Recording Completion for Finding and Certifying Proofs in Equational Logic

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29 May 2012

Outline

- Reminder: completion & conversions
- Extending completion to recording completion
- Certification
- Conclusion

Completion

- input: equational system and equation

$$E = \{\mathsf{ff} \approx \mathsf{f}, \mathsf{ggf} \approx \mathsf{g}\} \qquad \text{and} \qquad \mathsf{fgf} \overset{?}{\leftrightarrow^*_E} \mathsf{fgg}$$

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- result of completion: convergent rewrite system, equivalent to E

$$R = \{\mathsf{ff} \to \mathsf{f}, \mathsf{gf} \to \mathsf{g}, \mathsf{gg} \to \mathsf{g}\}$$

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$$\mathsf{fgf} \to^!_R \mathsf{fg} = \mathsf{fg} \;{}_R^!\!\leftarrow \mathsf{fgg}$$

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how to certify $\mathsf{fgf} \leftrightarrow^*_E \mathsf{fgg}$

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- two possibilities

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  1. convert normal form derivations of R into conversions of E

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- problem:

$$\text{how to certify } \mathsf{fgf} \leftrightarrow^*_E \mathsf{fgg}$$

- two possibilities
  1. convert normal form derivations of R into conversions of E
  2. prove that R is convergent and that $\leftrightarrow^*_E = \leftrightarrow^*_R$

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- problem:

how to certify $\mathsf{fgf} \leftrightarrow^*_E \mathsf{fgg}$

- two possibilities
  1. convert normal form derivations of R into conversions of E
  2. prove that R is convergent and that $\leftrightarrow^*_E = \leftrightarrow^*_R$
- both possibilities require more information from completion than R
- second possibility has the advantage that one can also certify $s \not\leftrightarrow^*_E t$

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$$E = \{\mathsf{ff} \approx \mathsf{f}, \mathsf{ggf} \approx \mathsf{g}\} \qquad \text{and} \qquad \mathsf{fgf} \overset{?}{\leftrightarrow}{}_E^* \mathsf{fgg}$$

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$$R = \{\mathsf{ff} \to \mathsf{f}, \mathsf{gf} \to \mathsf{g}, \mathsf{gg} \to \mathsf{g}\}$$

- answer question by comparing normal forms of lhs and rhs

$$\mathsf{fgf} \to_R^! \mathsf{fg} = \mathsf{fg} \;{}_R^!\!\leftarrow \mathsf{fgg}$$

- problem:

$$\text{how to certify } \mathsf{fgf} \leftrightarrow_E^* \mathsf{fgg}$$

- two possibilities
  1. convert normal form derivations of R into conversions of E
  2. prove that R is convergent and that $\leftrightarrow_E^* = \leftrightarrow_R^*$
- both possibilities require more information from completion than R
- second possibility has the advantage that one can also certify $s \not\leftrightarrow_E^* t$
- solution: extend completion to recording completion

Completion Rules

deduce
$$\frac{(E, R)}{(E \cup \{s \approx t\}, R)} \quad \text{if } s \;{}_R\!\leftarrow u \rightarrow_R t$$

orient
$$\frac{(E \cup \{s \dot{\approx} t\}, R)}{(E, R \cup \{s \rightarrow t\})} \quad \text{if } s > t$$

simplify
$$\frac{(E \cup \{s \dot{\approx} t\}, R)}{(E \cup \{u \dot{\approx} t\}, R)} \quad \text{if } s \rightarrow_R u$$

delete
$$\frac{(E \cup \{s \approx s\}, R)}{(E, R)}$$

compose
$$\frac{(E, R \cup \{s \rightarrow t\})}{(E, R \cup \{s \rightarrow u\})} \quad \text{if } t \rightarrow_R u$$

collapse
$$\frac{(E, R \cup \{s \rightarrow t\})}{(E \cup \{u \approx t\}, R)} \quad \text{if } s \overset{\sqsupset}{\rightarrow}_R u$$

Completion Rules

deduce
$$\frac{(E, R)}{(E \cup \{s \approx t\}, R)} \quad \text{if } s \;{}_R{\leftarrow}\; u \to_R t$$

orient
$$\frac{(E \cup \{s \mathrel{\dot{\approx}} t\}, R)}{(E, R \cup \{s \to t\})} \quad \text{if } s > t$$

simplify
$$\frac{(E \cup \{s \mathrel{\dot{\approx}} t\}, R)}{(E \cup \{u \mathrel{\dot{\approx}} t\}, R)} \quad \text{if } s \to_R u$$

delete
$$\frac{(E \cup \{s \approx s\}, R)}{(E, R)}$$

compose
$$\frac{(E, R \cup \{s \to t\})}{(E, R \cup \{s \to u\})} \quad \text{if } t \to_R u$$

collapse
$$\frac{(E, R \cup \{s \to t\})}{(E \cup \{u \approx t\}, R)} \quad \text{if } s \to_R u$$

- we will only be able to certify finite completion runs

$\Rightarrow$ new result: then strict-encompassment $\sqsupset$ can be dropped

Let $\rightsquigarrow$ be a step w.r.t. the completion rules (without the strict encompassment condition)

Theorem (Soundness of completion, formalized in IsaFoR)

If $(E, \emptyset) \rightsquigarrow^* (\emptyset, R)$ *where all critical pairs of* R *have been generated, then* R *is terminating, confluent, and* $\leftrightarrow_E^* = \leftrightarrow_R^*$.

IsaFoR: **Isa***belle* **F***ormalization* **o***f* **R***ewriting*

Completion

E	R	inference rule
① $\mathsf{ff} \approx \mathsf{f}$		
② $\mathsf{ggf} \approx \mathsf{g}$		

Completion

E	R	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		
② $\mathsf{ggf} \approx \mathsf{g}$		
	① $\mathsf{ff} \to \mathsf{f}$	orient ① $\to$

Completion

E	R	inference rule
~~① ff ≈ f~~		
~~② ggf ≈ g~~		
	① ff → f	orient ① →
	② ggf → g	orient ② →

Completion

E	R	inference rule
~~① ff ≈ f~~		
~~② ggf ≈ g~~		
	① ff → f	orient ① →
	② ggf → g	orient ② →
③ ggf ≈ gf		deduce ②, ①

Completion

E	R	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		
	① $\mathsf{ff} \to \mathsf{f}$	orient ① $\to$
	② $\mathsf{ggf} \to \mathsf{g}$	orient ② $\to$
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		deduce ②, ①
④ $\mathsf{g} \approx \mathsf{gf}$		simplify ③, ②

Completion

E	R	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		
	① $\mathsf{ff} \to \mathsf{f}$	orient ① $\to$
	② $\mathsf{ggf} \to \mathsf{g}$	orient ② $\to$
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		simplify ③, ②
	④ $\mathsf{gf} \to \mathsf{g}$	orient ④ $\leftarrow$

Completion

E	R	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		
	① $\mathsf{ff} \to \mathsf{f}$	orient ① $\to$
	② $\mathsf{ggf} \to \mathsf{g}$	orient ② $\to$
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		simplify ③, ②
	④ $\mathsf{gf} \to \mathsf{g}$	orient ④ $\leftarrow$
⑤ $\mathsf{gg} \approx \mathsf{g}$		deduce ②, ④

Completion

E	R	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		
	① $\mathsf{ff} \to \mathsf{f}$	orient ① $\to$
	② $\mathsf{ggf} \to \mathsf{g}$	orient ② $\to$
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		simplify ③, ②
	④ $\mathsf{gf} \to \mathsf{g}$	orient ④ $\leftarrow$
~~⑤ $\mathsf{gg} \approx \mathsf{g}$~~		deduce ②, ④
	⑤ $\mathsf{gg} \to \mathsf{g}$	orient ⑤ $\to$

Completion

E	*R*	inference rule
~~① ff $\approx$ f~~		
~~② ggf $\approx$ g~~		
	① ff $\rightarrow$ f	orient ① $\rightarrow$
	~~② ggf $\rightarrow$ g~~	orient ② $\rightarrow$
~~③ ggf $\approx$ gf~~		deduce ②, ①
~~④ g $\approx$ gf~~		simplify ③, ②
	④ gf $\rightarrow$ g	orient ④ $\leftarrow$
~~⑤ gg $\approx$ g~~		deduce ②, ④
	⑤ gg $\rightarrow$ g	orient ⑤ $\rightarrow$
⑥ gf $\approx$ g		collapse ②, ⑤

Completion

E	R	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		
	① $\mathsf{ff} \to \mathsf{f}$	orient ① $\to$
	~~② $\mathsf{ggf} \to \mathsf{g}$~~	orient ② $\to$
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		simplify ③, ②
	④ $\mathsf{gf} \to \mathsf{g}$	orient ④ $\leftarrow$
~~⑤ $\mathsf{gg} \approx \mathsf{g}$~~		deduce ②, ④
	⑤ $\mathsf{gg} \to \mathsf{g}$	orient ⑤ $\to$
~~⑥ $\mathsf{gf} \approx \mathsf{g}$~~		collapse ②, ⑤
⑦ $\mathsf{g} \approx \mathsf{g}$		simplify ⑥, ④

Completion

E	R	inference rule
~~① $ff \approx f$~~		
~~② $ggf \approx g$~~		
	① $ff \to f$	orient ① $\to$
	~~② $ggf \to g$~~	orient ② $\to$
~~③ $ggf \approx gf$~~		deduce ②, ①
~~④ $g \approx gf$~~		simplify ③, ②
	④ $gf \to g$	orient ④ $\leftarrow$
~~⑤ $gg \approx g$~~		deduce ②, ④
	⑤ $gg \to g$	orient ⑤ $\to$
~~⑥ $gf \approx g$~~		collapse ②, ⑤
~~⑦ $g \approx g$~~		simplify ⑥, ④
		delete ⑦

Completion

E	R	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		
	① $\mathsf{ff} \rightarrow \mathsf{f}$	orient ① $\rightarrow$
	~~② $\mathsf{ggf} \rightarrow \mathsf{g}$~~	orient ② $\rightarrow$
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		simplify ③, ②
	④ $\mathsf{gf} \rightarrow \mathsf{g}$	orient ④ $\leftarrow$
~~⑤ $\mathsf{gg} \approx \mathsf{g}$~~		deduce ②, ④
	⑤ $\mathsf{gg} \rightarrow \mathsf{g}$	orient ⑤ $\rightarrow$
~~⑥ $\mathsf{gf} \approx \mathsf{g}$~~		collapse ②, ⑤
~~⑦ $\mathsf{g} \approx \mathsf{g}$~~		simplify ⑥, ④
		delete ⑦

All other critical pairs can be deleted after simplification

result of completion

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

R

① $\mathsf{ff} \to \mathsf{f}$

④ $\mathsf{gf} \to \mathsf{g}$

⑤ $\mathsf{gg} \to \mathsf{g}$

Problem

- from completed rewrite system R one cannot infer how the rules have been derived from E

$\Rightarrow$ no possibility to convert $s \rightarrow^!_R t$ derivation into $s \leftrightarrow^*_E t$ conversion

$\Rightarrow$ no possibility to show $\leftrightarrow^*_E = \leftrightarrow^*_R$ (one can only show $\leftrightarrow^*_E \subseteq \leftrightarrow^*_R$)

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$\Rightarrow$ no possibility to show $\leftrightarrow_E^* = \leftrightarrow_R^*$ (one can only show $\leftrightarrow_E^* \subseteq \leftrightarrow_R^*$)

Solution: recording completion

idea:

- each rule and equation is indexed
- extent completion process by history

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Solution: recording completion

idea:

- each rule and equation is indexed
- extent completion process by history
- for each rule and equation there is a two step derivation in the history how the rule or equation has been derived
- initial history: $H_0 = \{i : s \xrightarrow{i} t \overset{0}{\approx} t \mid s \approx t \in E\}$

Completion Rules

deduce
$$\frac{(E, R)}{(E \cup \{s \approx t\}, R)} \quad \text{if } s \;{}_R\!\leftarrow u \rightarrow_R t$$

orient-l
$$\frac{(E \cup \{s \approx t\}, R)}{(E, R \cup \{s \rightarrow t\})} \quad \text{if } s > t$$

orient-r
$$\frac{(E \cup \{s \approx t\}, R)}{(E, R \cup \{t \rightarrow s\})} \quad \text{if } t > s$$

simplify-l
$$\frac{(E \cup \{s \approx t\}, R)}{(E \cup \{u \approx t\}, R)} \quad \text{if } s \rightarrow_R u$$

simplify-r
$$\frac{(E \cup \{s \approx t\}, R)}{(E \cup \{s \approx u\}, R)} \quad \text{if } t \rightarrow_R u$$

delete
$$\frac{(E \cup \{s \approx s\}, R)}{(E, R)}$$

compose
$$\frac{(E, R \cup \{s \rightarrow t\})}{(E, R \cup \{m : s \rightarrow u\})} \quad \text{if } t \rightarrow_R u$$

collapse
$$\frac{(E, R \cup \{s \rightarrow t\})}{(E \cup \{u \approx t\}, R)} \quad \text{if } s \rightarrow_R u$$

Recording Completion Rules

deduce

$$\frac{(E, R, H)}{(E \cup \{m : s \approx t\}, R, H \cup \{m : s \overset{j}{\leftarrow} u \overset{k}{\rightarrow} t\})} \quad \text{if } s \mathrel{{}_R\overset{j}{\leftarrow}} u \overset{k}{\rightarrow}_R t$$

orient-l

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$$\frac{(E \cup \{ s \approx t\}, R)}{(E, R \cup \{ t \rightarrow s\})} \quad \text{if } t > s$$

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orient-r
$$\frac{(E \cup \{j : s \approx t\}, R, H \cup \{j : s \circ u \bullet t\})}{(E, R \cup \{j : t \rightarrow s\}, H \cup \{j : t \bullet^{-1} u \circ^{-1} s\})} \quad \text{if } t > s$$

simplify-l
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$$\frac{(E, R, H)}{(E \cup \{m : s \approx t\}, R, H \cup \{m : s \overset{j}{\leftarrow} u \overset{k}{\rightarrow} t\})} \quad \text{if } s \; {}_R\overset{j}{\leftarrow} \; u \overset{k}{\rightarrow}_R \; t$$

orient-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E, R \cup \{j : s \rightarrow t\}, H)} \quad \text{if } s > t$$

orient-r
$$\frac{(E \cup \{j : s \approx t\}, R, H \cup \{j : s \circ u \bullet t\})}{(E, R \cup \{j : t \rightarrow s\}, H \cup \{j : t \bullet^{-1} u \circ^{-1} s\})} \quad \text{if } t > s$$

simplify-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : u \approx t\}, R, H \cup \{m : u \overset{k}{\leftarrow} s \overset{j}{\rightarrow} t\})} \quad \text{if } s \overset{k}{\rightarrow}_R \; u$$

simplify-r
$$\frac{(E \cup \{ \quad s \approx t\}, R \quad)}{(E \cup \{ \quad s \approx u\}, R \quad)} \quad \text{if } t \rightarrow_R \; u$$

delete
$$\frac{(E \cup \{ \quad s \approx s\}, R \quad)}{(E, R \quad)}$$

compose
$$\frac{(E, R \cup \{ \quad s \rightarrow t\} \quad)}{(E, R \cup \{m : s \rightarrow u\} \quad)} \quad \text{if } t \rightarrow_R \; u$$

collapse
$$\frac{(E, R \cup \{ \quad s \rightarrow t\} \quad)}{(E \cup \{ \quad u \approx t\}, R \quad)} \quad \text{if } s \rightarrow_R \; u$$

Recording Completion Rules

deduce
$$\frac{(E, R, H)}{(E \cup \{m : s \approx t\}, R, H \cup \{m : s \overset{j}{\leftarrow} u \overset{k}{\rightarrow} t\})} \quad \text{if } s \;{}_R\!\overset{j}{\leftarrow}\; u \overset{k}{\rightarrow}_R t$$

orient-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E, R \cup \{j : s \to t\}, H)} \quad \text{if } s > t$$

orient-r
$$\frac{(E \cup \{j : s \approx t\}, R, H \cup \{j : s \circ u \bullet t\})}{(E, R \cup \{j : t \to s\}, H \cup \{j : t \bullet^{-1} u \circ^{-1} s\})} \quad \text{if } t > s$$

simplify-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : u \approx t\}, R, H \cup \{m : u \overset{k}{\leftarrow} s \overset{j}{\rightarrow} t\})} \quad \text{if } s \overset{k}{\rightarrow}_R u$$

simplify-r
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : s \approx u\}, R, H \cup \{m : s \overset{j}{\rightarrow} t \overset{k}{\rightarrow} u\})} \quad \text{if } t \overset{k}{\rightarrow}_R u$$

delete
$$\frac{(E \cup \{ \quad s \approx s\}, R \qquad\qquad)}{(E, R \quad)}$$

compose
$$\frac{(E, R \cup \{ \quad s \to t\} \quad)}{(E, R \cup \{m : s \to u\} \qquad\qquad)} \quad \text{if } t \to_R u$$

collapse
$$\frac{(E, R \cup \{ \quad s \to t\} \quad)}{(E \cup \{ \quad u \approx t\}, R \qquad\qquad)} \quad \text{if } s \to_R u$$

Recording Completion Rules

deduce
$$\frac{(E, R, H)}{(E \cup \{m : s \approx t\}, R, H \cup \{m : s \xleftarrow{j} u \xrightarrow{k} t\})} \quad \text{if } s \ {}_R\!\xleftarrow{j} \ u \xrightarrow{k}_R \ t$$

orient-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E, R \cup \{j : s \to t\}, H)} \quad \text{if } s > t$$

orient-r
$$\frac{(E \cup \{j : s \approx t\}, R, H \cup \{j : s \circ u \bullet t\})}{(E, R \cup \{j : t \to s\}, H \cup \{j : t \bullet^{-1} u \circ^{-1} s\})} \quad \text{if } t > s$$

simplify-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : u \approx t\}, R, H \cup \{m : u \xleftarrow{k} s \xrightarrow{j} t\})} \quad \text{if } s \xrightarrow{k}_R u$$

simplify-r
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : s \approx u\}, R, H \cup \{m : s \xrightarrow{j} t \xrightarrow{k} u\})} \quad \text{if } t \xrightarrow{k}_R u$$

delete
$$\frac{(E \cup \{j : s \approx s\}, R, H \cup \{j : s \circ v \bullet s\})}{(E, R, H)}$$

compose
$$\frac{(E, R \cup \{\ s \to t\}\ )}{(E, R \cup \{m : s \to u\}\ )} \quad \text{if } t \to_R u$$

collapse
$$\frac{(E, R \cup \{\ s \to t\}\ )}{(E \cup \{\ u \approx t\}, R\ )} \quad \text{if } s \to_R u$$

Recording Completion Rules

deduce
$$\frac{(E, R, H)}{(E \cup \{m : s \approx t\}, R, H \cup \{m : s \overset{j}{\leftarrow} u \overset{k}{\rightarrow} t\})} \quad \text{if } s \,{}_R\!\overset{j}{\leftarrow}\, u \overset{k}{\rightarrow}_R t$$

orient-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E, R \cup \{j : s \rightarrow t\}, H)} \quad \text{if } s > t$$

orient-r
$$\frac{(E \cup \{j : s \approx t\}, R, H \cup \{j : s \circ u \bullet t\})}{(E, R \cup \{j : t \rightarrow s\}, H \cup \{j : t \bullet^{-1} u \circ^{-1} s\})} \quad \text{if } t > s$$

simplify-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : u \approx t\}, R, H \cup \{m : u \overset{k}{\leftarrow} s \overset{j}{\rightarrow} t\})} \quad \text{if } s \overset{k}{\rightarrow}_R u$$

simplify-r
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : s \approx u\}, R, H \cup \{m : s \overset{j}{\rightarrow} t \overset{k}{\rightarrow} u\})} \quad \text{if } t \overset{k}{\rightarrow}_R u$$

delete
$$\frac{(E \cup \{j : s \approx s\}, R, H \cup \{j : s \circ v \bullet s\})}{(E, R, H)}$$

compose
$$\frac{(E, R \cup \{j : s \rightarrow t\}, H)}{(E, R \cup \{m : s \rightarrow u\}, H \cup \{m : s \overset{j}{\rightarrow} t \overset{k}{\rightarrow} u\})} \quad \text{if } t \overset{k}{\rightarrow}_R u$$

collapse
$$\frac{(E, R \cup \{ \quad s \rightarrow t\} \quad)}{(E \cup \{ \quad u \approx t\}, R \quad)} \quad \text{if } s \rightarrow_R u$$

Recording Completion Rules

deduce
$$\frac{(E, R, H)}{(E \cup \{m : s \approx t\}, R, H \cup \{m : s \overset{j}{\leftarrow} u \overset{k}{\rightarrow} t\})} \quad \text{if } s \;{}_R\!\overset{j}{\leftarrow}\; u \overset{k}{\rightarrow}_R\; t$$

orient-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E, R \cup \{j : s \rightarrow t\}, H)} \quad \text{if } s > t$$

orient-r
$$\frac{(E \cup \{j : s \approx t\}, R, H \cup \{j : s \circ u \bullet t\})}{(E, R \cup \{j : t \rightarrow s\}, H \cup \{j : t \bullet^{-1} u \circ^{-1} s\})} \quad \text{if } t > s$$

simplify-l
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : u \approx t\}, R, H \cup \{m : u \overset{k}{\leftarrow} s \overset{j}{\rightarrow} t\})} \quad \text{if } s \overset{k}{\rightarrow}_R\; u$$

simplify-r
$$\frac{(E \cup \{j : s \approx t\}, R, H)}{(E \cup \{m : s \approx u\}, R, H \cup \{m : s \overset{j}{\rightarrow} t \overset{k}{\rightarrow} u\})} \quad \text{if } t \overset{k}{\rightarrow}_R\; u$$

delete
$$\frac{(E \cup \{j : s \approx s\}, R, H \cup \{j : s \circ v \bullet s\})}{(E, R, H)}$$

compose
$$\frac{(E, R \cup \{j : s \rightarrow t\}, H)}{(E, R \cup \{m : s \rightarrow u\}, H \cup \{m : s \overset{j}{\rightarrow} t \overset{k}{\rightarrow} u\})} \quad \text{if } t \overset{k}{\rightarrow}_R\; u$$

collapse
$$\frac{(E, R \cup \{j : s \rightarrow t\}, H)}{(E \cup \{m : u \approx t\}, R, H \cup \{m : u \overset{k}{\leftarrow} s \overset{j}{\rightarrow} t\})} \quad \text{if } s \overset{k}{\rightarrow}_R\; u$$

From Completion to Conversions

3 phases:

1. record (using Recording Completion)
2. compare
3. recall (two variants)

record phase (Recording Completion)

E	R	H	inference rule
① $\mathsf{ff} \approx \mathsf{f}$		① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$	
② $\mathsf{ggf} \approx \mathsf{g}$		② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$	

record phase (Recording Completion)

E	R	H	inference rule
~~① ff $\approx$ f~~		① ff $\xrightarrow{1}$ f $\overset{0}{\approx}$ f	
② ggf $\approx$ g		② ggf $\xrightarrow{2}$ g $\overset{0}{\approx}$ g	
	① ff $\to$ f		orient-l ①

record phase (Recording Completion)

E	R	H	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$	
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$	
	① $\mathsf{ff} \to \mathsf{f}$		orient-l ①
	② $\mathsf{ggf} \to \mathsf{g}$		orient-l ②

record phase (Recording Completion)

E	R	H	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$	
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$	
	① $\mathsf{ff} \to \mathsf{f}$		orient-l ①
	② $\mathsf{ggf} \to \mathsf{g}$		orient-l ②
③ $\mathsf{ggf} \approx \mathsf{gf}$		③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$	deduce ②, ①

record phase (Recording Completion)

E	R	H	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$	
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$	
	① $\mathsf{ff} \to \mathsf{f}$		orient-l ①
	② $\mathsf{ggf} \to \mathsf{g}$		orient-l ②
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$	deduce ②, ①
④ $\mathsf{g} \approx \mathsf{gf}$		④ $\mathsf{g} \xleftarrow{2} \mathsf{ggf} \xrightarrow{3} \mathsf{gf}$	simplify-l ③, ②

record phase (Recording Completion)

E	R	H	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$	
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$	
	① $\mathsf{ff} \to \mathsf{f}$		orient-l ①
	② $\mathsf{ggf} \to \mathsf{g}$		orient-l ②
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$	deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		~~④ $\mathsf{g} \xleftarrow{2} \mathsf{ggf} \xrightarrow{3} \mathsf{gf}$~~	simplify-l ③, ②
	④ $\mathsf{gf} \to \mathsf{g}$	④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$	orient-r ④

record phase (Recording Completion)

E	R	H	inference rule
~~① $ff \approx f$~~		① $ff \xrightarrow{1} f \overset{0}{\approx} f$	
~~② $ggf \approx g$~~		② $ggf \xrightarrow{2} g \overset{0}{\approx} g$	
	① $ff \to f$		orient-l ①
	② $ggf \to g$		orient-l ②
~~③ $ggf \approx gf$~~		③ $ggf \xleftarrow{1} ggff \xrightarrow{2} gf$	deduce ②, ①
~~④ $g \approx gf$~~		~~④ $g \xleftarrow{2} ggf \xrightarrow{3} gf$~~	simplify-l ③, ②
	④ $gf \to g$	④ $gf \xleftarrow{3} ggf \xrightarrow{2} g$	orient-r ④
⑤ $gg \approx g$		⑤ $gg \xleftarrow{4} ggf \xrightarrow{2} g$	deduce ②, ④

record phase (Recording Completion)

E	*R*	*H*	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$	
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$	
	① $\mathsf{ff} \to \mathsf{f}$		orient-l ①
	② $\mathsf{ggf} \to \mathsf{g}$		orient-l ②
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$	deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		~~④ $\mathsf{g} \xleftarrow{2} \mathsf{ggf} \xrightarrow{3} \mathsf{gf}$~~	simplify-l ③, ②
	④ $\mathsf{gf} \to \mathsf{g}$	④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$	orient-r ④
~~⑤ $\mathsf{gg} \approx \mathsf{g}$~~		⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$	deduce ②, ④
	⑤ $\mathsf{gg} \to \mathsf{g}$		orient-l ⑤

record phase (Recording Completion)

E	R	H	inference rule
~~① $ff \approx f$~~		① $ff \xrightarrow{1} f \overset{0}{\approx} f$	
~~② $ggf \approx g$~~		② $ggf \xrightarrow{2} g \overset{0}{\approx} g$	
	① $ff \to f$		orient-l ①
	~~② $ggf \to g$~~		orient-l ②
~~③ $ggf \approx gf$~~		③ $ggf \xleftarrow{1} ggff \xrightarrow{2} gf$	deduce ②, ①
~~④ $g \approx gf$~~		~~④ $g \xleftarrow{2} ggf \xrightarrow{3} gf$~~	simplify-l ③, ②
	④ $gf \to g$	④ $gf \xleftarrow{3} ggf \xrightarrow{2} g$	orient-r ④
~~⑤ $gg \approx g$~~		⑤ $gg \xleftarrow{4} ggf \xrightarrow{2} g$	deduce ②, ④
	⑤ $gg \to g$		orient-l ⑤
⑥ $gf \approx g$		⑥ $gf \xleftarrow{5} ggf \xrightarrow{2} g$	collapse ②, ⑤

record phase (Recording Completion)

E	*R*	*H*	inference rule
~~① $\mathsf{ff} \approx \mathsf{f}$~~		① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$	
~~② $\mathsf{ggf} \approx \mathsf{g}$~~		② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$	
	① $\mathsf{ff} \to \mathsf{f}$		orient-l ①
	~~② $\mathsf{ggf} \to \mathsf{g}$~~		orient-l ②
~~③ $\mathsf{ggf} \approx \mathsf{gf}$~~		③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$	deduce ②, ①
~~④ $\mathsf{g} \approx \mathsf{gf}$~~		~~④ $\mathsf{g} \xleftarrow{2} \mathsf{ggf} \xrightarrow{3} \mathsf{gf}$~~	simplify-l ③, ②
	④ $\mathsf{gf} \to \mathsf{g}$	④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$	orient-r ④
~~⑤ $\mathsf{gg} \approx \mathsf{g}$~~		⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$	deduce ②, ④
	⑤ $\mathsf{gg} \to \mathsf{g}$		orient-l ⑤
~~⑥ $\mathsf{gf} \approx \mathsf{g}$~~		⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$	collapse ②, ⑤
⑦ $\mathsf{g} \approx \mathsf{g}$		⑦ $\mathsf{g} \xleftarrow{4} \mathsf{gf} \xrightarrow{6} \mathsf{g}$	simplify-l ⑥, ④

record phase (Recording Completion)

E	R	H	inference rule
~~① $ff \approx f$~~		① $ff \xrightarrow{1} f \overset{0}{\approx} f$	
~~② $ggf \approx g$~~		② $ggf \xrightarrow{2} g \overset{0}{\approx} g$	
	① $ff \to f$		orient-l ①
	~~② $ggf \to g$~~		orient-l ②
~~③ $ggf \approx gf$~~		③ $ggf \xleftarrow{1} ggff \xrightarrow{2} gf$	deduce ②, ①
~~④ $g \approx gf$~~		~~④ $g \xleftarrow{2} ggf \xrightarrow{3} gf$~~	simplify-l ③, ②
	④ $gf \to g$	④ $gf \xleftarrow{3} ggf \xrightarrow{2} g$	orient-r ④
~~⑤ $gg \approx g$~~		⑤ $gg \xleftarrow{4} ggf \xrightarrow{2} g$	deduce ②, ④
	⑤ $gg \to g$		orient-l ⑤
~~⑥ $gf \approx g$~~		⑥ $gf \xleftarrow{5} ggf \xrightarrow{2} g$	collapse ②, ⑤
~~⑦ $g \approx g$~~		~~⑦ $g \xleftarrow{4} gf \xrightarrow{6} g$~~	simplify-l ⑥, ④
			delete ⑦

result of record phase

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

R

① $\mathsf{ff} \to \mathsf{f}$

④ $\mathsf{gf} \to \mathsf{g}$

⑤ $\mathsf{gg} \to \mathsf{g}$

H

① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$

② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$

③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$

④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

result of record phase

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

R

① $\mathsf{ff} \to \mathsf{f}$

④ $\mathsf{gf} \to \mathsf{g}$

⑤ $\mathsf{gg} \to \mathsf{g}$

H

~~① $\mathsf{ff} \xrightarrow{1} \mathsf{f} \overset{0}{\approx} \mathsf{f}$~~

~~② $\mathsf{ggf} \xrightarrow{2} \mathsf{g} \overset{0}{\approx} \mathsf{g}$~~

③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$

④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

compare phase

$$\mathsf{fgf} \overset{?}{\leftrightarrow}^*_E \mathsf{fgg}$$

compare phase

$\text{fgf} \overset{?}{\leftrightarrow}^*_E \text{fgg}$

use R to reduce lhs and rhs to normal form

R {

① ff $\to$ f

④ gf $\to$ g

⑤ gg $\to$ g

}

compare phase

$\mathsf{fgf} \overset{?}{\leftrightarrow}_E^* \mathsf{fgg}$

use R to reduce lhs and rhs to normal form

R {
① $\mathsf{ff} \to \mathsf{f}$
④ $\mathsf{gf} \to \mathsf{g}$
⑤ $\mathsf{gg} \to \mathsf{g}$
}

$\mathsf{f}\underline{\mathsf{gf}} \overset{4}{\to} \mathsf{fg} \overset{5}{\leftarrow} \mathsf{f}\underline{\mathsf{gg}}$

recall phase (first variant)

while there are rules with indices not in E replace them with the corresponding sequence from H

$$\mathsf{fgf} \xrightarrow{4} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$$

E	H
① $\mathsf{ff} \approx \mathsf{f}$	③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$
② $\mathsf{ggf} \approx \mathsf{g}$	④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$
	⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$
	⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

recall phase (first variant)

while there are rules with indices not in E replace them with the corresponding sequence from H

$\mathsf{fgf} \xrightarrow{4} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

$\mathsf{fgf} \xleftarrow{3} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

context

$\mathsf{f}\Box$

H

③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$

④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

recall phase (first variant)

while there are rules with indices not in E replace them with the corresponding sequence from H

$\mathsf{fgf} \xrightarrow{4} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

$\mathsf{fgf} \xleftarrow{3} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

$\mathsf{fgf} \xleftarrow{2} \mathsf{fggff} \xrightarrow{1} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

context

$\mathsf{f}\square$

H

③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$

④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

recall phase (first variant)

while there are rules with indices not in E replace them with the corresponding sequence from H

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

H

③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$

④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

context

$\mathsf{f}\Box$

$$\mathsf{fgf} \xrightarrow{4} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$$

$$\mathsf{fgf} \xleftarrow{3} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$$

$$\mathsf{fgf} \xleftarrow{2} \mathsf{fggff} \xrightarrow{1} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$$

$$\mathsf{fgf} \xleftarrow{2} \mathsf{fggff} \xrightarrow{1} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{2} \mathsf{fggf} \xrightarrow{4} \mathsf{fgg}$$

recall phase (first variant)

while there are rules with indices not in E replace them with the corresponding sequence from H

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

H

③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$

④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

context

$\mathsf{fg}\square$

$\mathsf{fgf} \xrightarrow{4} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

$\mathsf{fgf} \xleftarrow{3} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

$\mathsf{fgf} \xleftarrow{2} \mathsf{fggff} \xrightarrow{1} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$

$\mathsf{fgf} \xleftarrow{2} \mathsf{fggff} \xrightarrow{1} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{2} \mathsf{fggf} \xrightarrow{4} \mathsf{fgg}$

$\mathsf{fgf} \xleftarrow{2} \mathsf{fggff} \xrightarrow{1} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{2} \mathsf{fggf} \xleftarrow{3} \mathsf{fgggf} \xrightarrow{2} \mathsf{fgg}$

recall phase (first variant)

while there are rules with indices not in E replace them with the corresponding sequence from H

E

① $\mathsf{ff} \approx \mathsf{f}$

② $\mathsf{ggf} \approx \mathsf{g}$

context

$\mathsf{fg}\square$

H

③ $\mathsf{ggf} \xleftarrow{1} \mathsf{ggff} \xrightarrow{2} \mathsf{gf}$

④ $\mathsf{gf} \xleftarrow{3} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑤ $\mathsf{gg} \xleftarrow{4} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

⑥ $\mathsf{gf} \xleftarrow{5} \mathsf{ggf} \xrightarrow{2} \mathsf{g}$

$$\mathsf{fgf} \xrightarrow{4} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$$

$$\mathsf{fgf} \xleftarrow{3} \mathsf{fggf} \xrightarrow{2} \mathsf{fg} \xleftarrow{5} \mathsf{fgg}$$

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- expansion can lead to exponential blow up

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Solution: second variant of recall phase

- for every (required) history entry $i : s_i \circ u_i \bullet t_i$ derive $s_i \leftrightarrow_E^* t_i$
- to this end use $s_m \leftrightarrow^* t_m$ for all $m < i$ as hypothesis

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- finally have $R \subseteq \{s_i \to t_i \mid i \leq i_{max}\} \subseteq \leftrightarrow_E^*$ and hence $\leftrightarrow_R^* \subseteq \leftrightarrow_E^*$

$\Rightarrow$ linear size certificate, no problem for KBCV

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  - observation: infinite set of variables is essential
  - example: let R be a confluent TRS over $\mathcal{T}(\mathcal{F}, \mathcal{V})$,
    then it does not follow that R is confluent over $\mathcal{T}(\mathcal{F} \cup \mathcal{F}', \mathcal{V})$

  $\Rightarrow$ signature extensions for confluence require infinite set of variables

Experiments

- KBCV 1.6 using recording completion (first variant of recall phase)
- KBCV 1.7 using recording completion (second variant of recall phase)
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- CeTA is certifier, extracted from IsaFoR
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Tool	Completed	Time	Cert.	CeTA accept	time	timeout
KBCV 1.6	86	7767	84	80	1483	4
KBCV 1.7	86	7735	86	86	13	0
MKBTT	80	1514	80	80	92	0
Total	94		94	94		

Slothrop	71	without certification
MAXCOMP	86	without certification

Conclusion

- encompassment condition is not required for finite completion runs
- recording completion to derive $\leftrightarrow_R^* \subseteq \leftrightarrow_E^*$
- recall phase is linear if intermediate lemmas are used
- tools for certified completion are available (KBCV, MKBTT, IsaFoR + CeTA)

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- alternative approach to obtain conversions is available in C*i*ME3 for ordered completion
- future work: study relationship in more detail