A Characterization of Quasi-Decreasingness*

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Outline

Motivation

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Characterizing and proving operational termination of deterministic conditional term rewriting systems
F. Schernhammer, B. Gramlich,
doi: 10.1016/j.jlap.2009.08.001,
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CSRS $U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on original terms $\Longleftrightarrow$ quasi-decreasing DCTRS $\mathcal{R}$

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quasi-decreasing DCTRS $\mathcal{R}$

cs-quasi-reductive $U_{\mathsf{CS}}(\mathcal{R})$

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CSRS $U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on original terms $\overset{?}{\Longrightarrow}$ quasi-decreasing DCTRS $\mathcal{R}$

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Basics

DCTRS

$$\ell \to r \Leftarrow \underbrace{s_1 \approx t_1, \ldots, s_k \approx t_k}_{c}$$

- $\approx$ interpreted as $\to_{\mathcal{R}}^{*}$
- $\ell \notin \mathcal{V}$
- $\mathcal{V}(r) \subseteq \mathcal{V}(\ell, c)$
- $\mathcal{V}(s_i) \subseteq \mathcal{V}(\ell, t_1, \ldots, t_{i-1})$

Unraveling $U(\mathcal{R})$

$$\alpha\colon \ell \to r \Leftarrow s_1 \approx t_1, s_2 \approx t_2, \ldots, s_n \approx t_n \in \mathcal{R}\ (\mathcal{F})$$

Unraveling $U(\mathcal{R})$

$$\alpha : \ell \to r \Leftarrow s_1 \approx t_1, s_2 \approx t_2, \ldots, s_n \approx t_n \in \mathcal{R} \ (\mathcal{F})$$

$$\ell \to U_1^{\alpha}(s_1, \mathsf{v}(\ell))$$

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$$\ell \to U_1^\alpha(s_1, \mathsf{v}(\ell))$$
$$U_1^\alpha(t_1, \mathsf{v}(\ell)) \to U_2^\alpha(s_2, \mathsf{v}(\ell), \mathsf{ev}(t_1))$$

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$$U_1^\alpha(t_1, \mathsf{v}(\ell)) \to U_2^\alpha(s_2, \mathsf{v}(\ell), \mathsf{ev}(t_1))$$

$$\vdots$$

$$U_n^\alpha(t_n, \mathsf{v}(\ell), \mathsf{ev}(t_1, \ldots, t_{n-1})) \to r$$

Context-Sensitive Unraveling $U_{\mathsf{CS}}(\mathcal{R})$

$$\alpha\colon \ell \to r \Leftarrow s_1 \approx t_1, s_2 \approx t_2, \ldots, s_n \approx t_n \in \mathcal{R}\ (\mathcal{F})$$

$$\begin{aligned} \ell &\to U_1^\alpha(s_1, \mathsf{v}(\ell)) \\ U_1^\alpha(t_1, \mathsf{v}(\ell)) &\to U_2^\alpha(s_2, \mathsf{v}(\ell), \mathsf{ev}(t_1)) \\ &\vdots \\ U_n^\alpha(t_n, \mathsf{v}(\ell), \mathsf{ev}(t_1, \ldots, t_{n-1})) &\to r \end{aligned}$$

Replacement map

if $f/k \in \mathcal{F}$ then $\{1, \ldots, k\}$ else $\{1\}$

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$$\begin{aligned}
\ell &\to U_1^\alpha(s_1, \mathsf{v}(\ell)) \\
U_1^\alpha(t_1, \mathsf{v}(\ell)) &\to U_2^\alpha(s_2, \mathsf{v}(\ell), \mathsf{ev}(t_1)) \\
&\ \vdots \\
U_n^\alpha(t_n, \mathsf{v}(\ell), \mathsf{ev}(t_1, \ldots, t_{n-1})) &\to r
\end{aligned}$$

Replacement map

if $f/k \in \mathcal{F}$ then $\{1, \ldots, k\}$ else $\{1\}$

Theorem (Simulation completeness)

$\to_\mathcal{R} \subseteq \to^+_{U_{\mathsf{CS}}(\mathcal{R})}$

Context-Sensitive Rewriting

μ-termination

CSRS $(\mathcal{R}, \mu)$, $\rightarrow_{\mathcal{R},\mu}$ terminating:
$\mathcal{R}$ μ-*terminating*

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μ-termination on original terms

CSRS $(\mathcal{R}, \mu)$, no infinite $\to_{\mathcal{R},\mu}$-reductions from $\mathcal{T}(\mathcal{F}, \mathcal{V})$:
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μ-restricted proper subterm relation

$\rhd$ restricted to positions induced by μ: $\rhd_\mu$

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μ-termination on original terms

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Lemma

$\rhd_\mu \cdot \rightarrow_\mu \subseteq \rightarrow_\mu \cdot \rhd_\mu$

Quasi-Decreasingness

DCTRS $\mathcal{R}$ ($\mathcal{F}$) is *quasi-decreasing* if there is $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$:

1. well-founded $\succ$

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Quasi-Decreasingness

DCTRS $\mathcal{R}$ ($\mathcal{F}$) is *quasi-decreasing* if there is $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

1. well-founded $\succ$
2. $\succ = (\succ \cup \rhd)^+$
3. $\to_{\mathcal{R}} \subseteq \succ$
4. $\forall \ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n \in \mathcal{R}$,
   $\sigma \colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$, $0 \leqslant i < n$:
   $\forall 1 \leqslant j \leqslant i.\ s_j\sigma \to_{\mathcal{R}}^* t_j\sigma \longrightarrow \ell\sigma \succ s_{\mathsf{i+1}}\sigma$

Proof

Theorem

DCTRS $\mathcal{R}$:
$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F}, \mathcal{V}) \implies \mathcal{R}$ quasi-decreasing

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Proof outline

Assume $U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ and find $\succ$:

1. well-founded $\succ$
2. $\succ = (\succ \cup \rhd)^+$
3. $\to_{\mathcal{R}} \subseteq \succ$
4. $\forall \ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n \in \mathcal{R}$,
   $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$, $0 \leqslant i < n$:
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Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

$\succ \stackrel{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ $(\star)$

1 well-foundedness of $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

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1. well-foundedness of $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

(1) $t_1 \succ t_2 \succ t_3 \succ \ldots \qquad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

Proof I

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(1) $t_1 \succ t_2 \succ t_3 \succ \dots \qquad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

(2) $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})}$ well-founded on $\mathcal{T}(\mathcal{F},\mathcal{V})$ by $(\dagger)$

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

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(1) $t_1 \succ t_2 \succ t_3 \succ \dots \qquad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

(2) $\to_{U_{\mathsf{CS}}(\mathcal{R})}$ well-founded on $\mathcal{T}(\mathcal{F},\mathcal{V})$ by $(\dagger)$

(3) $s_1 \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \qquad s_1 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ $(\dagger)$

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(3) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \quad s_1 \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ assume

(4) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdots$ by $\rhd_\mu \cdot \rightarrow_\mu \subseteq \rightarrow_\mu \cdot \rhd_\mu$

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $\quad (\dagger)$

$\succ \overset{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ $\quad (\star)$

1. well-foundedness of $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

(1) $t_1 \succ t_2 \succ t_3 \succ \ldots \qquad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ — assume

(2) $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})}$ well-founded on $\mathcal{T}(\mathcal{F},\mathcal{V})$ — by $(\dagger)$

(3) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \qquad s_1 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ — assume

(4) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdots$ — by $\rhd_\mu \cdot \rightarrow_\mu \subseteq \rightarrow_\mu \cdot \rhd_\mu$

(5) $\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V})$. t is $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating — by $\lightning$ with (2)

Proof I

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(1) $t_1 \succ t_2 \succ t_3 \succ \ldots \quad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

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(3) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \quad s_1 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

(4) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdots$ by $\rhd_\mu \cdot \rightarrow_\mu \subseteq \rightarrow_\mu \cdot \rhd_\mu$

(5) $\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V})$. t is $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating by $\lightning$ with (2)

(6) $\rhd_\mu$ well-founded by definition

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

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(1) $t_1 \succ t_2 \succ t_3 \succ \dots \quad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

(2) $\to_{U_{\mathsf{CS}}(\mathcal{R})}$ well-founded on $\mathcal{T}(\mathcal{F},\mathcal{V})$ by $(\dagger)$

(3) $s_1 \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \quad s_1 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

(4) $s_1 \to_{U_{\mathsf{CS}}(\mathcal{R})} \cdot \to_{U_{\mathsf{CS}}(\mathcal{R})} \cdots$ by $\rhd_\mu \cdot \to_\mu \subseteq \to_\mu \cdot \rhd_\mu$

(5) $\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V}).\ t$ is $\to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating by $\lightning$ with (2)

(6) $\rhd_\mu$ well-founded by definition

(7) case analysis on # of $U_{\mathsf{CS}}(\mathcal{R})$-steps

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ (†)

$\succ \stackrel{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ (⋆)

1 well-foundedness of $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

(1) $t_1 \succ t_2 \succ t_3 \succ \ldots \qquad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

(2) $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})}$ well-founded on $\mathcal{T}(\mathcal{F},\mathcal{V})$ by (†)

(3) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \qquad s_1 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

(4) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdots$ by $\rhd_\mu \cdot \rightarrow_\mu \subseteq \rightarrow_\mu \cdot \rhd_\mu$

(5) $\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V})$. t is $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating by $\lightning$ with (2)

(6) $\rhd_\mu$ well-founded by definition

(7) case analysis on # of $U_{\mathsf{CS}}(\mathcal{R})$-steps

- finite:

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ ($\dagger$)

$\succ \stackrel{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ ($\star$)

1 well-foundedness of $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

(1) $t_1 \succ t_2 \succ t_3 \succ \ldots \quad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ — assume

(2) $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})}$ well-founded on $\mathcal{T}(\mathcal{F},\mathcal{V})$ — by ($\dagger$)

(3) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \quad s_1 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ — assume

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(5) $\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V}).\ t$ is $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating — by $\lightning$ with (2)

(6) $\rhd_\mu$ well-founded — by definition

(7) case analysis on $\#$ of $U_{\mathsf{CS}}(\mathcal{R})$-steps

- finite: $\lightning$ — by (6)

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

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1 well-foundedness of $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

(1) $t_1 \succ t_2 \succ t_3 \succ \dots \qquad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ assume

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(5) $\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V}).\ t$ is $\to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating by $\lightning$ with (2)

(6) $\rhd_\mu$ well-founded by definition

(7) case analysis on # of $U_{\mathsf{CS}}(\mathcal{R})$-steps

- finite: $\lightning$ by (6)
- infinite:

Proof I

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

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1 well-foundedness of $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$:

(1) $t_1 \succ t_2 \succ t_3 \succ \dots \quad \forall t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})$ — assume

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(3) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots \quad s_1 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ — assume

(4) $s_1 \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdot \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdots$ — by $\rhd_\mu \cdot \rightarrow_\mu \subseteq \rightarrow_\mu \cdot \rhd_\mu$

(5) $\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V}).\ t$ is $\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating — by $\lightning$ with (2)

(6) $\rhd_\mu$ well-founded — by definition

(7) case analysis on # of $U_{\mathsf{CS}}(\mathcal{R})$-steps

- finite: $\lightning$ — by (6)
- infinite: $\lightning$ — by (5)

Proof II

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

$\succ \overset{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ $(\star)$

2 $(\succ \cup \rhd)^+ \subseteq \succ$:

Proof II

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ $(\dagger)$

$\succ \stackrel{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V}))$ $(\star)$

2 $(\succ \cup \rhd)^+ \subseteq \succ$:

- $s \; (\succ \cup \rhd)^{n+1} \; t$ assume

Proof II

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

$\succ \stackrel{\text{def}}{=} (\to_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ $(\star)$

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- induction on n

Proof II

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2 $(\succ \cup \rhd)^+ \subseteq \succ$:

- $s\ (\succ \cup \rhd)^{n+1}\ t$ assume
- induction on n
- $s\ (\succ \cup \rhd)\ t$ base case $(n = 0)$

Proof II

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2 $(\succ \cup \rhd)^+ \subseteq \succ$:

- $s\ (\succ \cup \rhd)^{n+1}\ t$ assume
- induction on n
- $s\ (\succ \cup \rhd)\ t$ base case $(n = 0)$
  - $s \rhd t$

Proof II

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ $(\dagger)$

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- $s \ (\succ \cup \rhd)^{n+1} \ t$ assume
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- $s \ (\succ \cup \rhd) \ t$ base case $(n = 0)$
  - $s \rhd t$
  - $s \rhd_\mu t$ because $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$

Proof II

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

$\succ \overset{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ $(\star)$

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  - $s \succ t$ — by $(\star)$
- $s\ (\succ \cup \rhd)\ u\ (\succ \cup \rhd)^{k+1}\ t$ — step case $(n = k + 1)$
  - $s \succ u$ — see base case

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  - $u \succ t$ — by induction hypothesis

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  - $s \succ u$ — see base case
  - $u \succ t$ — by induction hypothesis
  - $s \succ t$ — by transitivity

Proof III

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ $(\dagger)$

$\succ \stackrel{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V}))$ $(\star)$

3 $\rightarrow_{\mathcal{R}} \subseteq \succ$

Proof III

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ $(\dagger)$

$\succ \overset{\text{def}}{=} (\rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ $(\star)$

3 $\rightarrow_{\mathcal{R}} \subseteq \succ$

- $s \rightarrow_{\mathcal{R}} t$ assume

Proof III

$U_{\mathsf{CS}}(\mathcal{R})$ μ-terminating on $\mathcal{T}(\mathcal{F},\mathcal{V})$ (†)

$\succ \stackrel{\text{def}}{=} (\to_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V}))$ ($\star$)

3 $\to_{\mathcal{R}} \subseteq \succ$

- $s \to_{\mathcal{R}} t$ — assume
- $s \to^+_{U_{\mathsf{CS}}(\mathcal{R})} t$ — by $\to_{\mathcal{R}} \subseteq \to^+_{U_{\mathsf{CS}}(\mathcal{R})}$

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4. $\forall \ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n \in \mathcal{R}$,
$\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$, $0 \leqslant i < n$:
$\forall 1 \leqslant j \leqslant i.\ s_j\sigma \to_{\mathcal{R}}^* t_j\sigma \longrightarrow \ell\sigma \succ s_{\mathsf{i+1}}\sigma$

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(1) $\ell\sigma \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} U_1^\rho(s_1, \mathsf{v}(\ell))\sigma \rightarrow^*_{\mathcal{R}} U_1^\rho(t_1, \mathsf{v}(\ell))\sigma \rightarrow_{U_{\mathsf{CS}}(\mathcal{R})} \cdots$

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(2) $\ell\sigma \to_{U_{\mathsf{CS}}(\mathcal{R})}^+ U_{i+1}^\rho(s_{i+1}, \mathsf{v}(\ell), \mathsf{ev}(t_1, \ldots, t_i))\sigma$ by $\to_{\mathcal{R}} \subseteq \to_{U_{\mathsf{CS}}(\mathcal{R})}^+$

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(4) $\ell\sigma \rightarrow^+_{U_{\mathsf{CS}}(\mathcal{R})} \cdot \rhd_\mu s_{i+1}\sigma$ from (2) and (3)

(5) $\ell\sigma \succ s_{i+1}\sigma$ by $(\star)$ because $\ell\sigma, s_{i+1}\sigma \in \mathcal{T}(\mathcal{F},\mathcal{V})$

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Automation & Experiments

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presently only VMTL supports μ-termination *on original terms*

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DCTRS $\mathcal{R}$:
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Remarks

- VMTL *cannot* show μ-non-termination on original terms
- MU-TERM currently only tool that can show non-quasi-decreasingness of DCTRS $\mathcal{R}$

Experiments

	$\mathcal{R}$			$U_{\mathsf{CS}}(\mathcal{R})$		
	AProVE	MU-TERM	VMTL	AProVE	MU-TERM	VMTL
YES	80	78	80	78	78	79
NO	–	12	–	–	–	–

	$U(\mathcal{R})$					
	AProVE	MU-TERM	NaTT	$\mathsf{T_TT_2}$	VMTL	total
YES	81	78	77	78	78	84
NO	–	–	–	–	–	12

Table: (Non-)quasi-decreasingness of 103 DCTRSs from Cops.

Conclusion

Summary

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Further talks

- Thursday @ AJSW: Other characterization of quasi-decreasingness
- Friday @ IWC: Application of quasi-decreasingness for confluence of CTRSs

Appendix

Context-Sensitive Quasi-Reductivity

CSRS $\mathcal{R}$ ($\mathcal{F}$) is *context-sensitively quasi-reductive* if there is

- $\mathcal{F}' \supseteq \mathcal{F}$
- μ (with $\mu(f) = \{1, \ldots, n\}$ for every n-ary $f \in \mathcal{F}$)
- partial order $\succ_\mu$ on $\mathcal{T}(\mathcal{F}', \mathcal{V})$

such that:

- well-founded $\succ_\mu$
- μ-monotonic $\succ_\mu$
- $\forall \ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_k \approx t_k,\ \sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V}),\ 0 \leqslant i \leqslant k-1$:
  - $\forall 1 \leqslant j \leqslant i.\ s_j\sigma \succeq_\mu t_j\sigma \longrightarrow \ell\sigma\ (\succ_\mu \cup \rhd_\mu)^+\ s_{i+1}\sigma$
  - $\forall 1 \leqslant j \leqslant k.\ s_j\sigma \succeq_\mu t_j\sigma \longrightarrow \ell\sigma \succ_\mu r\sigma$