Termination of term graph rewriting

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This is a widely applied standard way of computation

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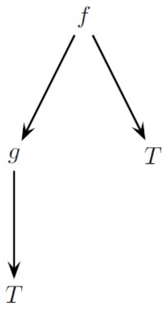

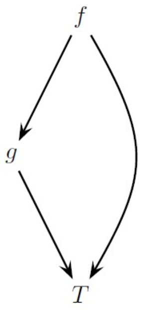

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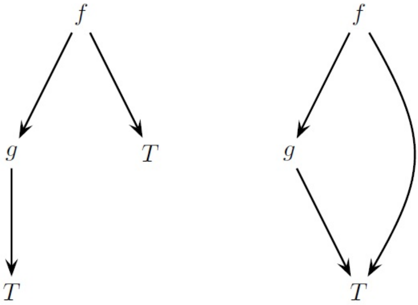

In this way terms are represented by *DAGs* (directed acyclic graphs) rather than by trees

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With the three rules

$$f(x, a, b) \to f(x, x, x), \; c \to a, \; c \to b$$

the infinite term reduction

$$f(c, a, b) \to f(c, c, c) \to f(c, a, c) \to f(c, a, b) \to \cdots$$

can not be mimicked in DAG rewriting without doing intermediate unsharing

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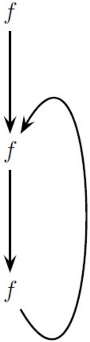
What happens if we allow graphs with cycles?

They may represent *infinite* terms

Example



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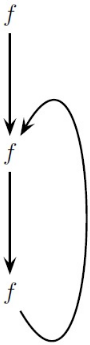

represents the infinite term

$$f^{\omega} = f(f(f(f(f(\cdots)))))$$

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More precisely:

Definition

A term graph over a signature Σ is a triple $(V, \mathrm{lab}, \mathrm{succ})$ in which

- V is a finite set of nodes (vertices)
- $\mathrm{lab} : V \to \Sigma$ is a partial labeling function
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Terms are interpreted as term graphs where lab and succ are undefined for variables

This numbering is essential since we want to distinguish the term graphs corresponding to the terms $f(a,b)$ and $f(b,a)$

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Precise definition is given by a *double push-out*:

$$\begin{tikzcd}
L \arrow[d, "g"'] & I \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, "i"] & R \arrow[d, "h"] \\
G & D \arrow[l, "dg"'] \arrow[r, "dh"] & H
\end{tikzcd}$$

where L, R are the term graphs of ℓ, r, I is the *interface* (describing which parts of L are connected to which parts of R), and G is the graph that is rewritten to H

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cycle rewriting $\approx$ term graph rewriting in basic version with unary symbols only

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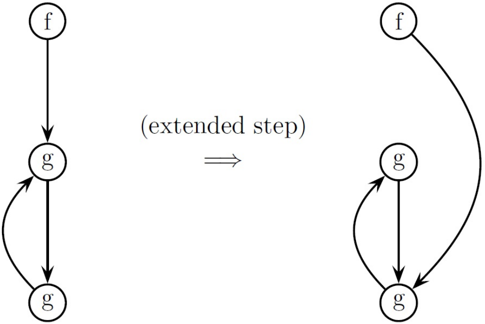

corresponding to rewriting $f(g^\omega)$ to itself

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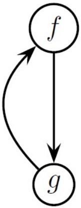

rewrites to itself both in the basic and extended version, hence is not terminating

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Main idea: transform TGRS to GTS in such a way that termination of resulting GTS implies termination of original TGRS

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We have to transform TG to G, and give two ways to do so: the *function encoding* and the *number encoding*

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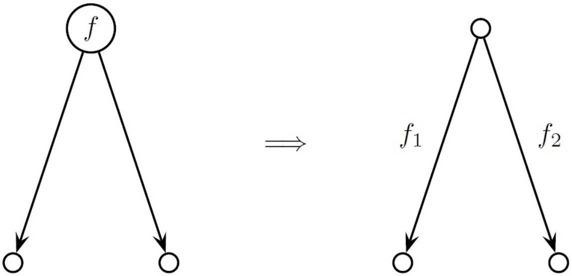

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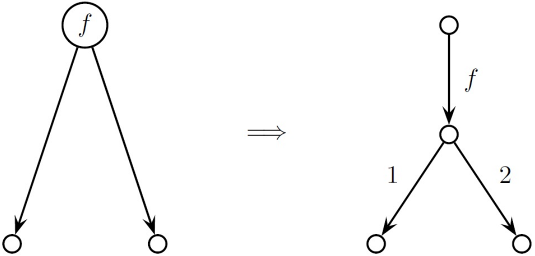

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Example requiring heavy techniques from Grez:

$$\begin{array}{rcl} f(x, a(b(y))) & \to & f(c(d(x)), y) \\ f(c(x), y) & \to & f(x, a(y)) \\ f(d(x), y) & \to & f(x, b(y)) \end{array}$$

Surprise: AProVE fails to prove the weaker property of TRS termination

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- Proving termination by transforming TGRS to GTS
- Drastically extends set of benchmarks for GTS termination proofs