Non-termination of String and Cycle Rewriting by Automata

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First we focus on *string rewriting*:

- a set $\mathcal{R}$ of rules $\ell \to r$ are given, for strings ℓ, r
- a step is a replacement of an occurrence of ℓ by r for a rule $\ell \to r$ in $\mathcal{R}$:
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Later we will consider *cycle rewriting*

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is non-terminating:

$$bLb \to bRb \to bLab \to bRab \to baRb \to baLab \to bLaab$$

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but does not admit a loop

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Even non-WN

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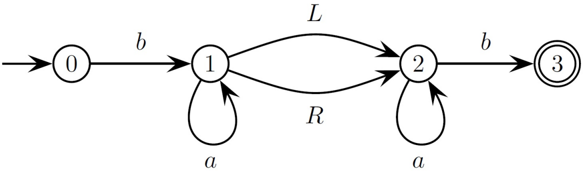

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Apply a SAT solver to this formula: if it is satisfiable then $\mathcal{L}$ satisfies the requirements proving non-termination

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  for every rule $\ell \rightarrow r$ and every two states i, j in A for which there is a path labeled by ℓ from i to j, there is also a path labeled by r from i to j
- $\mathcal{L}$ contains no normal forms:
  In a preprocessing build an automaton A' exactly accepting the normal forms, then in the product automaton $A \times A'$ there should be no path from the initial state to a state (f, f') for which f is final in A and f' is final in A'

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- Extend the approach to *cycle rewriting*

Simulations

A relation $\sim$ on the states of an automaton is called a *simulation* if for every p, q, r, a such that $p \sim q$ and $p \xrightarrow{a} r$, there exists a state s such that $q \xrightarrow{a} s$ and $r \sim s$:

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A *backward simulation* is a simulation in the automaton obtained by reversing all arrows

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- $\mathcal{L}$ contains no normal forms:
  No big product automaton required any more: if M accepts non-normal forms, then for $\mathcal{L} = L(A)$ we require $\mathcal{L} \subseteq L(M)$, to be expressed by simulating any path from initial to final in A by a similar path in M

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We adapt our automata based approach for cycle rewriting

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Then R is not cycle terminating

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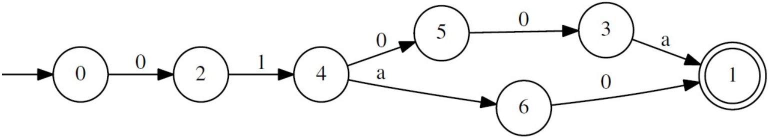

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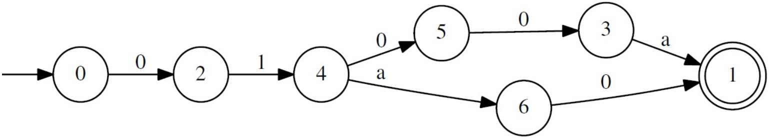

Indeed $L = \{0100a, 01a0\}$ satisfies all properties

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