Type Introduction for Confluence Proofs

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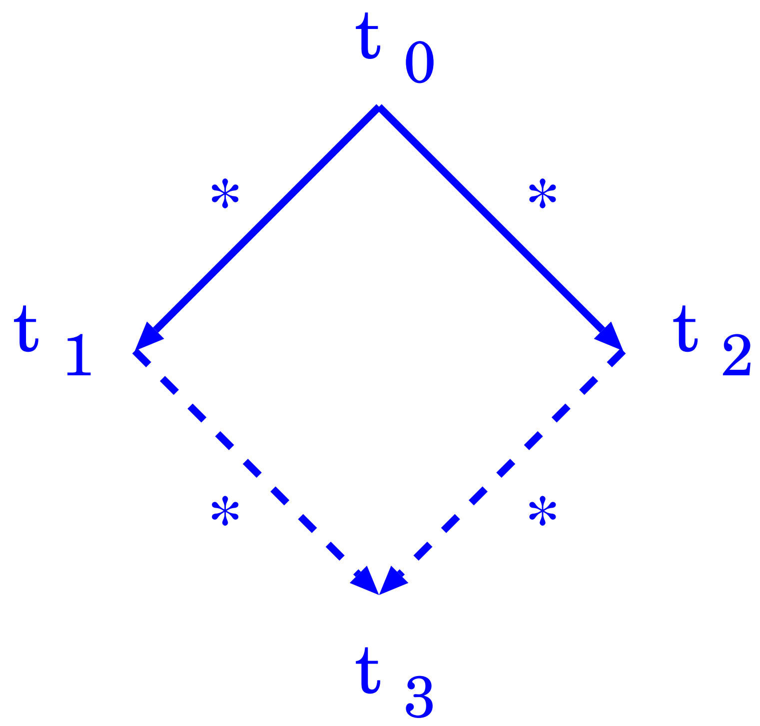
Joint Work with Tsubasa Suzuki and Takahito Aoto

Confluence



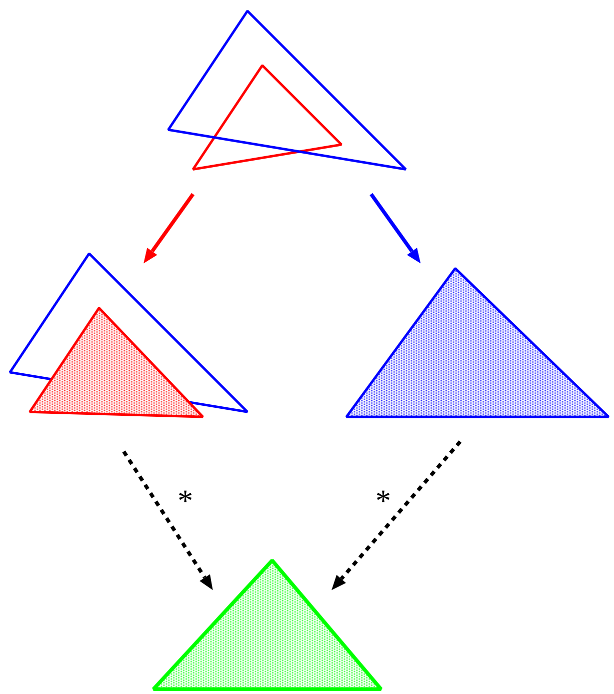
Confluence implies at most one normal form for any term. Thus, confluent term rewriting systems give flexible computation and effective deduction for equational systems.

Classical Criteria for Confluence

- Terminating TRS is confluent iff all critical pairs in it are joinable (Knuth and Bendix 1970).
- Left-linear non-overlapping TRS is confluent (Rosen 1973).
- Left-linear parallel-closed TRS is confluent (Huet 1980).

Confluence Criterion for Terminating TRS

- **Terminating TRS is confluent iff all critical pairs in it are joinable (Knuth and Bendix 1970).**



Thus confluence of terminating TRSs is decidable.

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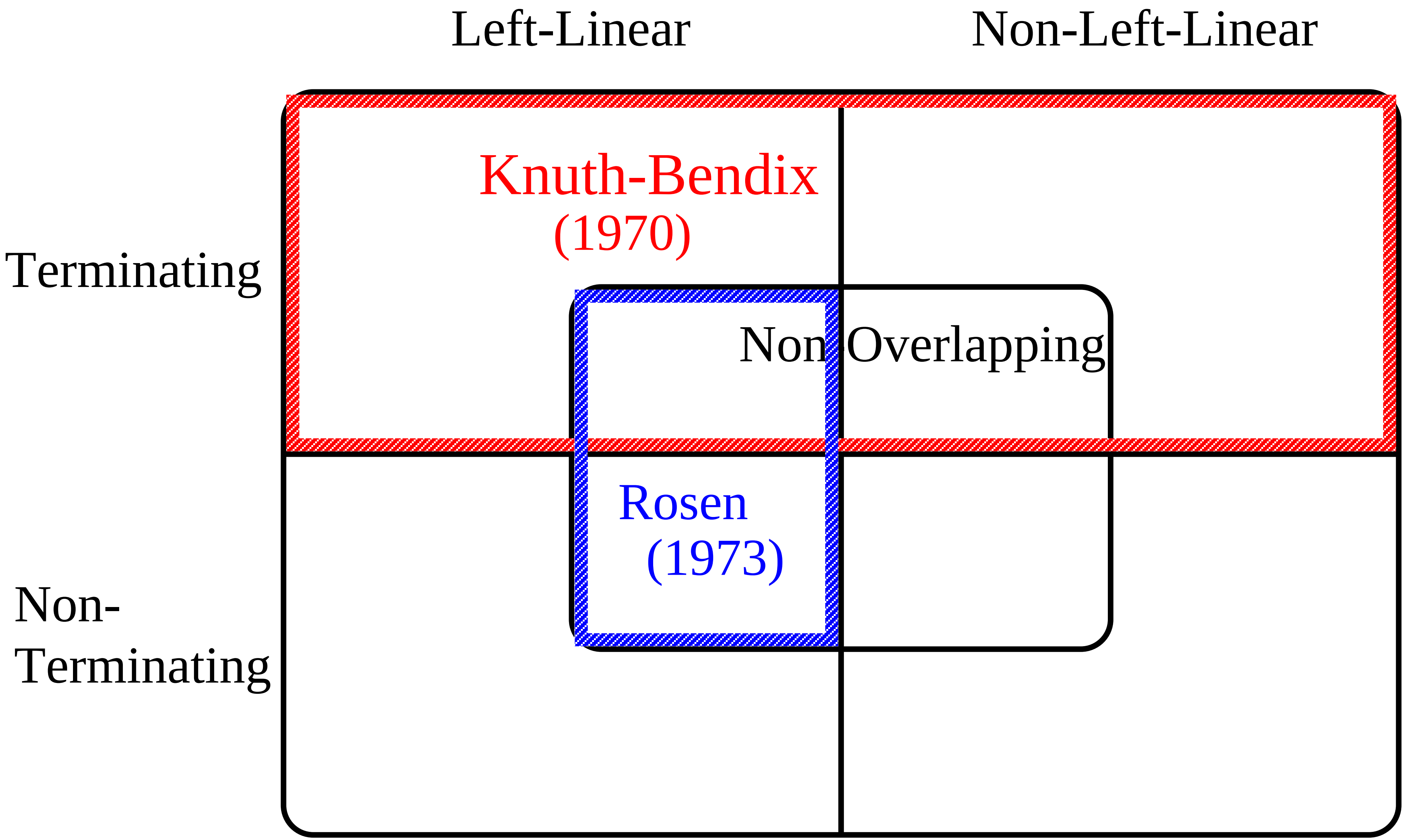
Term is linear if no variable occurs more than once.

TRS is left-linear if the left-hand side is linear for every rewrite rule.

TRS is non-overlapping if it has no critical pairs.

Confluence Criteria



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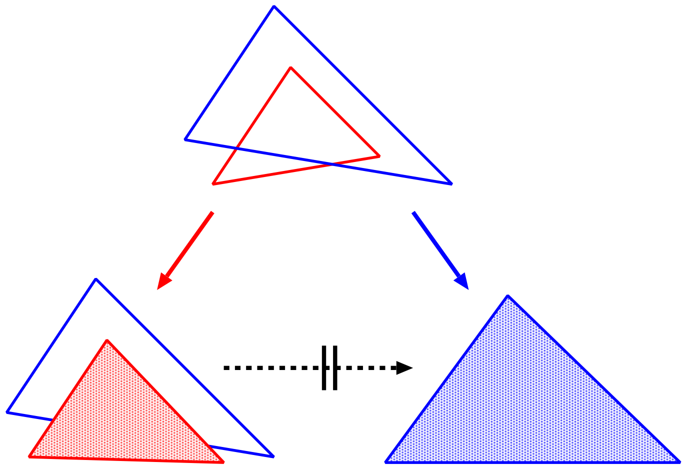
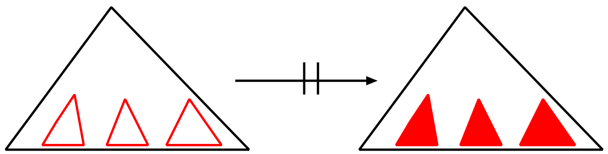
Huet criterion for left-linear TRS was extended by Toyama (1981, 1988), van Oostrom (1995), Gramlich (1996), Oyamaguchi and Ohta (1997, 2003), Okui (1998), et al.

Confluence Criterion (Huet 1980)

- Left-linear TRS is confluent if every critical pair satisfies parallel-closed.



Parallel reduction is defined by



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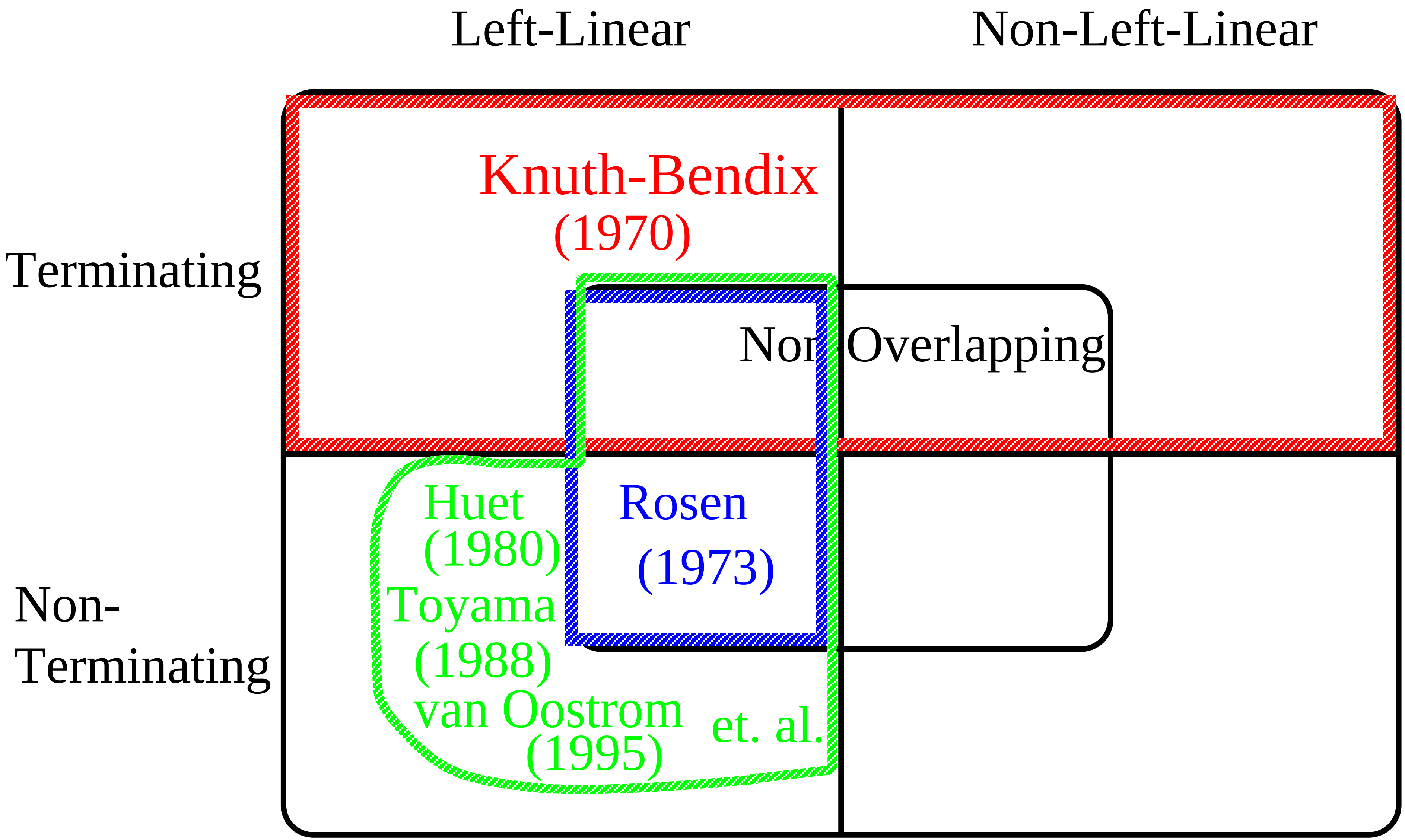

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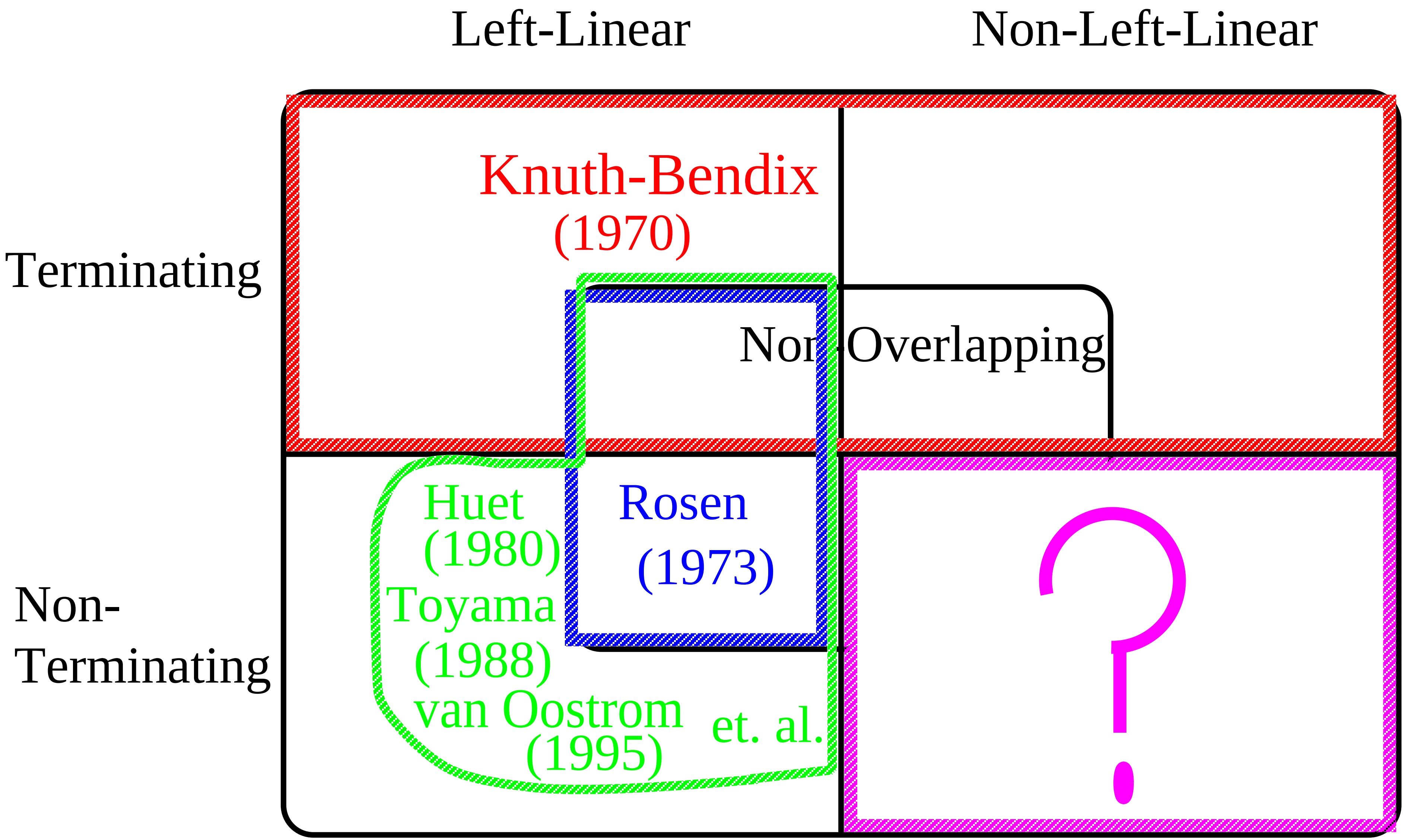

Criteria for Non-Left-Linear Non-Terminating TRS?

Questions:

Is a left-linear non-overlapping TRS + parallel-if confluence?

$$\text{parallel-if} \begin{cases} \mathsf{if}(\mathsf{true}, x, y) \to x \\ \mathsf{if}(\mathsf{false}, x, y) \to y \\ \mathsf{if}(z, x, x) \to x \end{cases}$$

(O'Donnell 1977)

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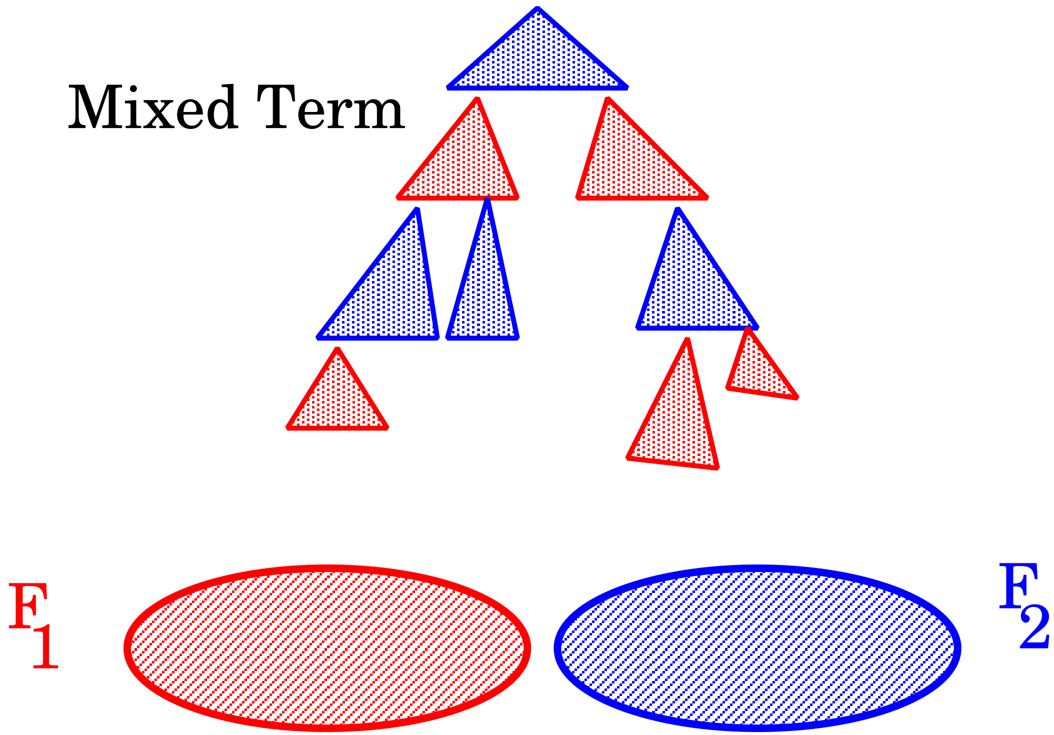
Note that we cannot apply all the confluence criteria which have been mentioned to this problem.

Direct Sum of TRSs

Let $\mathcal{R}_1$ on $\mathcal{F}_1$ and $\mathcal{R}_2$ on $\mathcal{F}_2$ be two TRSs with $\mathcal{F}_1 \cap \mathcal{F}_2 = \phi$. Then the direct sum $\mathcal{R}_1 \oplus \mathcal{R}_2$ is defined as the new TRS $\mathcal{R}_1 \cup \mathcal{R}_2$ on $\mathcal{F}_1 \cup \mathcal{F}_2$.



Modularity of Confluence

R_1 **and** R_2 **are confluent** $\iff$ $R_1 \oplus R_2$ **is confluent.**

(Toyama 1987)

Modularity of Confluence

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Example: Let $\mathcal{R}$ on $\mathcal{F}$ be a left-linear non-overlapping TRS, and let $\mathcal{F} \cap \{\mathsf{if}, \mathsf{true}, \mathsf{false}\} = \phi$. Then $\mathcal{R}$ + parallel-if is confluent.

$$\text{parallel-if} \begin{cases} \mathsf{if}(\mathsf{true}, x, y) \to x \\ \mathsf{if}(\mathsf{false}, x, y) \to y \\ \mathsf{if}(z, x, x) \to x \end{cases}$$

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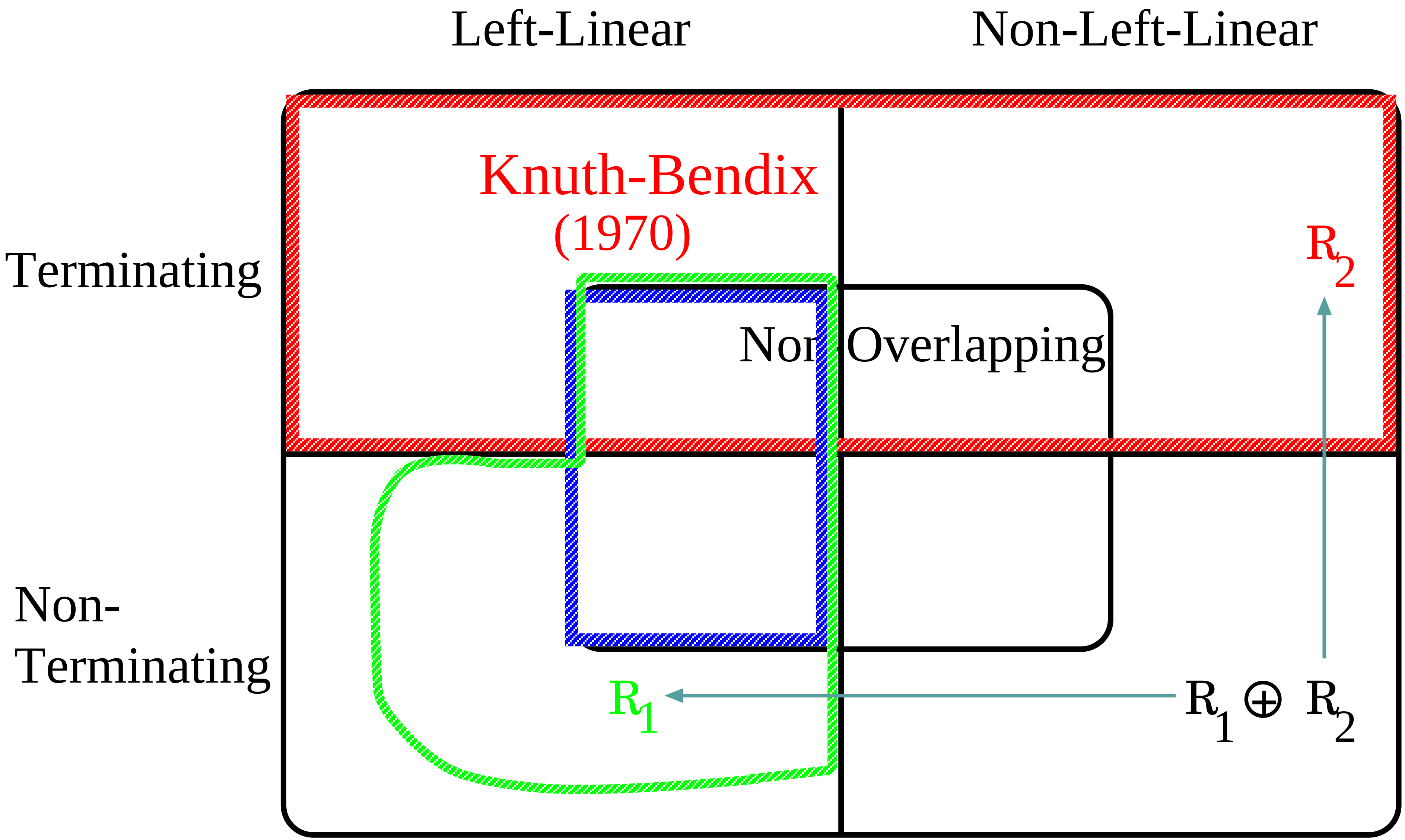
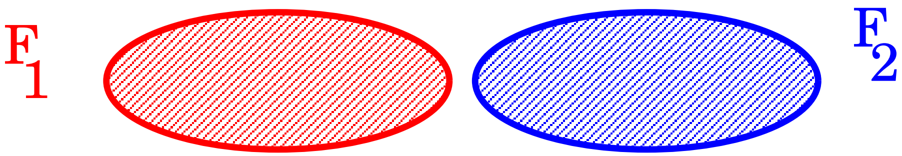
Note that $\mathcal{R}$ is confluent from Rosen criterion, and parallel-if is confluent from Knuth-Bendix criterion.

Divide and Conquer by Modularity



Confluence Criteria for Non-Disjoint Union

R_1 and R_2 are confluent $\Longleftrightarrow$ $R_1 \oplus R_2$ is confluent.



Drawback:

The disjointness requirement $\mathcal{F}_1 \cap \mathcal{F}_2 = \phi$ is too strong.

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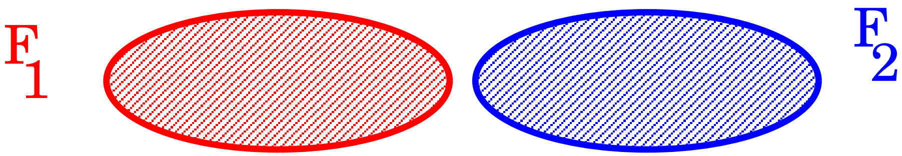

Drawback:
The disjointness requirement $\mathcal{F}_1 \cap \mathcal{F}_2 = \phi$ is too strong.

Persistency (Zantema 1994) generalizes the modularity w.r.t. direct sum.

For any simple type τ,
R^τ has property $P \iff R$ has property P.

Introducing Simple Type τ

- Typed Terms

$$\frac{f : \sigma_1 \times \cdots \times \sigma_n \to \sigma, \ t_1 : \sigma_1, \cdots, t_n : \sigma_n}{f(t_1, \cdots, t_n) : \sigma}$$

- T^σ is the set of all terms with type σ.
- For every rule $l \to r \in R$, l and r have the same type.
- If $x : \sigma$ then $x\theta : \sigma$.
- $R^\tau_\sigma \subseteq R^\tau$ is the set of all rules applicable to some term with type σ.

Persistency of Confluence

R^τ is confluent for some simple type $\tau \Rightarrow R$ is confluent.

(Aoto and Toyama 1997)

$$R \quad \begin{cases} f(x) \to g(x) \\ a(x,y) \to a(f(x), f(x)) \\ b(f(x), x) \to b(x, f(x)) \\ b(g(x), x) \to b(x, g(x)) \end{cases}$$

We have the most general type

$$\tau \quad \begin{cases} f : 0 \to 0 \\ g : 0 \to 0 \\ a : 0 \times 0 \to 1 \\ b : 0 \times 0 \to 2 \end{cases}$$

Then R^τ is confluent because ......

Persistency of Confluence

$$R_0^\tau \quad \left\{ f(x) \to g(x) \right.$$

$$R_1^\tau \quad \left\{ \begin{array}{l} f(x) \to g(x) \\ a(x,y) \to a(f(x), f(x)) \end{array} \right.$$

$$R_2^\tau \quad \left\{ \begin{array}{l} f(x) \to g(x) \\ b(f(x),x) \to b(x, f(x)) \\ b(g(x),x) \to b(x, g(x)) \end{array} \right.$$

$$\tau \quad \left\{ \begin{array}{l} f : 0 \to 0 \\ g : 0 \to 0 \\ a : 0 \times 0 \to 1 \\ b : 0 \times 0 \to 2 \end{array} \right.$$

R_0^τ and R_1^τ are confluent by Rosen's criterion, and R_2^τ is confluent by Knuth-Bendix's criterion.

Undecomposable TRSs

$$R \quad \begin{cases} f(x,x) \to f(g(x),x) \\ f(g(x),x) \to f(h(x),h(x)) \\ h(g(x)) \to g(g(h(x))) \end{cases}$$

We have the most general type

$$\tau \quad \begin{cases} f : 0 \times 0 \to 1 \\ g : 0 \to 0 \\ h : 0 \to 0 \end{cases}$$

But $R_1^\tau = R^\tau$.

Elimination of Left-Non-Linearity

Let $C_V = \{c_x \mid x \in V\}$.

$$f(g(x, y), y) \in T(F, V) \Leftrightarrow f(g(c_x, c_y), c_y) \in T(F \cup C_V)$$

From now on we consider R over $T(F \cup C_V)$ instead of R over $T(F, V)$.

Note that R over $T(F \cup C_V)$ is confluent iff so is R over $T(F, V)$.

Elimination of Left-Non-Linearity

$$R \quad \begin{cases} f(x,x) \to f(g(x),x) \\ f(g(x),x) \to f(h(x),h(x)) \\ h(g(x)) \to g(g(h(x))) \end{cases}$$

$$\tau \quad \begin{cases} f : 0 \times 0 \to 1 \\ g : 0 \to 0 \\ h : 0 \to 0 \end{cases}$$

$$R_0^\tau \quad \{\ h(g(x)) \to g(g(h(x)))$$

Note that every left-non-linear variable in R^τ has type 0 and R_0^τ is terminating.

Replace left-non-linear variables in R^τ with ground normal terms with type 0.

Elimination of Left-Non-Linearity

$$R \quad \begin{cases} f(x,x) \to f(g(x),x) \\ f(g(x),x) \to f(h(x),h(x)) \\ h(g(x)) \to g(g(h(x))) \end{cases}$$

$$\tau \quad \begin{cases} f : 0 \times 0 \to 1 \\ g : 0 \to 0 \\ h : 0 \to 0 \end{cases}$$

Then we have

$$R^{\tau}_{nf} \quad \begin{cases} f(n,n) \to f(g(n),n) \\ f(g(n),n) \to f(h(n),h(n)) \\ h(g(n)) \to g(g(h(n))) \\ \qquad\qquad \text{for every } n \in T(F \cup C_V)^0 \cap NF(R^{\tau}) \end{cases}$$

Note that R^{τ}_{nf} is left-linear but infinite.

Relation between R^τ and R^τ_{nf}

$$f(s, s) \xrightarrow[R^\tau]{} f(g(s), s)$$

Relation between R^τ and R^τ_{nf}

$$
\begin{tikzcd}
f(s, s) \arrow[r, "R^\tau"'] \arrow[d, shift right=2, "R^\tau"', "\text{in} \ *"] \arrow[d, shift left=2, "\text{in} \ *"] & f(g(s), s) \\
f(n, n) &
\end{tikzcd}
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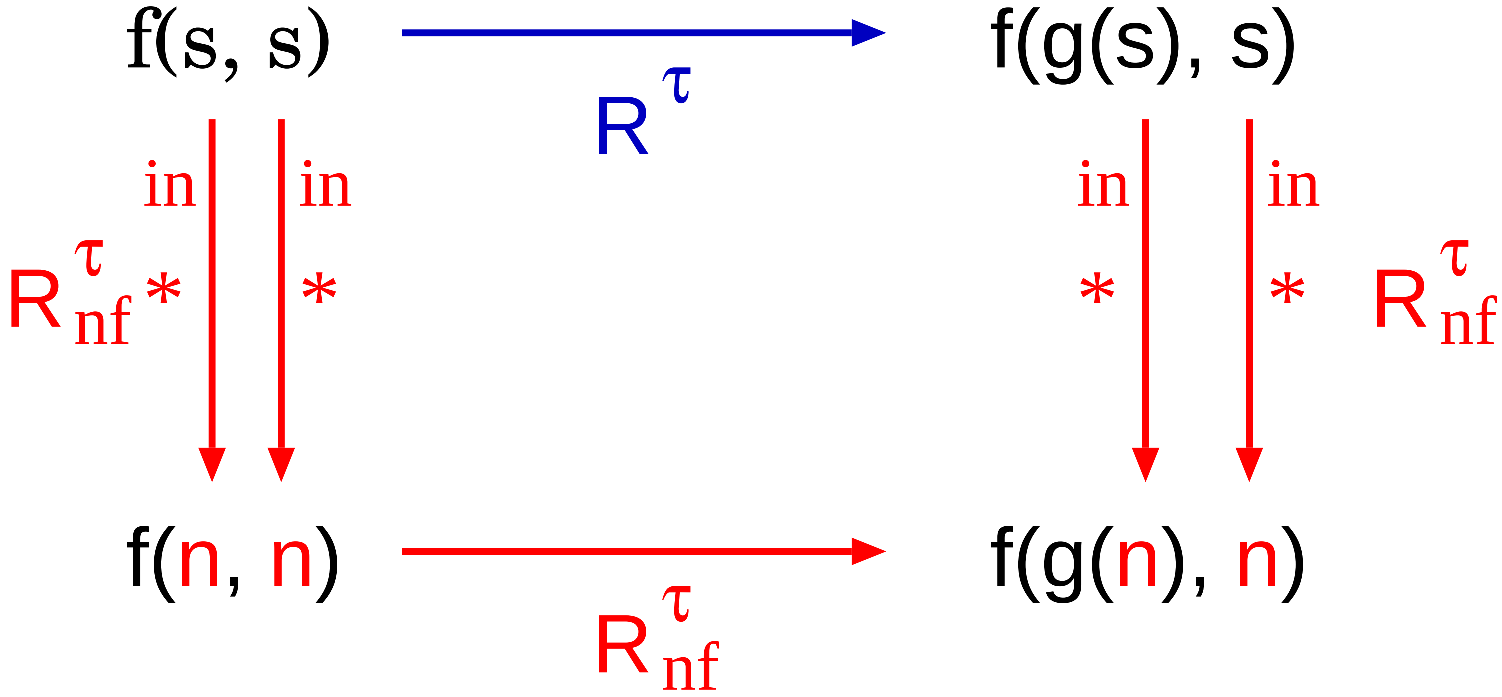
Relation between R^{τ} and R^{τ}_{nf}



Since $\rightarrow_{R^{\tau}_{nf}} \subseteq \rightarrow_{R^{\tau}} \subseteq \overset{*}{\leftrightarrow}_{R^{\tau}_{nf}}$**, we have** $\overset{*}{\leftrightarrow}_{R^{\tau}} = \overset{*}{\leftrightarrow}_{R^{\tau}_{nf}}$**.**

Confluence Criterion of R^τ

$$s \overset{*}{\underset{R^\tau}{\longleftrightarrow}} t$$

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Thus, if R^τ_{nf} is confluent then R^τ is confluent.

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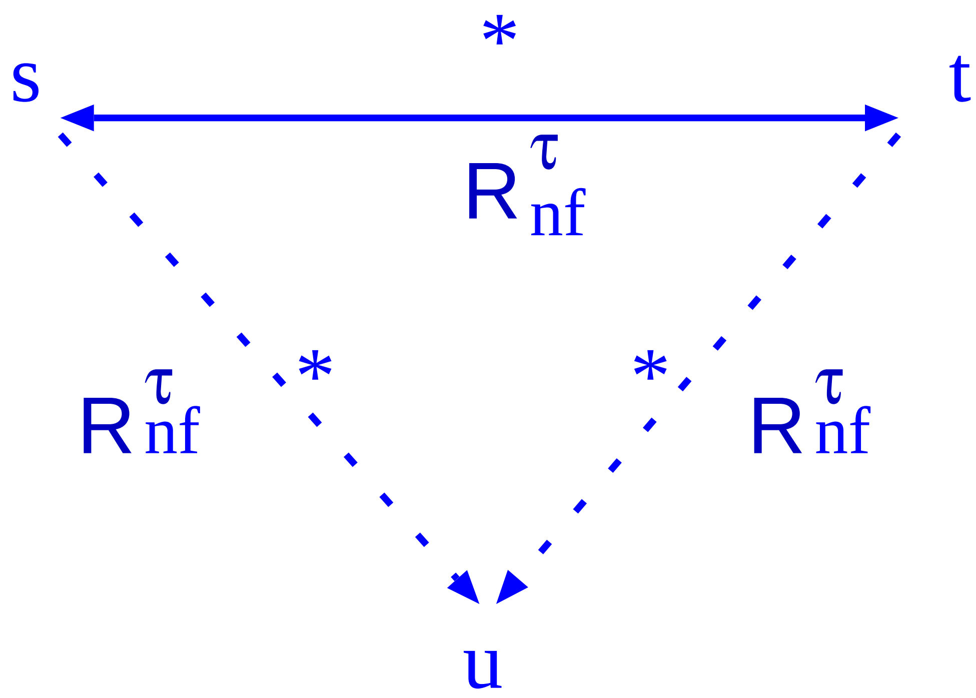

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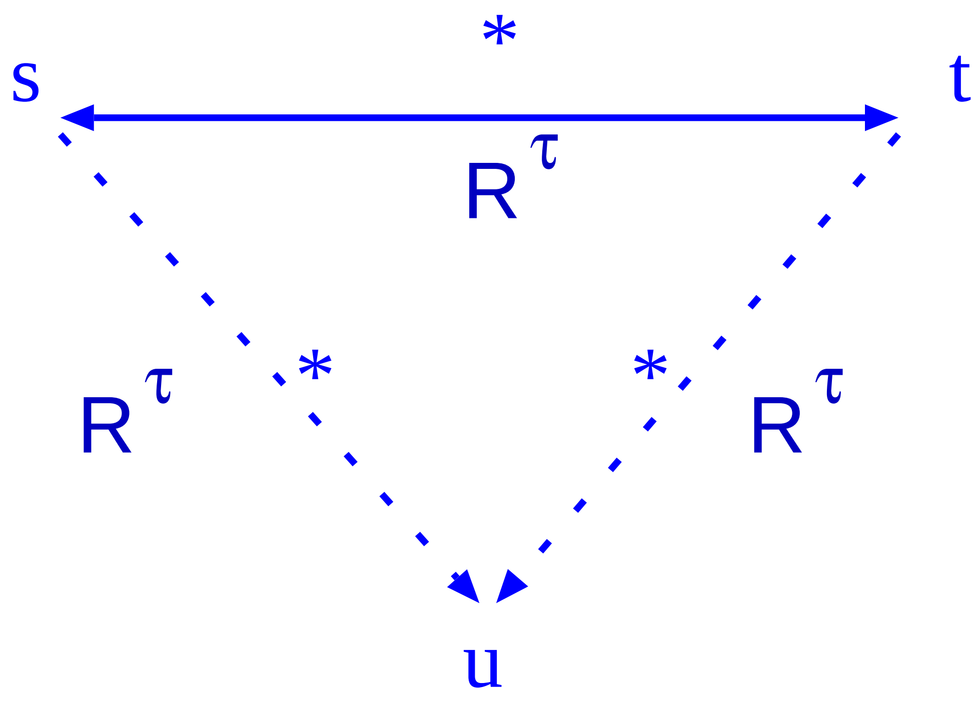

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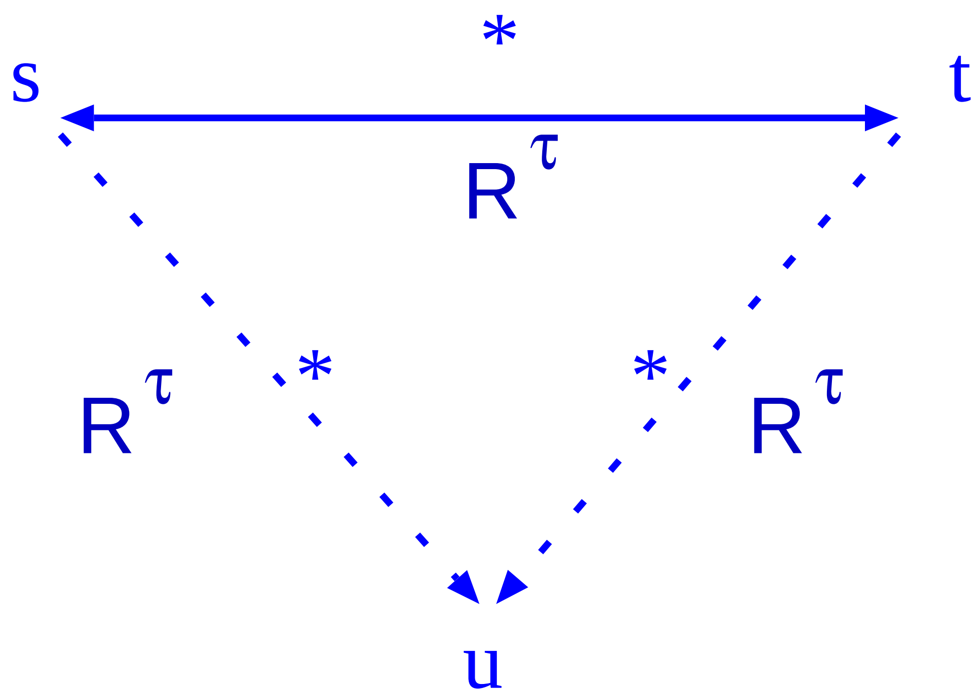

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Thus, if R^τ_{nf} is confluent then R^τ is confluent.

From persistency of confluence,
if R^τ_{nf} is confluent then R is confluent!

Confluence Criterion of R^{τ}_{nf}

- R^{τ}_{nf} is left-linear (and infinite).
- R^{τ}_{nf} is non-overlapping.

From Rosen's criterion, R^{τ}_{nf} is confluent.

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$$l[x, x, y] \to r[x, y, y] \ \in R$$

$$l[x^0, x^0, y^1] \to r[x^0, y^1, y^1] \ \in R^{\tau}$$

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$l[n^0, n^0, y^1] \to r[n^0, y^1, y^1] \ \in R^{\tau}_{nf}$ where n^0 is ground normal.

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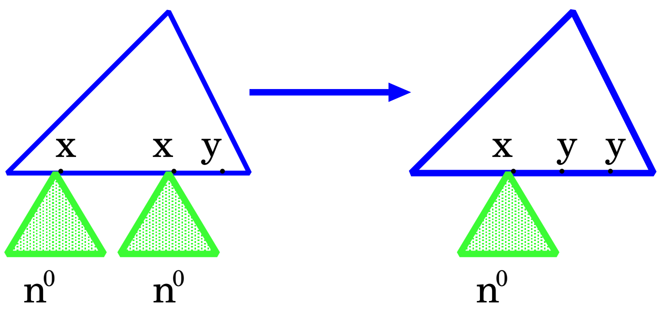

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- R^{τ}_{nf} **is left-linear (and infinite).**
- R **is non-overlapping** $\Rightarrow$ R^{τ}_{nf} **is non-overlapping.**

$l[x, x, y] \to r[x, y, y] \ \in R$

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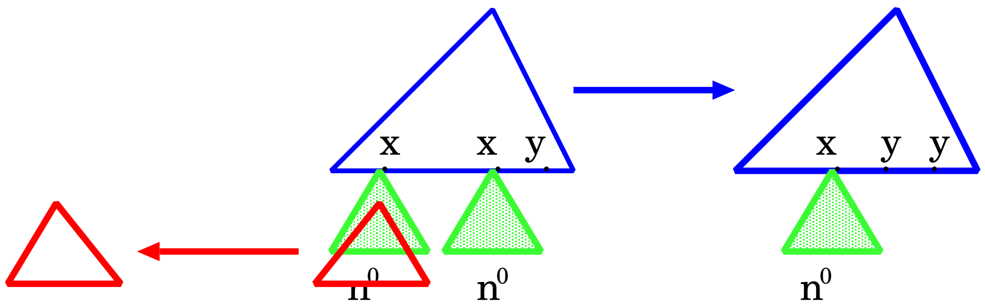

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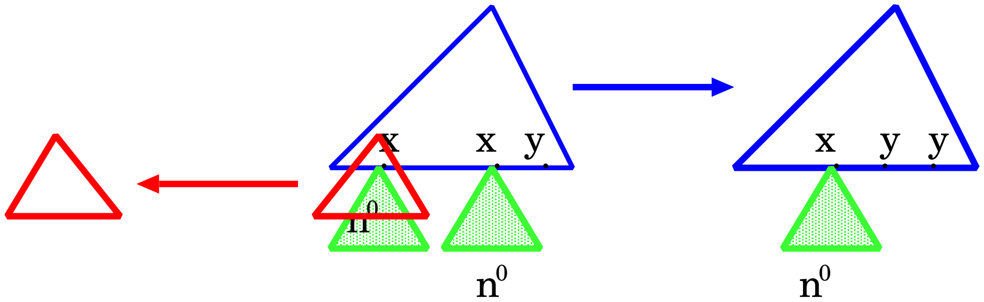

Confluence Criterion of R^{τ}_{nf}

- R^{τ}_{nf} is left-linear (and infinite).
- R is non-overlapping $\Rightarrow$ R^{τ}_{nf} is non-overlapping.

$l[x, x, y] \rightarrow r[x, y, y] \ \in R$

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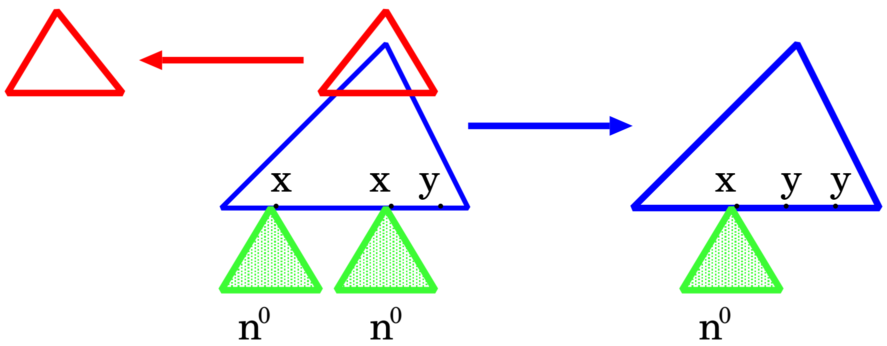

Confluence Criterion of R

- R^{τ}_{nf} is left-linear (and infinite).
- R is non-overlapping $\Rightarrow$ R^{τ}_{nf} is non-overlapping.

From Rosen's criterion, R^{τ}_{nf} is confluent.

Confluence Criterion of R

- R^τ_{nf} is left-linear (and infinite).
- R is non-overlapping $\Rightarrow$ R^τ_{nf} is non-overlapping.

From Rosen's criterion, R^τ_{nf} is confluent.

Theorem. Let R be non-overlapping and have some type τ such that for every left-non-linear variable x^σ in R^τ, R^τ_σ is terminating. Then R is confluent.

Confluence Criterion of R^{τ}_{nf}

- R^{τ}_{nf} is left-linear (and infinite).
- R^{τ}_{nf} is parallel-closed.

From Huet's criterion, R^{τ}_{nf} is confluent.

Confluence Criterion of R^{τ}_{nf}

- R^{τ}_{nf} **is left-linear (and infinite).**
- R **is parallel-closed** $\not\Rightarrow$ R^{τ}_{nf} **is parallel-closed.**

Confluence Criterion of R^{τ}_{nf}

- R^{τ}_{nf} is left-linear (and infinite).
- R is parallel-closed $\Rightarrow$ R^{τ}_{nf} is parallel-closed.
  with left-linear rules

Confluence Criterion of R

- R^{τ}_{nf} is left-linear (and infinite).
- R is parallel-closed $\Rightarrow$ R^{τ}_{nf} is parallel-closed.
  with left-linear rules

From Huet's criterion, R^{τ}_{nf} is confluent.

Theorem. Let R be parallel-closed with left-linear rules and have some type τ such that for every left-non-linear variable x^{σ}, R^{τ}_{σ} is terminating. Then R is confluent.

Conclusion

- **Elimination of left-non-linearity by type introduction**
- **Confluence criteria for typed infinite left-linear TRSs**
- **Implementation of automated procedure**
- **Experiments for several examples**