Proofs of Termination of Rewrite Systems for Polytime Functions

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Abstract. We define a new path order $\prec_{\text{POP}}$ so that for a finite rewrite system R compatible with $\prec_{\text{POP}}$, the *complexity* or *derivation length function* Dl_R^f for each function symbol f is guaranteed to be bounded by a polynomial in the length of the inputs. Our results yield a simplification and clarification of the results obtained by Beckmann and Weiermann (Archive for Mathematical Logic, 36:11–30, 1996).

Keywords: Termination, term rewriting characterisation, derivation length, complexity theory.

1 Introduction

Suppose $\mathcal{C}$ denotes an inductively defined class of recursive number-theoretic functions and suppose each $f \in \mathcal{C}$ is defined via an equation (or more generally a system of equations) of the form

$$f(\mathbf{x}) = t(\lambda \mathbf{y}.f(\mathbf{y}), \mathbf{x}) \; , \tag{1}$$

where t may involve previously defined functions. In a term-rewriting context these defining equations are oriented from left to right and the canonical *term-rewriting characterisation* $R_{\mathcal{C}}$ of $\mathcal{C}$ can be defined as follows: The signature Σ of $R_{\mathcal{C}}$ includes for each function f in $\mathcal{C}$ a corresponding function symbol f. In order to represent natural numbers Σ includes a constant 0 and a unary function symbol S. I.e. numbers are represented by their numerals. (Later we represent natural numbers in the form of binary strings.) For each function $f \in \mathcal{C} - \{0, S\}$, defined by (1), the rule

$$f(\mathbf{x}) \to t(\lambda \mathbf{y}.f(\mathbf{y}), \mathbf{x}) \; ,$$

is added to $R_{\mathcal{C}}$. In all non-pathological cases the term rewrite system (TRS) $R_{\mathcal{C}}$ is terminating and confluent. $R_{\mathcal{C}}$ is best understood as a constructor TRS, where the constructors are 0 and S. Hence $R_{\mathcal{C}}$ may be conceived as a *functional program* implementing the functions in $\mathcal{C}$.

Term-rewriting characterisations have been studied e.g. in [1,2,3,4]. The analysis of $R_{\mathcal{C}}$ provides insight into the structure of $\mathcal{C}$ or renders us with a delineation of a class of rewrite systems whose complexity (measured by the length of derivations) is guaranteed to belong to the class $\mathcal{C}$. Term-rewriting characterisations turn the emphasis form the *definition* of a function f to its *computation*. An essential property of term-rewriting characterisations $R_{\mathcal{C}}$ is its *feasibility*: $R_{\mathcal{C}}$ is called *feasible*, if for each n-ary function $f \in \mathcal{C}$, there exists a function symbol g in the signature of $R_{\mathcal{C}}$ such that $g(\overline{m}_1, \ldots, \overline{m}_n)$ computes the value of $f(m_1, \ldots, m_n)$ and the derivation length of this computation is bounded by a function from $\mathcal{C}$.

We study *term-rewriting characterisations* of the complexity class **FP**. In particular, our starting point is a clever characterisation R'_B of **FP** introduced by Beckmann and Weiermann. In [1] the feasibility of R'_B is established and conclusively shown that any reduction strategy for R'_B yields an algorithm for $f \in \mathbf{FP}$ that runs in polytime. We provide a slight generalisation of the fact that R'_B is feasible. Moreover, we flesh out the crucial ingredients of the TRS R'_B by defining a *path order for* **FP**, denoted as $\prec_{\text{POP}}$. We show that for a finite TRS R, compatible with $\prec_{\text{POP}}$, the *derivation length function* Dl_R^f is bounded by a polynomial in the length of the inputs for any defined function symbol f. Furthermore $\prec_{\text{POP}}$ is *complete* in the sense that for any function $f \in \mathbf{FP}$, there exists a TRS R computing f such that termination of R can be shown by $\prec_{\text{POP}}$.

2 A rewrite system for FP

In the following we need some notions from term rewriting and assume (at least nodding) acquaintance with term rewriting. (For background information, please see [5].) Let $\mathcal{V}$ denote a countably infinite set of variables and Σ a signature. The set of terms over Σ and $\mathcal{V}$ is denoted as $T(\Sigma, \mathcal{V})$, while the set of ground terms is written as $\mathcal{T}(\Sigma)$. The rewrite relation induced by a rewrite system R is denoted as $\rightarrow_R$, and its transitive closure by $\rightarrow_R^*$. We write $\tau(t)$ to denote the *size* of a term t, i.e. the number of symbols in t.

Conventions: Terms are denoted by r, s, t, possibly extended by subscripts. We write $\mathbf{t}$, to denote sequences of terms $t_1, \ldots, t_k \in T(\Sigma, \mathcal{V})$ and $\mathbf{g}$ to denote sequences of function symbols $g_1, \ldots, g_k$, respectively. The letters i, j, k, l, m, n, possible extended by subscripts will always refer to natural numbers. The set of natural numbers is denoted as usual by $\mathbb{N}$.

We consider the class **FP** of *polytime computable functions*, i.e. those functions computable by a deterministic Turing machine M, such that M runs in time $\leq p(n)$ for all inputs of length n, where p denotes a polynomial. We consider equivalent formulations of the class of polytime computable functions in terms of recursion schemes.

Recursion schemes such as *bounded recursion* due to Cobham [6] generate exactly the functions computable in polytime. In contrast to this, Bellantoni-Cook [7] introduce certain *unbounded* recursion schemes that distinguish between arguments as to their position in a function. This separation of variables gives rise to the following definition of the *predicative recursive functions* $\mathcal{B}$; for further details see [7]. We fix a suitable signature of *predicative recursive function symbols* B.

Definition 1. *For $k,l \in \mathbb{N}$ we define $B^{k,l}$ inductively.*

- $S_i^{0,1} \in B^{0,1}$, *where* $i \in [0,1]$.
- $O^{k,l} \in B^{k,l}$.
- $U_r^{k,l} \in B^{k,l}$, *for all* $r \in [1, k+l]$.
- $P^{0,1} \in B^{0,1}$.
- $C^{0,3} \in B^{0,3}$.
- *If* $f \in B^{k',l'}$, $g_1, \dots, g_{k'} \in B^{k,0}$, *and* $h_1, \dots, h_{l'} \in B^{k,l}$, *then* $\mathrm{SUB}_{k',l'}^{k,l}[f, \mathbf{g}, \mathbf{h}] \in B^{k,l}$.
- *If* $g \in B^{k,l}$, $h_0, h_1 \in B^{k+1,l+1}$, *then* $\mathrm{PREC}^{k+1,l}[g, h_1, h_2] \in B^{k+1,l}$.

Set $B := \bigcup_{k,l \in \mathbb{N}} B^{k,l}$.

To simplify notation we usually drop the superscripts, when denoting predicative recursive function symbols. Occasionally, we even write SUB (, PREC), instead of $\mathrm{SUB}^{k,l}[f, \mathbf{g}]$ (,$\mathrm{PREC}^{n+1}[g,h]$). No confusion will arise from this.

The binary successor function $m \mapsto 2m+i$, $i \in \{0,1\}$ is denoted as $\mathcal{S}_i$. Every natural number can be buildt up from 0 with repeated applications of $\mathcal{S}_i$. The binary length of a number m is defined as follows: $|0| := 0$ and $|\mathcal{S}_i(m)| := |m|+1$.

We write $\mathbb{N}^{k,l}$ for $\mathbb{N}^k \times \mathbb{N}^l$ and for $f \colon \mathbb{N}^{k,l} \to \mathbb{N}$, write $f(m_1, \dots, m_k; n_1, \dots, n_l)$ instead of $f(\langle m_1, \dots, m_k \rangle, \langle n_1, \dots, n_l \rangle)$. The arguments occurring to the left of the semi-colon are called *normal*, while the arguments to the right are called *safe*.

We define the following functions: $\mathcal{S}_i^{0,1}$, $i \in \{0,1\}$ denotes the function $\langle ; m \rangle \mapsto 2m+i$. $\mathcal{O}^{k,l}$ denotes the function $\langle \mathbf{m}; \mathbf{n} \rangle \mapsto 0$. $\mathcal{U}_r^{k,l}$ denotes the function $\langle m_1, \dots, m_k; m_{k+1}, \dots, m_{k+l} \rangle \mapsto m_r$. $\mathcal{P}^{0,1}$ denotes the unique number-theoretic function satisfying the following equations: $f(;0) = 0$, $f(;\mathcal{S}_i(m)) = m$. $\mathcal{C}^{0,3}$ denotes the unique function satisfying: $f(;0, m_0, m_1) = m_0$, $f(;\mathcal{S}_i(m), m_0, m_1) = m_i$.

If $f \colon \mathbb{N}^{k',l'} \to \mathbb{N}$, $g_i \colon \mathbb{N}^{k,0} \to \mathbb{N}$ for $i \in [1,k']$, $h_j \colon \mathbb{N}^{k,l} \to \mathbb{N}$ for $j \in [1,l']$, then $\mathcal{SUB}_{k',l'}^{k,l}[f, \mathbf{g}, \mathbf{h}]$ denotes the function $\langle \mathbf{m}; \mathbf{n} \rangle \mapsto f(g_1(\mathbf{m};), \dots, g_{k'}(\mathbf{m};); h_1(\mathbf{m}; \mathbf{n}), \dots, h_{l'}(\mathbf{m}; \mathbf{n}))$.

If $g \colon \mathbb{N}^{k,l} \to \mathbb{N}$, $h_i \colon \mathbb{N}^{k+1,l+1} \to \mathbb{N}$ for $i \in [0,1]$ then $\mathcal{PREC}^{k+1,l}[g, h_1, h_2]$ denotes the unique number-theoretic function f satisfying: $f(0, \mathbf{m}; \mathbf{n}) = g(\mathbf{m}; \mathbf{n})$ and $f(\mathcal{S}_i(m), \mathbf{m}; \mathbf{n}) = h_i(m, \mathbf{m}; \mathbf{n}, f(m, \mathbf{m}; \mathbf{n}))$.

Definition 2. *For $k,l \in \mathbb{N}$ we define $\mathcal{B}^{k,l}$ inductively.*

- $\mathcal{S}_i^{0,1} \in \mathcal{B}^{0,1}$, *where* $i \in [0,1]$.
- $\mathcal{O}^{k,l} \in \mathcal{B}^{k,l}$.
- $\mathcal{U}_r^{k,l} \in \mathcal{B}^{k,l}$, *for all* $r \in [1, k+l]$.
- $\mathcal{P}^{0,1} \in \mathcal{B}^{0,1}$.
- $\mathcal{C}^{0,3} \in \mathcal{B}^{0,3}$.
- *If* $f \in \mathcal{B}^{k',l'}$, $g_1, \dots, g_{k'} \in \mathcal{B}^{k,0}$, *and* $h_1, \dots, h_{l'} \in \mathcal{B}^{k,l}$, *then* $\mathcal{SUB}_{k',l'}^{k,l}[f, \mathbf{g}, \mathbf{h}] \in \mathcal{B}^{k,l}$.

Table 1. A Feasible Term-Rewriting Characterisation of the Predicative Recursive Functions

$O^{k,l}(\mathbf{x};\mathbf{a}) \to 0$,	**[zero]**
$U^{k,l}(x_1,\dots,x_k;x_{k+1},\dots,x_{k+l}) \to x_r$,	**[projection]**
$P^{0,1}(;0) \to 0$,	**[predecessor]**
$P^{0,1}(;S_i(;a)) \to a$,	
$C^{0,3}(;0,a_0,a_1) \to a_0$,	**[conditional]**
$C^{0,3}(;S_i(;a),a_1,a_0) \to a_{2-i}$,	
$\mathrm{SUB}^{k,l}[f,\mathbf{g},\mathbf{h}](\mathbf{x};\mathbf{n}) \to f(\mathbf{g}(\mathbf{x};);\mathbf{h}(\mathbf{x};\mathbf{n}))$,	**[safe composition]**
$\mathrm{PREC}^{k+1,l}[g,h_1,h_2](0,\mathbf{x};\mathbf{n}) \to g(\mathbf{x};\mathbf{n})$,	**[predicative recursion**
$\mathrm{PREC}^{k+1,l}[g,h_1,h_2](S_i(;b),\mathbf{x};\mathbf{n}) \to$	**on notation]**
$\to h_i(b,\mathbf{x};\mathbf{n},\mathrm{PREC}^{k+1,l}[g,h_1,h_2](b,\mathbf{x};\mathbf{n}))$.	

We use the following notation: $i \in [0,1]$ and $r \in [1,k+l]$.

– *If* $g \in \mathcal{B}^{k,l}$, $h_0, h_1 \in \mathcal{B}^{k+1,l+1}$, *then* $\mathcal{PREC}^{k+1,l}[g,h_1,h_2] \in \mathcal{B}^{k+1,l}$.

The set of predicative recursive functions *is defined as* $\mathcal{B} = \bigcup_{k,l} \mathcal{B}^{k,l}$.

It follows from the definitions that for each $f \in B$, there exists a unique predicative recursive function $f^{\mathcal{B}}$; the latter is called the *interpretation* of f in $\mathcal{B}$. For every number m we define its *numeral* $\overline{m} \in T(B,\mathcal{V})$ as follows: $\overline{0} := 0$, $\overline{\mathcal{S}_i(;m)} := S_i(;m)$ for $i \in [0,1]$. We write $\overline{\mathbf{m}}$ to denote a sequence of numerals $\overline{m}_1,\dots,\overline{m}_k$. Now the polytime computable functions **FP** can be defined as follows, see [7]:

$$\mathbf{FP} = \bigcup_k \mathcal{B}^{k,0} \ .$$

In [1] a clever *feasible* term-rewriting characterisation R'_B of the predicative recursive functions $\mathcal{B}$ is given. By Bellantoni's result this yields a feasible term-rewriting characterisation of the class of polytime computable functions **FP**. The (infinite) TRS is given in Table 1.

The TRS R'_B is terminating and confluent. Termination follows by recursive path order (RPO). Confluence is a consequence of the fact that R'_B is orthogonal. Note the restriction in the rewrite rules for *safe composition* and *predicative recursion*. These rules only apply if all *safe* arguments are numerals, i.e. in normal-form. This peculiar restriction is necessary as the canonical term-rewriting characterisation R_B of $\mathcal{B}$, admits exponential lower-bounds, hence R_B is *non-feasible*, compare. [1].

Let R denote a TRS. A *derivation* is a sequence of terms t_i, $i \in \mathbb{N}$, such that for all i, $t_i \to_R t_{i+1}$. The $(i+1)^{th}$ element of a sequence a is denoted as $(a)_i$. We write $\frown$ for the concatenation of sequences and define the length

$|a|$ of a sequence a as usually. We define a partial order $\subseteq$ on pairs of sequences. $a \subseteq b$, if b is an *extension* of a, i.e. $|a| \leq |b|$ and for all $i < |a|$ we have $(a)_i = (b)_i$. A derivation d with $(d)_0 = t$ is called *derivation starting with* t. The *derivation tree* $\mathcal{T}_R(t)$ of t is defined as the structure $(T(t), \subseteq)$, where $T(t) := \{d | d \text{ is a derivation starting with } t\}$. The root of $\mathcal{T}_R(t)$ is denoted by t (instead of (t)).

We measure the *complexity* or *derivation length* of the computation of $f(\overline{\mathbf{m}})$ by the *height* of $\mathcal{T}_R(f(\overline{\mathbf{m}}))$; more concisely we define the *derivation length function* $\mathrm{Dl}_R^f \colon \mathcal{T}(\Sigma) \to \mathbb{N}$:

$$\mathrm{Dl}_R^f(\overline{\mathbf{m}}) := \max\{n \mid \exists\, t_0, \ldots, t_n \in \mathcal{T}(\Sigma)\, (t_n \leftarrow_R \ldots \leftarrow_R t_0 = f(\overline{\mathbf{m}}))\} \, .$$

Based on these definitions we make the notion of *feasible* term-rewriting characterisation precise. A term-rewriting characterisation $R_\mathcal{C}$ of a function class $\mathcal{C}$ is called *feasible*, if for each n-ary function $f \in \mathcal{C}$, there exists a function symbol g in the signature of $R_\mathcal{C}$ such that $g(\overline{m}_1, \ldots, \overline{m}_n)$ computes the value of $f(m_1, \ldots, m_n)$ and $\mathrm{Dl}_{R_\mathcal{C}}^f$ is bounded by a function from $\mathcal{C}$. For the rewrite system R'_B we have the following proposition.

Proposition 1. *For every $f \in \mathcal{B}$, $\mathrm{Dl}_{R'_B}^f$ is bounded by a monotone polynomial in the length of the normal inputs. Specifically for each f we can find a number $\ell(f)$ so that $\mathrm{Dl}_{R'_B}^f(\overline{\mathbf{m}}; \overline{\mathbf{n}}) \leq (2+|\mathbf{m}|)^{\ell(f)}$, where $|\mathbf{m}|$ denotes the sum of the length normal inputs m_i.*

Proof. See [8] for a proof, essentially we employ the observation that the derivation trees $\mathcal{T}_{R'_B}(f(\mathbf{m}; \mathbf{n}))$ are *isomorphic* no matter how the safe input numerals $\mathbf{n}$ vary, to drop the dependency on the length of the normal inputs. □

3 A path ordering for FP

To extend the above results and to facilitate the study of the polytime computable functions in a term-rewriting framework, we introduce in this section a new *path order for* **FP**, which is a *miniaturisation* of the recursive path order, cf. [5], see also [9].

In the definition we make use of an auxiliary varyadic function symbol 'list' of arbitrary, but finite arity, to denote sequences $s_0, \ldots, s_n$ of terms. Instead of $\mathrm{list}(s_0, \ldots, s_n)$ we write $(s_0, \ldots, s_n)$. We write $a \frown b$ for sequences $a = (s_0, \ldots, s_n)$, $b = (s_{n+1}, \ldots, s_{n+m})$ to denote the concatenation $(s_0, \ldots, s_{n+m})$ of a and b.

Let Σ be a signature. We write $T^*(\Sigma, \mathcal{V})$ to denote the set of all finite sequences of terms in $T(\Sigma, \mathcal{V})$. To ensure that $T(\Sigma, \mathcal{V}) \subset T^*(\Sigma, \mathcal{V})$, any term is identified with the sequence $\mathrm{list}(t) = (t)$. We denote sequences by a, b, c, both possible extended with subscripts. Sometimes we write fa as abbreviations of $f(t_0, \ldots, t_n)$, if $a = (t_0, \ldots, t_n)$.

We suppose a partial well-founded relation on S, the *precedence*, denoted as $<$. We write $f \sim g$ if $(f \lesssim g) \wedge (g \lesssim f)$ and we write $f > g$ and $g < f$

interchangeably. Further, we suppose that the signature Σ contains two unary symbols S_0, S_1 of lowest rank in the precedence. I.e. $\Sigma = \{S_0, S_1\} \cup \Sigma'$ and $S_0 \sim S_1$ and for all $f \in \Sigma'$, $S_0, S_1 < f$. Moreover, we define $0 := ()$. For every number m we define its *numeral* $\overline{m} \in T(\Sigma, \mathcal{V})$ as follows: $\overline{0} := ()$; $\overline{\mathcal{S}_i(m)} := S_i(\overline{m})$ for $i \in [0, 1]$.

The definition of the path order for **FP** (POP) $\prec_{\text{POP}}$ (induced by $<$) is based on an auxiliary order $\sqsubset$. The separation in two orders is necessary to break the strength of the recursive path order that induces primitive recursive derivation length, cf. [10].

Definition 3. *Inductive definition of $\sqsubseteq$ induced by $<$.*

1. $\exists j \in [1, n]\, (s \sqsubseteq t_j) \Longrightarrow s \sqsubset f(t_1, \dots, t_n)$,
2. $t = f(t_1, \dots, t_n)$ & $s = g(s_1, \dots, s_m)$ *with* $g < f$ & $\forall i \in [1, m]\, (s_i \sqsubset t)$
   $\Longrightarrow s \sqsubset t$.

Definition 4. *Inductive definition of $\prec_{\text{POP}}$ induced by $<$; $\prec_{\text{POP}}$ is based on $\sqsubset$.*

1. $s \sqsubset t \Longrightarrow s \prec_{\text{POP}} t$,
2. $\exists j \in [1, n]\ (s \preceq_{\text{POP}} t_j) \Longrightarrow s \prec_{\text{POP}} f(t_1, \dots, t_n)$ & $s \prec_{\text{POP}} (t_1, \dots, t_n)$,
3. $t = f(t_1, \dots, t_n)$ & $(m = 0$ *or* $(\exists i_0\ (\forall i \neq i_0\ (s_i \sqsubset t)$ & $s_i \prec_{\text{POP}} t))$
   $\Longrightarrow (s_1, \dots, s_m) \prec_{\text{POP}} t$,
4. $t = f(t_0, \dots, t_n)$ & $s = g(s_0, \dots, s_m)$ *with* $f \sim g$ & $(s_0, \dots, s_m) \prec_{\text{POP}} (t_0, \dots, t_n)$
   $\Longrightarrow s \prec_{\text{POP}} t$,
5. $a \approx a_0 \frown \cdots \frown a_n$ & $\forall i \leq n\, (a_i \preceq_{\text{POP}} b_i)$ & $\exists i \leq n\, (a_i \prec_{\text{POP}} b_i)$
   $\Longrightarrow a \prec_{\text{POP}} (b_0, \dots, b_n)$ *if* $n \geq 1$,

   $a \approx a_0 \frown \cdots \frown a_n$ denotes the fact that the sequence a of terms is obtained from the concatenated $a_0 \frown \cdots \frown a_n$ by permutation.

Note that due to rule 3 $() \prec_{\text{POP}} a$ for any sequence $a \in T^*(\Sigma, \mathcal{V})$. Further, we write $s \succ_{\text{POP}} t$ for $t \prec_{\text{POP}} s$. It is not difficult to argue that $\prec_{\text{POP}}$ is a reduction order. A number of relations are missing; we mention only the following:

- $t = f(t_1, \dots, t_n)$ & $s = g(s_1, \dots, s_m)$ with $g < f$ & $\forall i \in [1, m]\, (s_i \prec_{\text{POP}} t) \Longrightarrow s \prec_{\text{POP}} t$.

We indicate the reasons for the omission of this clause.

Example 1. Consider the following TRS, where Σ contains additionally the symbols a, g, h, f with precedence $a, h < f$, $g < h$.

$$f(0) \to a \qquad f(S_i(x)) \to h(f(x)) \qquad h(x) \to g(x, x) .$$

It is easy to see that $\prec_{\text{POP}}$ cannot handle the TRS in the example, but would if rule above is included. However, note that the TRS admits an *exponential lower-bound* on the derivation length function.

We introduce suitable *approximations* $\prec_k$ of $\prec_{\text{POP}}$.

Definition 5. *Inductive definition of $\sqsubset_k^l$ induced by $<$; we write $\sqsubset_k$ to abbreviate $\sqsubset_k^k$.*

1. $\exists j \in [1,n]\,(s \sqsubseteq_k^l t_j) \Longrightarrow s \sqsubset_k^l f(t_0,\dots,t_n)$,
2. $t = f(t_0,\dots,t_n)$ & $s = g(s_0,\dots,s_m)$ *with* $g < f$ & $m < k$ & $\forall i\,(s_i \sqsubset_k^l t)$ $\Longrightarrow s \sqsubset_k^{l+1} t$.

Definition 6. *Inductive definition of $\prec_k$ induced by $<$; $\prec_k$ is based on $\sqsubset_k$.*

1. $s \sqsubset_k t \Longrightarrow s \prec_k t$,
2. $\exists j \in [1,n]\ (s \preceq_k t_j) \Longrightarrow s \prec_k f(t_1,\dots,t_n)$,
3. $t = f(t_1,\dots,t_n)$ & $(m = 0$ *or* $\exists i_0 \in [1,m]\,(\forall i \neq i_0\,(s_i \sqsubset_k t)$ & $s_{i_0} \prec_k t))$ & $m < k \Longrightarrow (s_1,\dots,s_m) \prec_k t$,
4. $t = f(t_0,\dots,t_n)$ & $s = g(s_0,\dots,s_m)$ *with* $f \sim g$ & $(s_0,\dots,s_m) \prec_k (t_0,\dots,t_n)$ & $m < \max\{k,n\} \Longrightarrow s \prec_k t$,
5. $a \approx a_0 \frown \cdots \frown a_n$ & $\forall i \leq n\ (a_i \preceq_k b_i)$ & $\exists i \leq n\,(a_i \prec_k b_i) \Longrightarrow a \prec_k (b_0,\dots,b_n)$ *if* $n \geq 1$.

In the following we prove that if for a finite rewrite system R, $R \subseteq \prec_{\text{POP}}$, then it even holds that $\to_R \subseteq \prec_k$, where k depends on R only.

Lemma 1. *If $s \prec_k t$ and $k < l$, then $s \prec_l t$.*

We introduce the auxiliary measure $|.|\colon \mathcal{T}^*(\Sigma,\mathcal{V}) \to \mathbb{N}$: (i) $|x| := 1$, $x \in \mathcal{V}$, (ii) $|(s_1,\dots,s_n)| := \max\{n, |s_1|,\dots,|s_n|\}$, (iii) $|fa| := |a| + 1$.

Lemma 2. *If $s \prec_{\text{POP}} t$, then for any substitution σ, $s\sigma \prec_{|s|} t\sigma$.*

Lemma 3. *If $t = f(t_1,\dots,v,\dots,t_n)$, $s = f(t_1,\dots,u,\dots,t_n)$ with $u \prec_k v$, where $k \geq \max\{\mathrm{ar}(f)\colon f \in \Sigma\}$, then $s \prec_k t$.*

Recall that $\prec_{\text{POP}}$ is a reduction order. Hence the assumption $R \subseteq \prec_{\text{POP}}$ implies $\to_R \subseteq \prec_{\text{POP}}$.

Lemma 4. *If $t \to_R s$, then $s \prec_k t$, where $k = \max\{\max\{\tau(r) | (l \to r) \in R\}, \max\{\mathrm{ar}(f) | f \in S\}\}$.*

We set

$$\mathrm{G}_k(\sigma) := \max\{n \in \mathbb{N} \mid \exists (a_0,\dots,a_n)\ (a_n \prec_k \cdots \prec_k a_0 = a)\} ,$$
$$F_{k,p}(n) := \max\{\mathrm{G}_k(fa)\colon \mathrm{rk}(f) = p \ \&\ \mathrm{G}_k(a) \leq n\} ,$$

where $\mathrm{rk}(f)\colon \Sigma \to \mathbb{N}$ is defined inductively: $\mathrm{rk}(f) := \max\{\mathrm{rk}(g)+1\colon g \in \Sigma \wedge g \prec f\}$. We collect some properties of the function G_k in the next lemma.

Lemma 5. *1.* $\mathrm{G}_k((s_0,\dots,s_n)) = \sum_{i=0}^n \mathrm{G}_k(a_i)$.
2. $\mathrm{G}_k(\overline{m}) = |m|$ *for any natural number m.*

Lemma 6. *Inductively we define $d_{k,0} := 2$ and $d_{k,p-1} := (d_{k,p})^k + 1$. Then there exists a constant c (depending only on k and p) such that $F_{k,p}(n) \leq c \cdot n^{d_{k,p}} + c$.*

Proof. The lemma is proven by main induction on p and side induction on σ.

Set $a := (t_0, \dots, t_n)$ and let $w \prec_k f(t_0, \dots, t_n) =: t$, $\mathrm{rk}(f) = p$ and w maximal. By assumption $\mathrm{G}_k(a) \leq n$. We prove

$$\mathrm{G}_k(w) < cn^{d_{k,p}} \quad \text{for almost all } n ,$$

by case-distinction on the definition of $\prec_k$. It suffices to consider the case $w = (r_0, \dots, r_m)$.

CASE. $p = 0$ and $\forall i \leq m\ (r_i \sqsubset_k t)$. By definition of $\prec_{\text{POP}}$ we have $\forall i \leq m\ \exists j \leq n\ (r_i \preceq_k t_j)$. Then $\mathrm{G}_k(w) \leq \mathrm{G}_k(a) = n$. Hence

$$\mathrm{G}_k(w) \leq kn < cn^2 ,$$

where we set $c := k$.

CASE. $p = 0$, $\forall i \neq i_0\ (r_i \sqsubset_k t)$, and $r_{i_0} \prec_k t$. By definition of $\prec_{\text{POP}}$ we have $\forall i \leq m\ \exists j \leq n\ (r_i \preceq_k t_j)$ and $r_{i_0} = f(s_0, \dots, s_l)$, $\mathrm{rk}(f) = 0$, with $(s_0, \dots, s_l) \prec_k a$. Hence by induction hypothesis (IH) on a, there exists a constant c, such that $G_k(r_{i_0}) \leq c(n-1)^2$ a.e. Employing Lemma 5.1 we obtain:

$$\mathrm{G}_k(w) = \mathrm{G}_k((r_0, \dots, r_m)) = \sum_{i=0}^{m} \mathrm{G}_k(r_i) \leq c(n-1)^2 + (k-1)n < cn^2 ,$$

as we can assume $c > k$.

CASE. $p > 0$ and $\forall i \leq m\ (r_i \sqsubset_k t)$. Let i be arbitrary. We can assume $r_i = g(s_0, \dots, s_l)$, $g \prec f$, and $\forall i \leq l\ (s_i \sqsubset_k^{k-1} t)$. Otherwise, if $r_i = g(s_0, \dots, s_l)$ with $g \succ f$ s.t. there $\exists j \leq n\ (r_i \sqsubseteq t_j)$ we proceed as in the first case. By IH there exists c and $d = d_{k,p}$ s.t. $F_{k,p}(n) \leq cn^d$ a.e.

We show the existence of a constant c' s.t. $F_{k,p+1}(n) \leq c'n^{d'}$, where $d' = d_{k,p+1}$. We define $f(a) := ca^d$ and $g^{(0)}(a) := a$, $g^{(l+1)}(a) = f(g^{(l)}(a) \cdot k)$; we obtain:

$$s \sqsubset_k^l t \Longrightarrow \mathrm{G}_k(s) \leq g^{(l)}(n) \text{ a.e.} \qquad (\star)$$

To see $(\star)$ we show by induction on l, that $s \sqsubset_k^l t$ implies $\mathrm{G}_k(s) \leq g^{(l)}(n)$, where $g^{(l)}(n) = c_0 a^{d^{(l)}}$ with $c_0 = c^{\sum_{i=0}^{l-1} d^i} k^{\sum_{i=1}^{l} d^i}$. Suppose $l > 0$, then we obtain by IH on the claim and $F_{k,p}(n) \leq cn^d$ we obtain:

$$\mathrm{G}_k(s) \leq c[(c_0 n^{d^l}) \cdot k]^d = c_1 n^{d^{l+1}} \text{a.e.} ,$$

where $c_1 = c^{\sum_{i=0}^{l} d^i} k^{\sum_{i=1}^{l+1} d^i}$. This accomplishes the claim.

Now the upper-bound for $\mathrm{G}_k(w)$ follows:

$$\mathrm{G}_k(w) \leq kg^{(k)}(n) < c'n^{d'} \text{ a.e.} ,$$

where $c' = c^{\sum_{i=0}^{k-1} d^i} k^{\sum_{i=0}^{k} d^i}$ and $d' = d^{k+1} + 1 = d_{k,p+1}$.

CASE. $p > 0$, $\forall i \neq i_0$ ($r_i \sqsubset_k t$), and $r_{i_0} \prec_k t$. By definition $\forall i \leq m \, \exists j \leq n$ ($r_i \preceq_k t_j$), and $r_{i_0} = f(s_0, \ldots, s_l)$ so that $(s_0, \ldots, s_l) \prec_k a$. Let c, c', d' be defined as above. By IH on σ we obtain $\mathrm{G}_k(r_{i_0}) \leq c'(n-1)^{d'}$ and thus

$$\mathrm{G}_k(w) \leq c'(n-1)^{d'} + (k-1) \cdot c \cdot n^{d^k} < c'n^{d'} \ .$$

□

Recall the definition of the derivation length function:

$$\mathrm{Dl}_R^f(\overline{\mathbf{m}}) = \max\{l \mid \exists \, t_0, \ldots, t_n \in \mathcal{T}(\Sigma) \, (t_n \leftarrow_R \ldots \leftarrow_R t_0 = f(\overline{\mathbf{m}}))\}$$

We have established the following theorem.

Theorem 1. *If for a finite TRS R defined over $T(\Sigma, \mathcal{V})$, $R \subseteq \prec_{\text{POP}}$ then for each $f \in \Sigma$, Dl_R^f is bounded by a* monotone polynomial *in the sum of the binary length of the inputs.*

Proof. Let R be a finite TRS defined over $T(\Sigma, \mathcal{V})$, such that for every rule $(l \to r) \in R$, $r \prec_{\text{POP}} l$ holds. This implies that for any two terms t, s, $t \to_R s$ implies $s \prec_{\text{POP}} t$. Hence by Lemma 4 there exists $k \in \mathbb{N}$, s.t. $\leftarrow_R \subseteq \prec_k$. Suppose f is an n-ary function symbol and set $t := f(\overline{m}_1, \ldots, \overline{m}_n)$. By definition it follows that

$$\mathrm{Dl}_R^f(\overline{m}_1, \ldots, \overline{m}_n) \leq \mathrm{G}_k(f(\overline{m}_1, \ldots, \overline{m}_n)) \ .$$

By Lemma 6 there exists a polynomial p, depending only on k and the rank of f, s.t.

$$\mathrm{G}_k(f(\overline{m}_1, \ldots, \overline{m}_n)) \leq p(\mathrm{G}_k((\overline{m}_1, \ldots, \overline{m}_n)) \ .$$

Employing with Lemma 5, we obtain $\mathrm{Dl}_R^f(\overline{m}_1, \ldots, \overline{m}_n) \leq p(\sum_{i=1}^n |m_i|)$. □

4 Predicative Recursion and POP

In the previous section we have shown that if for a finite TRS R, defined over $T^*(\Sigma, \mathcal{V})$, $R \subseteq \prec_{\text{POP}}$, then the derivation length function Dl_R^f is bounded by a monotone polynomial in the binary length of the inputs. As an application of Theorem 1, we prove in this section that $\mathrm{Dl}_{R'_B}^f$ is bounded by a monotone polynomial in the binary length of the normal inputs. I.e. we give an alternative proof of Prop. 1. As R'_B exactly characterises the functions in **FP** this yields that $\prec_{\text{POP}}$—via the mapping S defined below—exactly characterises the class of polytime computable functions **FP**.

It suffices to define a mapping $\mathrm{S}\colon T(B) \to T^*(\Sigma)$, such that S is a monotone interpretation such that $\mathrm{S}(l\sigma) \succ_{\text{POP}} \mathrm{S}(r\sigma)$ holds for all $(l \to r) \in R'_B$. We suppose the signature Σ is defined such that for any function symbol $f \in B^{k,l}$ there is a function symbol $f' \in \Sigma$ of arity k. Moreover, Σ includes two constants S_0, S_1 and a varyadic function symbol $\bullet$ of lowest rank. We need a few auxiliary notions: $\mathrm{sn}(\overline{n}) := n$ for numerals $\overline{n}$; $\mathrm{sn}(f(\mathbf{t}; \mathbf{s})) = \sum_j(\mathrm{sn}(s_j))$, otherwise. For every number m we define its representation $\widehat{m} \in T(\Sigma, \mathcal{V})$ as follows: $\widehat{0} := \bullet$; $\widehat{\mathcal{S}_i(m)} := \bullet(S_i) * \widehat{m}$ for $i \in [0,1]$, where $\bullet(s_0, \dots, s_i) * \bullet(s_{i+1}, \dots, s_n) := \bullet(s_0, \dots, s_n)$. We define $\mathrm{S}\colon T(B) \to T^*(\Sigma)$ by mutual induction together with the interpretation $\mathrm{N}\colon T(B) \to T^*(\Sigma)$.

Definition 7.

- $\mathrm{S}(\overline{n}) := ()$ *and* $\mathrm{S}(S_i(;t)) := (S_i) \frown \mathrm{S}(t)$ *for* $t \not\equiv \overline{n}$ *(i.e. t is not a numeral).*
- *For* $f \neq S_i$*, define* $\mathrm{S}(f(\mathbf{t};\mathbf{s})) := (f(\mathrm{N}(t_0), \dots, \mathrm{N}(t_n)), \mathrm{S}(s_0), \dots, \mathrm{S}(s_m))$.
- $\mathrm{N}(t) := \bullet\, \mathrm{S}(t) * \widehat{sn(t)}$.

First we show that for $\mathsf{Q} \in \{\mathrm{S}, \mathrm{N}\}$, $\mathsf{Q}(l\sigma) \succ_{\text{POP}} \mathsf{Q}(r\sigma)$. More precisely we show the following lemma.

Lemma 7. *Let* $(l \to r) \in R'_B$*,* σ *a ground substitution, such that* $l\sigma, r\sigma \in T(B)$*. Then there exists* k*, depending on the rule* $(l \to r)$*, such that* $\mathsf{Q}(r\sigma) \prec_k \mathsf{Q}(l\sigma)$.

Proof. Let $(l \to r)$ and σ as in the assumptions of the lemma. We sketch the proof by considering the rule:

$$\mathrm{PREC}^{p+1,q}[g, h_1, h_2](S_i(;t), \mathbf{t}; \mathbf{n}) \to h_i(t, \mathbf{t}; \mathbf{n}, \mathrm{PREC}[g, h_1, h_2](t, \mathbf{t}; \mathbf{n}))\ .$$

We abbreviate $F := \mathrm{PREC}^{p+1,q}[g, h_1, h_2]$ and set $k := 1 + \max\{3, p+1, q+1\}$. Let $\mathrm{lh}(f)$, $f \in B$ be defined as follows: $\mathrm{lh}(f) := 1$, for $f \in \{S_i, O, U, P\}$. $\mathrm{lh}(\mathrm{SUB}[f, \mathbf{g}, \mathbf{h}]) := 1 + \mathrm{lh}(f) + \mathrm{lh}(g_1) + \cdots + \mathrm{lh}(g_{k'}) + \mathrm{lh}(h_1) + \cdots + \mathrm{lh}(h_{l'})$. $\mathrm{lh}(\mathrm{PREC}[g, h_1, h_2]) := 1 + \mathrm{lh}(g) + \mathrm{lh}(h_1) + \mathrm{lh}(h_2)$. Then we define the precedence $<$ over Σ compatible with lh, i.e. $f' < g'$ if $\mathrm{lh}(f) < \mathrm{lh}(g)$. For $\mathsf{Q} = \mathrm{S}$, we employ the following sequence of comparisons:

$$\begin{aligned}
&\mathrm{S}(F(S_i(;t), \mathbf{t}; \mathbf{n})) \\
&= (F'(\mathrm{N}(S_i(;t)), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p)), \mathrm{S}(\overline{n}_1), \dots, \mathrm{S}(\overline{n}_q)) \\
&= F'(\mathrm{N}(S_i(;t)), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p)) \\
&= F'(\bullet(S_i) * \mathrm{N}(t)), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p))\ .
\end{aligned}$$

By definition $\mathrm{S}(\overline{n}_i) = ()$ and for each $t \in T(\Sigma, \mathcal{V})$, $t = (t)$. Moreover it is a direct consequence of the definitions that $N(S_i(;t)) = \bullet(S_i) * N(t)$. Further:

$$\begin{aligned}
&F'(\bullet(S_i) * \mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p)) \\
&\succ_k (h'_i(\mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p)), F'(\mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p)))\ ,
\end{aligned}$$

By Definition 6.4 we obtain $\bullet(S_i) * \mathrm{N}(t) \succ_k \mathrm{N}(t)$. This yields by rules 6.4 and 6.5 using $k > p+1$: $F'(\bullet(S_i) * \mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p)) \succ_k F'(\mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p))$. Finally applying Definition 6.3 together with rule 6.2 and 5.2 yields the inequality. In these rule applications we employ $k > q+1$ and $F' > h'_i$.

$$\begin{aligned}
&(h'_i(\mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p)), F'(\mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p))) \\
&= (h'_i(\mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p))), \mathrm{S}(n_1), \dots, \mathrm{S}(n_l), F'(\mathrm{N}(t), \mathrm{N}(t_1), \dots, \mathrm{N}(t_p))) \\
&= \mathrm{S}(h_i(t, \mathbf{t}; \mathbf{n}, F(t, \mathbf{t}; \mathbf{n})))\ .
\end{aligned}$$

Finally, it is easy to see that $\mathrm{N}(F(S_i(;t),\mathbf{t};\mathbf{n})) \succ_k \mathrm{N}(h_i(t,\mathbf{t};\mathbf{n},F(t,\mathbf{t};\mathbf{n}))$. We established the lemma for the rule $F(S_i(;t),\mathbf{t};\mathbf{n}) \rightarrow h_i(t,\mathbf{t};\mathbf{n},F(t,\mathbf{t};\mathbf{n}))$. The other rules follow similar.

Note that the definition of k in all cases depends on the arity-information encoded in the head function symbol on the left-hand side. Moreover at most 3 iterated applications of $\sqsubset_k$ are necessary. □

The next lemma establish monotonicity for the interpretations S, N.

Lemma 8. *For $k \in \mathbb{N}$ and for $u, v \in T(\Sigma)$, $\mathsf{Q}(u) \prec_k \mathsf{Q}(v)$ for $\mathsf{Q} \in \{\mathrm{S}, \mathrm{N}\}$. Suppose $f \in B^{p,q}$ and $\overline{t}, \overline{s} \in T(\Sigma)$. Then*

- $\mathsf{Q}(f(t_1,\ldots,u,\ldots,t_p;\overline{s}) \prec_k \mathsf{Q}(f(t_1,\ldots,v,\ldots,t_p;\overline{s})$ *for* $\mathsf{Q} \in \{\mathrm{S},\mathrm{N}\}$, *and*
- $\mathsf{Q}(f(\overline{t};s_1,\ldots,u,\ldots,s_q) \prec_k \mathsf{Q}(f(\overline{t};s_1,\ldots,v,\ldots,s_q))$ *for* $\mathsf{Q} \in \{\mathrm{S},\mathrm{N}\}$.

We define the derivation length function $\mathrm{Dl}^f_{R'_B}$ over the ground term-set $T(\Sigma)$:

$$\mathrm{Dl}^f_{R'_B}(\overline{\mathbf{m}};\overline{\mathbf{n}}) := \max\{n \mid \exists\, t_0,\ldots,t_n \in T(B)\, \big(t_n \leftarrow_{R'_B} \cdots \leftarrow_{R'_B} t_0 = f(\overline{\mathbf{m}};\overline{\mathbf{n}})\big)\}\ .$$

Recall the definition of the derivation tree $\mathcal{T}_{R'_B}$. Note that for each $t \in T(B,\mathcal{V})$, $\mathcal{T}_{R'_B}(t)$ is finite. This follows from the fact that R'_B is terminating and $\mathcal{T}_{R'_B}(t)$ is finitely branching. The latter is shown by well-founded induction on $\rightarrow_{R'_B}$. Let $f \in B$ be a fixed predicative recursive function symbol. As the derivation tree $\mathcal{T}_{R'_B}(f(\overline{\mathbf{m}};\overline{\mathbf{n}}))$ is finite only finitely many function symbols occur in $\mathcal{T}_{R'_B}(f(\overline{\mathbf{m}};\overline{\mathbf{n}}))$. This allows to define a finite subset $F \subset B$, such that all terms occurring in $\mathcal{T}_{R'_B}(f(\overline{\mathbf{m}};\overline{\mathbf{n}}))$ belong to $T(F)$. We define

$$k := 1 + \max(\{3\} \cup \{p, q+1 | f^{p,q} \in B \text{ occurs in } \mathcal{T}_{R'_B}(f(\overline{\mathbf{m}};\overline{\mathbf{n}}))\})\ .$$

Let R' denote the restriction of R'_B to $T(F)$. Then, we have $\mathrm{Dl}^f_{R'_B}(\overline{\mathbf{m}};\overline{\mathbf{n}}) = \mathrm{Dl}^f_{R'}(\overline{\mathbf{m}};\overline{\mathbf{n}})$. From these observations together with Lemma 7 and 8 we conclude

Lemma 9. *Let $s, t \in T(F)$ such that $t \rightarrow_R s$. Then $\mathrm{S}(s) \prec_k \mathrm{S}(t)$.*

In summary we obtain, by following the pattern of the proof of Thm. 1:

Theorem 2. *For every $f \in B$, $\mathrm{Dl}^f_{R'_B}(\overline{m_1},\ldots,\overline{m_p};\overline{n_1},\ldots,\overline{n_q})$ is bounded by a monotone polynomial in the sum of the length of the normal inputs $m_1,\ldots,m_p$.*

5 Conclusion

The main contribution of this paper is the definition of a *path order for* **FP**, denoted as $\prec_{\text{POP}}$. This path order has the property that for a finite TRS R compatible with $\prec_{\text{POP}}$, the *derivation length function* Dl^f_R is bounded by a polynomial in the length of the inputs for any defined function symbol f in the signature of R. Moreover $\prec_{\text{POP}}$ is *complete* in the sense that for a function $f \in$ **FP**, there exists a TRS R computing f such that such that termination of R follows by

$\prec_{\text{POP}}$. Another feature of $\prec_{\text{POP}}$ is, that its definition is devoid of the separation of normal and safe arguments, present in the definition of the predicative recursive functions and therefore in the definition of the term-rewriting characterisation R'_B.

We briefly relate our findings to the notion of the *light multiset path order*, denoted as $\prec_{\text{LMPO}}$, introduced by Marion in [11]. It is possible to define a variant of $\prec_{\text{POP}}$—denoted as $\prec_{\text{POPV}}$—such that Theorem 1 remains true for $\prec_{\text{POPV}}$ when suitably reformulated. While Definition 3 and 4 are based on an arbitrary signature, the definition of $\prec_{\text{POPV}}$ assumes that normal and safe arguments are separated as in Section 2. It is easy to see that $\prec_{\text{POPV}} \subset \prec_{\text{LMPO}}$ and this inclusion is strict as $\prec_{\text{LMPO}}$ proves termination of the non-feasible rewrite system R_B, while $\prec_{\text{POPV}}$ clearly does not. On the other hand let R be a functional program (i.e. a constructor TRS) computing a number-theoretic function f. A termination proof of R via $\prec_{\text{LMPO}}$ guarantees the existence of a polytime algorithm for f. However, a termination proof of R via or the introduced path order $\prec_{\text{POPV}}$ (or $\prec_{\text{POP}}$) guarantees that R itself is already a polytime algorithm for f. It seems clear to us that the latter property is of more practical value.

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