Automatically Finding Non-confluent Examples in Abstract Rewriting

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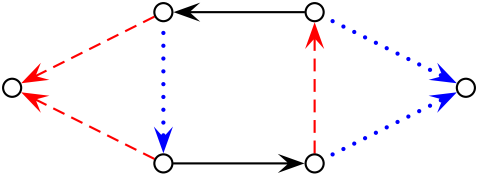
Solution for $n = 6$:



in which the three relations are indicated by solid black, dashed red and dotted blue arrows, respectively

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- Apply a SAT solver on the result
- In case of satisfiability: extract the solution from the resulting satisfying assignment

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For expressing most of these properties auxiliary relations will be needed

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Hence R^+ is irreflexive, hence R is acyclic = terminating

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- S is transitive by $\bigwedge_{i,j,k}((S_{ij} \wedge S_{jk}) \rightarrow S_{ik})$

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Similarly, we express

$$peak(R,S) = R^{-1} \cdot S$$

and

$$valley(R,S) = R \cdot S^{-1}$$

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Then R_k describes the desired relation R^*

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$$\mathit{peak}(S, S) \ \subseteq \ \mathit{valley}(S, S)$$

and local confluence by

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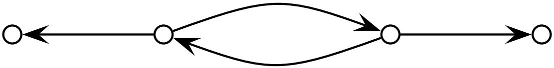
and local confluence for $n = 4$ yields the well-known example



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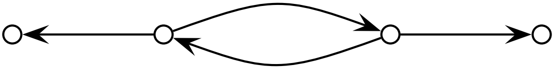

Adding termination of R yields an unsatisfiable formula, as expected due to Newman's Lemma

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It can be done much more efficient using only two auxiliary relations by

Theorem

A binary relation R on a finite set is complete if and only if two binary relations S and T exist such that

- $R \subseteq S$
- S *is transitive and irreflexive*
- $\bigwedge_i (T_{ii} \vee \bigvee_j R_{ij})$
- $\bigwedge_{i,j}((S_{ij} \wedge T_{jj}) \rightarrow T_{ij})$
- $\bigwedge_{i,j,k,j \neq k} \neg(T_{ij} \wedge T_{ik})$

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- `union`, `comp`(osition), `peak`, `val`, transitive closure, $\cdots$

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- transforms the result back to the desired example, or reports that no solution exists

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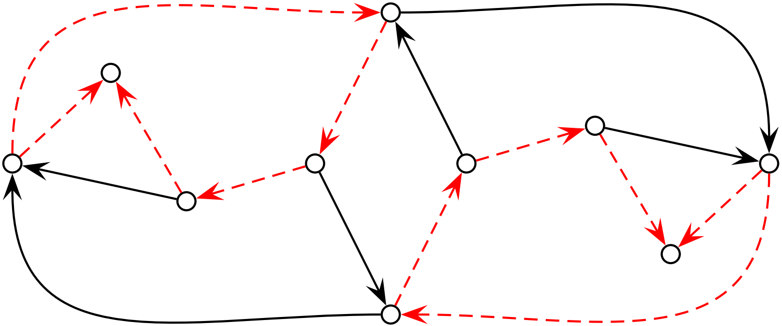

R steps: solid black arrows; S steps: dashed red arrows

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- More examples are welcome