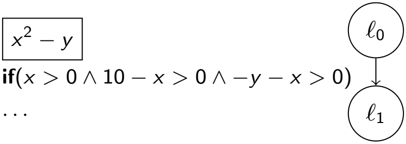

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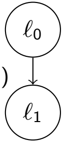

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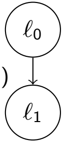

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$\mathcal{O}(n^3)$	2	–	–	3	–	–	–
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- **nontermination**

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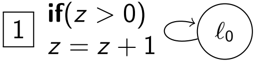

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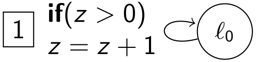

$\boxed{1}$ **if**$(z > 0)$

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ℓ_0

- **unbounded runtime**

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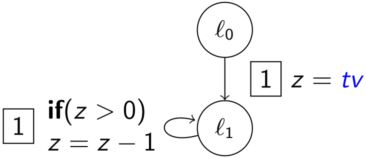

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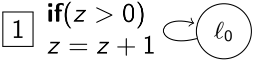

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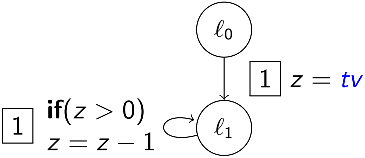

⇒ e.g. to model nondeterminism, user input, . . .