Derivational Complexity of Knuth-Bendix Orders revisited

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Abstract. We study the derivational complexity of rewrite systems $\mathcal{R}$ compatible with KBO, if the signature of $\mathcal{R}$ is infinite. We show that the known bounds on the derivation height are preserved, if $\mathcal{R}$ fulfils some mild conditions. This allows us to obtain bounds on the derivational height of non simply terminating TRSs. Furthermore, we re-establish the 2-recursive upper-bound on the derivational complexity of finite rewrite systems $\mathcal{R}$ compatible with KBO.

1 Introduction

One of the main themes in rewriting is *termination*. Over the years powerful methods have been introduced to establish termination of a given term rewriting system (TRS) $\mathcal{R}$. Earlier research mainly concentrated on inventing suitable reduction orders—for example simplification orders, see Chapter 6, authored by Zantema in [1]—capable of proving termination directly. In recent years the emphasis shifted towards transformation techniques like the *dependency pair method* or *semantic labelling*, see [1]. The dependency pair method is easily automatable and lies at the heart of many successful termination provers like TTT [2] or AProVE [3]. Semantic labelling with infinitely labels was conceived to be unsuitable for automation. Hence, only the variant with finitely many elements was incorporated (for example in AProVE [3] or TORPA [4]). Very recently this belief was proven wrong. TPA [5] implements semantic labelling with natural numbers, in combination with recursive path orders (RPOs) efficiently. As remarked in [6] a sensible extension of this implementation is the combination of semantic labelling with Knuth–Bendix orders (KBOs).

In order to assess the power and weaknesses of different termination techniques it is natural to look at the length of derivation sequences, induced by different techniques. This program has been suggested in [7]. The best known result is that for finite rewrite systems, RPOs induce primitive recursive derivational complexity. This bound is essentially optimal, see [8,9]. Similar optimal results have been obtained for lexicographic path orders (LPOs) and KBOs. Weiermann [10] showed that LPOs induce multiply recursive derivational complexity. In [11] Lepper showed that for term rewriting systems (TRSs) compatible with KBOs, the derivational complexity is bounded by the Ackermann function.

These results not only assess different proof techniques for termination, but constitute an a priori complexity analysis for term rewriting systems provably terminating by RPOs, LPOs, or KBOs. The application of termination provers as basis for the termination analysis of logic or functional programs is currently a very hot topic. Applicability of an a priori complexity analysis for TRSs in this direction seems likely.

While the aforementioned program has spawned a number of impressive results, not much is known about the derivational complexity induced by the dependency pair method or semantic labelling (for fixed base orders, obviously). We indicate the situation with an example.

Example 1. Consider the TRS $(\mathcal{F}, \mathcal{R})$ [12] consisting of the following rewrite rules:

$$f(h(x)) \to f(i(x)) \qquad h(a) \to b$$
$$g(i(x)) \to g(h(x)) \qquad i(a) \to b\ .$$

It is not difficult to see that termination of $\mathcal{R}$ cannot be established directly with path orders or KBOs. On the other hand, termination is easily shown via the dependency pair method or via semantic labelling. For the sake of the argument we show termination via semantic labelling with KBOs.

We use natural numbers as semantics and as labels. As interpretation for the function symbols we use $a_\mathbb{N} = b_\mathbb{N} = g_\mathbb{N}(n) = f_\mathbb{N}(n) = 1$, $i_\mathbb{N}(n) = n$, and $h_\mathbb{N}(n) = n + 1$. The resulting algebra $(\mathbb{N}, >)$ is a quasi-model for $\mathcal{R}$. It suffices to label the symbol f. We define the labelling function $\ell_f \colon \mathbb{N} \to \mathbb{N}$ as $\ell_f(n) = n$. Replacing

$$f(h(x)) \to f(i(x))\ ,$$

by the infinitely many rules

$$f_{n+1}(h(x)) \to f_n(i(x))\ ,$$

we obtain the labelled TRS, $(\mathcal{F}_{\text{lab}}, \mathcal{R}_{\text{lab}})$. Further the TRS $(\mathcal{F}_{\text{lab}}, \mathcal{D}ec)$ consists of all rules

$$f_{n+1}(x) \to f_n(x)\ .$$

Now we can show termination of $\mathcal{R}' := \mathcal{R}_{\text{lab}} \cup \mathcal{D}ec$ by an instance $\succ_{\mathsf{kbo}}$ of Knuth-Bendix order (KBO). We set the weight for all occurring function symbols to 1. Further, the precedence is defined as

$$f_{n+1} \succ f_n \succ \cdots \succ f_0 \succ i \succ h \succ g \succ a \succ b\ .$$

It is easy to see that $\mathcal{R}' \subseteq \succ_{\mathsf{kbo}}$. Thus termination of $\mathcal{R}$ is guaranteed.

As the rewrite system $\mathcal{R}'$ is infinite we cannot directly apply the aforementioned result on the derivational complexity induced by Knuth-Bendix order. A careful study of [11] reveals that the crucial problem is not that $\mathcal{R}'$ is infinite, but that the signature $\mathcal{F}_{\text{lab}}$ is infinite, as Lepper's proof makes explicit use of the finiteness of the signature: To establish an upper-bound on the derivational complexity of a TRS $\mathcal{R}$, compatible with KBO, an interpretation function $\mathcal{I}$ is defined, where the cardinality of the underlying signature is hard-coded into $\mathcal{I}$, cf. [11].

We study the situation by giving an alternative proof of Lepper's result compare [11]. The outcome of this study is that the assumption of finiteness of the rewrite system can be weakened. By enforcing conditions that are still weak enough to treat interesting rewrite systems, we show that for (possibly infinite) TRSs $\mathcal{R}$ over infinite signatures, compatible with KBO, the derivation height of $\mathcal{R}$ can be bounded by the Ackermann function. Using an example that stems from [8] we show that this upper-bound is essentially optimal.

Specialised to Example 1, our results provide an upper bound on the derivation height function with respect to $\mathcal{R}$: For every $t \in \mathcal{T}(\mathcal{F})$ there exists a constant c (depending only on t, $\mathcal{R}'$, and $\succ_{\mathsf{kbo}}$) such that the derivation height $\mathsf{dh}_{\mathcal{R}}(t)$ with respect to $\mathcal{R}$ is $\leq \mathsf{Ack}(c^n, 0)$. As the constant c can be made precise, the method is capable of automation.

This paper is organised as follows: In Section 2 and 3 some basic facts on rewriting, set theory and KBOs are recalled. In Section 4 we define an embedding from $\succ_{\mathsf{kbo}}$ into $>^{\text{lex}}$, the lexicographic comparison of sequences of natural numbers. This embedding renders an alternative description of the derivation height of a term, based on the partial order $>^{\text{lex}}$. This description is discussed in Section 5 and linked to the Ackermann function in Section 6. The above mentioned central result is contained in Section 7. Moreover in Section 7 we apply our result to a non simply terminating TRS, whose derivational complexity cannot be primitive recursively bounded.

2 Preliminaries

We assume familiarity with term rewriting. For further details see [1]. Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a signature. We assume that $\mathcal{F}$ contains at least one constant. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted as $\mathcal{T}(\mathcal{F}, \mathcal{V})$, while the set of ground terms is written as $\mathcal{T}(\mathcal{F})$. The set of variables occurring in a term t is denoted as $\mathsf{Var}(t)$. The set of function symbols occurring in t is denoted as $\mathsf{FS}(t)$. The *size* of a term t, written as $\mathrm{Size}(t)$, is the number of variables and functions symbols in it. The number of occurrences of a symbol $a \in \mathcal{F} \cup \mathcal{V}$ in t is denoted as $|t|_a$. A *TRS* $(\mathcal{F}, \mathcal{R})$ over $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a set of rewrite rules. The smallest rewrite relation that contains $\mathcal{R}$ is denoted as $\rightarrow_{\mathcal{R}}$. The transitive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\rightarrow_{\mathcal{R}}^{+}$, and its transitive and reflexive closure by $\rightarrow_{\mathcal{R}}^{*}$. A TRS $(\mathcal{F}, \mathcal{R})$ is called *terminating* if there is no infinite rewrite sequence. As usual, we frequently drop the reference to the signature $\mathcal{F}$.

A *partial order* $\succ$ is an irreflexive and transitive relation. The converse of $\succ$ is written as $\prec$. A partial order $\succ$ on a set A is *well-founded* if there exists no infinite descending sequence $a_1 \succ a_2 \succ \cdots$ of elements of A. A rewrite relation that is also a partial order is called *rewrite order*. A well-founded rewrite order is called *reduction order*. A TRS $\mathcal{R}$ and a partial order $\succ$ are *compatible* if $\mathcal{R} \subseteq \succ$.

We also say that $\mathcal{R}$ is compatible with $\succ$ or vice versa. A TRS $\mathcal{R}$ is terminating iff it is compatible with a reduction order $\succ$.

Let $(\mathcal{A}, >)$ denote a well-founded weakly monotone $\mathcal{F}$-algebra. $(\mathcal{A}, >)$ consists of a carrier A, interpretations $f_{\mathcal{A}}$ for each function symbol in $\mathcal{F}$, and a well-founded partial order $>$ on A such that every $f_{\mathcal{A}}$ is weakly monotone in all arguments. We define a quasi-order $\geqslant_{\mathcal{A}}$: $s \geqslant_{\mathcal{A}} t$ if for all assignments $\alpha \colon \mathcal{V} \to A$ $[\alpha]_{\mathcal{A}}(s) \geqslant [\alpha]_{\mathcal{A}}(t)$. Here $\geqslant$ denotes the reflexive closure of $>$. The algebra $(\mathcal{A}, >)$ is a *quasi-model* of a TRS $\mathcal{R}$, if $\mathcal{R} \subseteq \geqslant_{\mathcal{A}}$.

A *labelling* ℓ for $\mathcal{A}$ consists of a set of labels L_f together with mappings $\ell_f \colon A^n \to L_f$ for every $f \in \mathcal{F}$, f n-ary. A labelling is called *weakly monotone* if all labelling functions ℓ_f are weakly monotone in all arguments. The labelled signature $\mathcal{F}_{\text{lab}}$ consists of n-ary functions symbols f_a for every $f \in \mathcal{F}$, $a \in L_f$, together with all $f \in \mathcal{F}$, such that $L_f = \emptyset$. The TRS $\mathcal{D}ec$ consists of all rules

$$f_{a+1}(x_1, \ldots, x_n) \to f_a(x_1, \ldots, x_n) \;,$$

for all $f \in \mathcal{F}$. The x_i denote pairwise different variables. Our definition of $\mathcal{D}ec$ is motivated by a similar definition in [6]. Note that the rewrite relation $\to^*_{\mathcal{D}ec}$ is not changed by this modification of $\mathcal{D}ec$. For every assignment α, we inductively define a mapping $\mathsf{lab}_\alpha \colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}_{\text{lab}}, \mathcal{V})$:

$$\mathsf{lab}_\alpha(t) := \begin{cases} t & \text{if } t \in \mathcal{V} \;, \\ f(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \emptyset \;, \\ f_a(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{otherwise} \;. \end{cases}$$

The label a in the last case is defined as $l_f([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n))$. The *labelled* TRS $\mathcal{R}_{\text{lab}}$ over $\mathcal{F}_{\text{lab}}$ is defined as

$$\{\mathsf{lab}_\alpha(l) \to \mathsf{lab}_\alpha(r) \mid l \to r \in \mathcal{R} \text{ and } \alpha \text{ an assignment}\} \;.$$

Theorem 1 (Zantema [13]). *Let $\mathcal{R}$ be a TRS, $(\mathcal{A}, >)$ a well-founded weakly monotone quasi-model for $\mathcal{R}$, and ℓ a weakly monotone labelling for $(\mathcal{A}, >)$. Then $\mathcal{R}$ is terminating iff $\mathcal{R}_{\text{lab}} \cup \mathcal{D}ec$ is terminating.*

The proof of the theorem uses the following lemma.

Lemma 1. *Let $\mathcal{R}$ be a TRS, $(\mathcal{A}, >)$ a quasi-model of $\mathcal{R}$, and ℓ a weakly monotone labelling for $(\mathcal{A}, >)$. If $s \to_{\mathcal{R}} t$, then $\mathsf{lab}_\alpha(s) \to^*_{\mathcal{D}ec} \cdot \to_{\mathcal{R}_{\text{lab}}} \mathsf{lab}_\alpha(t)$ for all assignments α.*

We briefly review a few basic concepts from set-theory in particular ordinals, see [14]. We write $>$ to denote the well-ordering of ordinals. Any ordinal $\alpha \neq 0$, smaller than ϵ_0, can uniquely be represented by its *Cantor Normal Form (CNF)*:

$$\omega^{\alpha_1} n_1 + \ldots \omega^{\alpha_k} n_k \qquad \text{with } \alpha_1 > \cdots > \alpha_k \;.$$

To each well-founded partial order $\succ$ on a set A we can associate a (set-theoretic) ordinal, its *order type*. First we associate an ordinal to each element a of A by setting $\mathrm{otype}_{\succ}(a) := \sup\{\mathrm{otype}_{\succ}(b) + 1 \colon b \in A \text{ and } b \succ a\}$. The *order type* of $\succ$, denoted by $\mathrm{otype}(\succ)$, is the supremum of $\mathrm{otype}_{\succ}(a) + 1$ with $a \in A$. For two partial orders $\succ$ and $\succ'$ on A and A', respectively, a mapping $o \colon A \to A'$ *embeds* $\succ$ into $\succ'$ if for all $p, q \in A$, $p \succ q$ implies $o(p) \succ' o(q)$. Such a mapping is an *order-isomorphism* if it is bijective and the partial orders $\succ$ and $\succ'$ are linear .

3 The Knuth Bendix Orders

A *weight function* for $\mathcal{F}$ is a pair (w, w_0) consisting of a function $\mathrm{w} \colon \mathcal{F} \to \mathbb{N}$ and a minimal weight $w_0 \in \mathbb{N}$, $w_0 > 0$ such that $\mathrm{w}(c) \geq w_0$ if c is a constant. A weight function (w, w_0) is called *admissible* for a precedence $\succ$ if $f \succ g$ for all $g \in \mathcal{F}$ different from f, when f is unary with $\mathrm{w}(f) = 0$. The function symbol f (if present) is called *special*. The *weight* of a term t, denoted as $\mathrm{w}(t)$ is defined inductively. Assume t is a variable, then set $\mathrm{w}(t) := w_0$, otherwise if $t = g(t_1, \ldots, t_n)$, we define $\mathrm{w}(t) := \mathrm{w}(g) + \mathrm{w}(t_1) + \cdots + \mathrm{w}(t_n)$.

The following definition of KBO is tailored to our purposes. It is taken from [11]. We write $s = f^a s'$ if $s = f^a(s')$ and the root symbol of s' is distinct from the special symbol f. Let $\succ$ be a precedence. The *rank* of a function symbol is defined as: $\mathsf{rk}(f) := \max\{\mathsf{rk}(g) + 1 \mid f \succ g\}$. (To assert well-definedness we stipulate $\max(\emptyset) = 0$.)

Definition 1. *Let* (w, w_0) *denote an admissible weight function for* $\mathcal{F}$ *and let* $\succ$ *denote a precedence on* $\mathcal{F}$*. We write* f *for the special symbol. The* Knuth Bendix order $\succ_{\text{KBO2}}$ *on* $\mathcal{T}(\mathcal{F}, \mathcal{V})$ *is inductively defined as follows:* $s \succ_{\text{KBO2}} t$ *if* $|s|_x \geq |t|_x$ *for all* $x \in \mathcal{V}$ *and*

1. $\mathrm{w}(s) > \mathrm{w}(t)$*, or*
2. $\mathrm{w}(s) = \mathrm{w}(t)$*,* $s = f^a s'$*,* $t = f^b t'$*, where* $s' = g(s_1, \ldots, s_n)$*,* $t' = h(t_1, \ldots, t_m)$*, and one of the following cases holds.*
   (a) $a > b$*, or*
   (b) $a = b$ *and* $g \succ h$*, or*
   (c) $a = b$*,* $g = h$*, and* $(s_1, \ldots, s_n) \succ_{\text{KBO2}}^{\text{lex}} (t_1, \ldots, t_n)$*.*

Let $\succ_{\mathsf{kbo}}$ denote the KBO on terms in its usual definition, see [1]. The following lemma, taken from [11], states that both orders are interchangeable.

Lemma 2 (Lepper [11]). *The orders* $\succ_{\mathsf{kbo}}$ *and* $\succ_{\text{KBO2}}$ *coincide.*

In the literature *real-valued* KBOs and other generalisations of KBOs are studied as well, cf. [15,16]. However, as established in [17] any TRS shown to be terminating by a real-valued KBO can be shown to be terminating by a integer-valued KBO.

4 Exploiting the order-type of KBOs

We write $\mathbb{N}^*$ to denote the set of finite sequences of natural numbers. Let $p \in \mathbb{N}^*$, we write $|p|$ for the *length* of p, i.e. the number of positions in the sequence p.

The i^{th} element of the sequence a is denoted as $(p)_{i-1}$. We write $p \frown q$ to denote the concatenation of the sequences p and q. The next definition is standard but included here, for sake of completeness.

Definition 2. *We define the* lexicographic order *on* $\mathbb{N}^*$. *If* $p, q \in \mathbb{N}^*$, *then* $p >^{\text{lex}} q$ *if,*

- $|p| > |q|$, *or*
- $|p| = |q| = n$ *and there exists* $i \in [0, n-1]$, *such that for all* $j \in [0, i-1]$ $(p)_j = (q)_j$ *and* $(p)_i > (q)_i$.

It is not difficult to see that $\text{otype}(>^{\text{lex}}) = \omega^\omega$, moreover in [11] it is shown that $\text{otype}(\succ_{\mathsf{kbo}}) = \omega^\omega$. Hence $\text{otype}(>^{\text{lex}}) = \text{otype}(\succ_{\mathsf{kbo}})$, a fact we exploit below. However, to make this work, we have to restrict our attention to signatures $\mathcal{F}$ with bounded arities. The maximal arity of $\mathcal{F}$ is denoted as $\mathsf{Ar}(\mathcal{F})$.

Definition 3. *Let the signature* $\mathcal{F}$ *and a weight function* (w, w_0) *for* $\mathcal{F}$ *be fixed. We define an embedding* $\mathsf{tw}\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathbb{N}^*$. *Set* $b := \max\{\mathsf{Ar}(\mathcal{F}), 3\} + 1$.

$$\mathsf{tw}(t) := \begin{cases} (w_0, a, 0) \frown 0^m & \text{if } t = f^a x,\ x \in \mathcal{V}\ , \\ (\mathrm{w}(t), a, \mathsf{rk}(g)) \frown \mathsf{tw}(t_1) \frown \cdots \frown \mathsf{tw}(t_n) \frown 0^m & \text{if } t = f^a g(t_1, \ldots, t_n)\ . \end{cases}$$

The number m *is set suitably, so that* $|\mathsf{tw}(t)| = b^{\mathrm{w}(t)+1}$.

The mapping tw flattens a term t by transforming it into a concatenation of triples. Each triple holds the weight of the considered subterm r, the number of leading special symbols and the rank of the first non-special function symbol of r. In this way all the information necessary to compare two terms via $\succ_{\mathsf{kbo}}$ is expressed as a very simple data structure: a list of natural numbers.

Lemma 3. tw *embeds* $\succ_{\mathsf{kbo}}$ *into* $>^{\text{lex}}$: *If* $s \succ_{\mathsf{kbo}} t$, *then* $\mathsf{tw}(s) >^{\text{lex}} \mathsf{tw}(t)$.

Proof. The proof follows the pattern of the proof of Lemma 9 in [11].

Firstly, we make sure that the mapping tw is *well-defined*, i.e., we show that the length restriction can be met. We proceed by induction on t; let $t = f^a t'$. We consider two cases (i) $t' \in \mathcal{V}$ or (ii) $t' = g(t_1, \ldots, t_n)$. Suppose the former:

$$|(w_0, a, 0)| = 3 \leq b^{\mathrm{w}(t)+1}\ .$$

Now suppose case (ii): Let $j = \mathsf{rk}(g)$, we obtain

$$\begin{aligned} |(\mathrm{w}(t), a, j) \frown \mathsf{tw}(t_1) \frown \cdots \frown \mathsf{tw}(t_n)| &= 3 + b^{\mathrm{w}(t_1)+1} + \cdots + b^{\mathrm{w}(t_n)+1} \\ &\leq 3 + n \cdot b^{\mathrm{w}(t)} \leq b^{\mathrm{w}(t)+1}\ . \end{aligned}$$

Secondly, we show the following, slight generalisation of the lemma:

$$s \succ_{\mathsf{kbo}} t \land |\mathsf{tw}(s) \frown r| = |\mathsf{tw}(t) \frown r'| \Longrightarrow \mathsf{tw}(s) \frown r >^{\text{lex}} \mathsf{tw}(t) \frown r'\ . \qquad (1)$$

To prove (1) we proceed by induction on $s \succ_{\mathsf{kbo}} t$. Set $p = \mathsf{tw}(s) \frown r$, $q = \mathsf{tw}(t) \frown r'$.

CASE $\mathrm{w}(s) > \mathrm{w}(t)$: By definition of the mapping tw, we have: If $\mathrm{w}(s) > \mathrm{w}(t)$, then $(\mathsf{tw}(s))_0 > (\mathsf{tw}(t))_0$. Thus $p >^{\mathrm{lex}} q$ follows.

CASE $\mathrm{w}(s) = \mathrm{w}(t)$: We only consider the sub-case where $s = f^a g(s_1, \ldots, s_n)$ and $t = f^a g(t_1, \ldots, t_n)$ and there exists $i \in [1, n]$ such that $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$, and $s_i \succ_{\mathsf{kbo}} t_i$. (The other cases are treated as in the case above.) The induction hypotheses (IH) expresses that if $|\mathsf{tw}(s_i) \frown v| = |\mathsf{tw}(t_i) \frown v'|$, then $\mathsf{tw}(s_i) \frown v >^{\mathrm{lex}} \mathsf{tw}(t_i) \frown v'$. For $j = \mathsf{rk}(g)$, we obtain

$$\begin{aligned} p &= \overbrace{(\mathrm{w}(s), a, j) \frown \mathsf{tw}(s_1) \frown \cdots \frown \mathsf{tw}(s_{i-1})}^{w} \frown \mathsf{tw}(s_i) \frown \cdots \frown \mathsf{tw}(s_n) \frown r \; , \\ q &= \underbrace{(\mathrm{w}(s), a, j) \frown \mathsf{tw}(s_1) \frown \cdots \frown \mathsf{tw}(s_{i-1})}_{w} \frown \mathsf{tw}(t_i) \frown \cdots \frown \mathsf{tw}(t_n) \frown r' \; . \end{aligned}$$

Due to $|p| = |q|$, we conclude

$$|\mathsf{tw}(s_i) \frown \cdots \frown \mathsf{tw}(s_n) \frown r| = |\mathsf{tw}(t_i) \frown \cdots \frown \mathsf{tw}(t_n) \frown r'| \; .$$

Hence IH is applicable and we obtain

$$\mathsf{tw}(s_i) \frown \cdots \frown \mathsf{tw}(s_n) \frown r >^{\mathrm{lex}} \mathsf{tw}(t_i) \frown \cdots \frown \mathsf{tw}(t_n) \frown r' \; ,$$

which yields $p >^{\mathrm{lex}} q$. This completes the proof of (1).

Finally, to establish the lemma, we assume $s \succ_{\mathsf{kbo}} t$. By definition either $\mathrm{w}(s) > \mathrm{w}(t)$ or $\mathrm{w}(s) = \mathrm{w}(t)$. In the latter case $\mathsf{tw}(s) >^{\mathrm{lex}} \mathsf{tw}(t)$ follows by (1). While in the former $\mathsf{tw}(s) >^{\mathrm{lex}} \mathsf{tw}(t)$ follows as $\mathrm{w}(s) > \mathrm{w}(t)$ implies $|\mathsf{tw}(s)| > |\mathsf{tw}(t)|$. □

5 Derivation Height of Knuth-Bendix Orders

Let $\mathcal{R}$ be a TRS and $\succ_{\mathsf{kbo}}$ a KBO such that $\succ_{\mathsf{kbo}}$ is compatible with $\mathcal{R}$. The TRS $\mathcal{R}$ and the KBO $\succ_{\mathsf{kbo}}$ are fixed for the remainder of the paper. We want to extract an upper-bound on the length of derivations in $\mathcal{R}$. We recall the central definitions. Note that we can restrict the definition to the set ground terms. The *derivation height* function $\mathsf{dh}_{\mathcal{R}}$ (with respect to $\mathcal{R}$ on $\mathcal{T}(\mathcal{F})$) is defined as follows.

$$\mathsf{dh}_{\mathcal{R}}(t) := \max(\{n \mid \exists (t_0, \ldots, t_n) \; t = t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots \to_{\mathcal{R}} t_n\}) \; .$$

We introduce a couple of *measure functions* for term and sequence complexities, respectively. The first measure $\mathsf{sp}\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathbb{N}$ bounds the maximal nesting of special symbols in the term:

$$\mathsf{sp}(t) := \begin{cases} a & \text{if } t = f^a x,\; x \in \mathcal{V} \; , \\ \max(\{a\} \cup \{\mathsf{sp}(t_j) \mid j \in [1, n]\}) & \text{if } t = f^a g(t_1, \ldots, t_n) \; . \end{cases}$$

The second and third measure $\mathsf{rk}\colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathbb{N}$ and $\mathsf{mrk}\colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathbb{N}$ collect information on the ranks of non special function symbols occurring:

$$\mathsf{rk}(t) := \begin{cases} 0 & \text{if } t = f^a x,\ x \in \mathcal{V}\ , \\ j & \text{if } t = f^a g(t_1,\dots,t_n) \text{ and } \mathsf{rk}(g) = j\ , \end{cases}$$

$$\mathsf{mrk}(t) := \begin{cases} 0 & \text{if } t = f^a x,\ x \in \mathcal{V}\ , \\ \max(\{j\} \cup \{\mathsf{mrk}(t_i) \mid i \in [1,n]\}) & \text{if } t = f^a g(t_1,\dots,t_n),\ \mathsf{rk}(g) = j\ . \end{cases}$$

The fourth measure $\mathbf{max}\colon \mathbb{N}^* \to \mathbb{N}$ considers sequences p and bounds the maximal number occurring in p:

$$\mathbf{max}(p) := \max(\{(p)_i \mid i \in [0, |p|-1]\})\ .$$

It is immediate from the definitions that for any term t: $\mathsf{sp}(t), \mathsf{rk}(t), \mathsf{mrk}(t) \leq \mathbf{max}(\mathsf{tw}(t))$. We write $r \trianglelefteq t$ to denote the fact that r is a subterm of t.

Lemma 4. *If $r \trianglelefteq t$, then $\mathbf{max}(\mathsf{tw}(t)) \geq \mathbf{max}(\mathsf{tw}(r))$.*

We informally argue for the correctness of the lemma. Suppose r is a subterm of t. Then clearly $\mathrm{w}(r) \leq \mathrm{w}(t)$. The maximal occurring nesting of special symbols in r is smaller (or equal) than in t. And the maximal rank of a symbol in r is smaller (or equal) than in t. The mapping tw transforms r to a sequence p whose coefficients are less than $\mathrm{w}(t)$, less than the maximal nesting of special symbols and less than the maximal rank of non-special function symbol in r . Hence $\mathbf{max}(\mathsf{tw}(t)) \geq \mathbf{max}(\mathsf{tw}(r))$ holds.

Lemma 5. *If $p = \mathsf{tw}(t)$ and $q = \mathsf{tw}(f^a t)$, then $\mathbf{max}(p) + a \geq \mathbf{max}(q)$.*

Proof. The proof of the lemma proceeds by a case distinction on t. ☐

Lemma 6. *We write $m \dot{-} n$ to denote $\max(\{m-n, 0\})$. Assume $s \succ_{\mathsf{kbo}} t$ with $\mathsf{sp}(t) \leq K$ and $(\mathsf{mrk}(t) \dot{-} \mathsf{rk}(s)) \leq K$. Let σ be a substitution and set $p = \mathsf{tw}(s\sigma)$, $q = \mathsf{tw}(t\sigma)$. Then $p >^{\mathrm{lex}} q$ and $\mathbf{max}(p) + K \geq \mathbf{max}(q)$.*

Proof. It suffices to show $\mathbf{max}(p) + K \geq \mathbf{max}(q)$ as $p >^{\mathrm{lex}} q$ follows from Lemma 3. We proceed by induction on t; let $t = f^a t'$.

CASE $t' \in \mathcal{V}$: Set $t' = x$. We consider two sub-cases: Either (i) $x\sigma = f^b y$, $y \in \mathcal{V}$ or (ii) $x\sigma = f^b g(u_1,\dots,u_m)$. It suffices to consider sub-case (ii), as sub-case (i) is treated in a similar way. From $s \succ_{\mathsf{kbo}} t$, we know that for all $y \in \mathcal{V}$, $|s|_y \geq |t|_y$, hence $x \in \mathsf{Var}(s)$ and $x\sigma \trianglelefteq s\sigma$. Let $l := \mathsf{rk}(g)$; by Lemma 4 we conclude $\mathbf{max}(\mathsf{tw}(x\sigma)) \leq \mathbf{max}(p)$. I.e. $b, l, \mathbf{max}(\mathsf{tw}(u_1)), \dots, \mathbf{max}(\mathsf{tw}(u_m)) \leq \mathbf{max}(p)$. We obtain

$$\begin{aligned} \mathbf{max}(q) &= \max(\{w_0, a+b, l\} \cup \{\mathbf{max}(\mathsf{tw}(u_j)) \mid i \in [1,m]\}) \\ &\leq \max(\{\mathrm{w}(s\sigma), \mathsf{sp}(t) + \mathbf{max}(p), \mathbf{max}(p)\} \cup \{\mathbf{max}(p)\}) \\ &\leq \max(\{\mathrm{w}(s\sigma), \mathbf{max}(p) + K\} \cup \{\mathbf{max}(p)\}) = \mathbf{max}(p) + K\ . \end{aligned}$$

CASE $t' = g(t_1, \dots, t_n)$: Let $j = \mathsf{rk}(g)$. By Definition 1 we obtain $s \succ_{\mathsf{kbo}} t_i$. Moreover $\mathsf{sp}(t_i) \leq \mathsf{sp}(t) \leq K$ and $\mathsf{mrk}(t_i) \leq \mathsf{mrk}(t)$. Hence for all i: $\mathsf{sp}(t_i) \leq K$ and $(\mathsf{mrk}(t_i) \dot{-} \mathsf{rk}(s)) \leq K$ holds. Thus IH is applicable: For all i: $\mathbf{max}(\mathsf{tw}(t_i\sigma)) \leq \mathbf{max}(p) + K$. By using the assumption $(\mathsf{mrk}(t) \dot{-} \mathsf{rk}(s)) \leq K$ we obtain:

$$\begin{aligned}
\mathbf{max}(q) &= \max(\{\mathrm{w}(t\sigma), a, j\} \cup \{\mathbf{max}(\mathsf{tw}(t_i\sigma)) \mid i \in [1,n]\}) \\
&\leq \max(\{\mathrm{w}(t\sigma), \mathsf{sp}(t), \mathsf{rk}(s) + K\} \cup \{\mathbf{max}(p) + K\}) \\
&\leq \max(\{\mathrm{w}(s\sigma), \mathsf{sp}(t), \mathsf{rk}(s\sigma) + K\} \cup \{\mathbf{max}(p) + K\}) \\
&\leq \max(\{\mathrm{w}(s\sigma), K, \mathbf{max}(p) + K\} \cup \{\mathbf{max}(p) + K\}) = \mathbf{max}(p) + K\ .
\end{aligned}$$

□

In the following, we assume that the set

$$M := \{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}\} \tag{2}$$

is finite. We set $K := \max(M)$ and let K be fixed for the remainder.

Example 2. With respect to the TRS $\mathcal{R}' := \mathcal{R}_{\mathrm{lab}} \cup \mathcal{D}ec$ from Example 1, we have $M = \{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}'\}$. Note that the signature of $\mathcal{R}'$ doesn't contain a special symbol.

Clearly M is finite and it is easy to see that $\max(M) = 1$. Exemplary, we consider the rule schemata $f_{n+1}(h(x)) \to f_n(i(x))$. Note that the rank of i equals 4, the rank of h is 3, and the rank of f_n is given by $n + 5$. Hence $\mathsf{mrk}(f_n(i(x))) = n + 5$ and $\mathsf{rk}(f_{n+1}(h(x))) = n + 6$. Clearly $(n + 5 \dot{-} n + 6) \leq 1$.

Lemma 7. *If $s \to_{\mathcal{R}} t$, $p = \mathsf{tw}(s)$, $q = \mathsf{tw}(t)$, then $p >^{\mathrm{lex}} q$ and $u(\mathbf{max}(p), K) \geq \mathbf{max}(q)$, where u denotes a monotone polynomial such that $u(n,m) \geq 2n + m$.*

Proof. By definition of the rewrite relation there exists a context C, a substitution σ and a rule $l \to r \in R$ such that $s = C[l\sigma]$ and $t = C[r\sigma]$. We prove $\mathbf{max}(q) \leq u(\mathbf{max}(p), K)$ by induction on C. Note that C can only have the form (i) $C = f^a[\Box]$ or (ii) $C = f^a g(u_1, \dots, C'[\Box], \dots, u_n)$.

CASE $C = f^a[\Box]$: By Lemma 6 we see $\mathbf{max}(\mathsf{tw}(r\sigma)) \leq \mathbf{max}(\mathsf{tw}(l\sigma)) + K$. Employing in addition Lemma 5 and Lemma 4, we obtain:

$$\begin{aligned}
\mathbf{max}(q) = \mathbf{max}(\mathsf{tw}(f^a r\sigma)) &\leq \mathbf{max}(\mathsf{tw}(r\sigma)) + a \\
&\leq \mathbf{max}(\mathsf{tw}(l\sigma)) + K + a \\
&\leq \mathbf{max}(p) + K + \mathbf{max}(p) \leq u(\mathbf{max}(p), K)\ .
\end{aligned}$$

CASE $C = f^a g(u_1, \dots, C'[\Box], \dots, u_n)$: As $C'[l\sigma] \to_{\mathcal{R}} C'[r\sigma]$, IH is applicable: Let $p' = \mathsf{tw}(C'[l\sigma])$, $q' = \mathsf{tw}(C'[r\sigma])$. Then $\mathbf{max}(q') \leq u(\mathbf{max}(p'), K)$. For $\mathsf{rk}(g) = l$, we obtain by application of IH and Lemma 4:

$$\begin{aligned}
\mathbf{max}(q) &= \max(\{\mathrm{w}(t), a, l\} \cup \{\mathbf{max}(\mathsf{tw}(u_1)), \dots, \mathbf{max}(q'), \dots, \mathbf{max}(\mathsf{tw}(u_n))\}) \\
&\leq \max(\{\mathrm{w}(s), a, l\} \cup \\
&\quad \cup \{\mathbf{max}(\mathsf{tw}(u_1)), \dots, u(\mathbf{max}(p'), K), \dots, \mathbf{max}(\mathsf{tw}(u_n))\}) \\
&\leq \max(\{\mathrm{w}(s), a, l\} \cup \{\mathbf{max}(p), u(\mathbf{max}(p), K)\}) = u(\mathbf{max}(p), K)\ .
\end{aligned}$$

□

We define *approximations* of the partial order $>^{\text{lex}}$.

$$p >^{\text{lex}}_n q \quad \text{iff} \quad p >^{\text{lex}} q \text{ and } u(\mathbf{max}(p), n) \geq \mathbf{max}(q) \;,$$

where u is defined as in Lemma 7. Now Lemma 6 can be concisely expressed as follows, for K as above.

Proposition 1. *If $s \to_{\mathcal{R}} t$, then $\mathsf{tw}(s) >^{\text{lex}}_K \mathsf{tw}(t)$.*

In the spirit of the definition of derivation height, we define a family of functions $\mathsf{Ah}_n \colon \mathbb{N} \to \mathbb{N}$:

$$\mathsf{Ah}_n(p) := \max(\{m \mid \exists (p_0, \ldots, p_m)\; p = p_0 >^{\text{lex}}_n p_1 >^{\text{lex}}_n \cdots >^{\text{lex}}_n p_m\}) \;.$$

The following proposition is an easy consequence of the definitions and Proposition 1.

Theorem 2. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS, compatible with KBO. Assume the set $M := \{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}\}$ is finite and the arities in of the symbols in $\mathcal{F}$ are bounded; set $K := \max(M)$. Then $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{Ah}_K(\mathsf{tw}(t))$.*

In the next section we show that Ah_n is bounded by the Ackermann function Ack. Thus providing the sought upper-bound on the derivation height of $\mathcal{R}$.

6 Bounding the growth of Ah_n

Instead of directly relating the functions Ah_n to the Ackermann function, we make use of the fast-growing *Hardy* functions, cf. [18]. The Hardy functions form a hierarchy of unary functions $\mathsf{H}_\alpha \colon \mathbb{N} \to \mathbb{N}$ indexed by ordinals. We will only be interested in a small part of this hierarchy, namely in the set of functions $\{\mathsf{H}_\alpha \mid \alpha < \omega^\omega\}$.

Definition 4. *We define the embedding $o \colon \mathbb{N}^* \to \omega^\omega$ as follows:*

$$o(p) := \omega^{\ell-1}(p)_0 + \ldots \omega(p)_{\ell-2} + (p)_{\ell-1} \;,$$

where $\ell = |p|$.

The next lemma follows directly from the definitions.

Lemma 8. *If $p >^{\text{lex}} q$, then $o(p) > o(q)$.*

We associate with every $\alpha < \omega^\omega$ in *CNF* an ordinal α_n, where $n \in \mathbb{N}$. The sequence $(\alpha_n)_n$ is called *fundamental sequence* of α. (For the connection between rewriting and fundamental sequences see e.g. [19].)

$$\alpha_n := \begin{cases} 0 & \text{if } \alpha = 0 \;, \\ \beta & \text{if } \alpha = \beta + 1 \;, \\ \beta + \omega^{\gamma+1} \cdot (k-1) + \omega^\gamma \cdot (n+1) & \text{if } \alpha = \beta + \omega^{\gamma+1} \cdot k \;. \end{cases}$$

Based on the definition of α_n, we define $\mathsf{H}_\alpha \colon \mathbb{N} \to \mathbb{N}$, for $\alpha < \omega^\omega$ by transfinite induction on α:

$$\mathsf{H}_0(n) := n \qquad \mathsf{H}_\alpha(n) := \mathsf{H}_{\alpha_n}(n+1) \ .$$

Let $>_{(n)}$ denote the transitive closure of $(.)_n$, i.e. $\alpha >_{(n)} \beta$ iff $\alpha_n >_{(n)} \beta$ or $\alpha_n = \beta$. Suppose $\alpha, \beta < \omega^\omega$. Let $\alpha = \omega^{\alpha_1} n_1 + \dots \omega^{\alpha_k} n_k$ and $\beta = \omega^{\beta_1} m_1 + \dots \omega^{\beta_l} m_l$. Recall that any ordinal $\alpha \neq 0$ can be uniquely written in CNF, hence we can assume that $\alpha_1 > \cdots > \alpha_k$ and $\beta_1 > \cdots > \beta_l$. Furthermore by our assumption that $\alpha, \beta < \omega^\omega$, we have $\alpha_i, \beta_j \in \mathbb{N}$. We write $\mathsf{NF}(\alpha, \beta)$ if $\alpha_k \geq \beta_1$.

Before we proceed in our estimation of the functions Ah_n, we state some simple facts that help us to calculate with the function H_α.

Lemma 9. *1. If $\alpha >_{(n)} \beta$, then $\alpha >_{(n+1)} \beta + 1$ or $\alpha = \beta + 1$.*
2. If $\alpha >_{(n)} \beta$ and $n \geq m$, then $\mathsf{H}_\alpha(n) > \mathsf{H}_\beta(m)$.
3. If $n > m$, then $\mathsf{H}_\alpha(n) > \mathsf{H}_\alpha(m)$.
4. If $\mathsf{NF}(\alpha, \beta)$, then $\mathsf{H}_{\alpha+\beta}(n) = \mathsf{H}_\alpha \circ \mathsf{H}_\beta(n)$; $\circ$ denotes function composition.

We relate the Hardy functions with the Ackermann function. The stated upper-bound is a gross one, but a more careful estimation is not necessary here.

Lemma 10. *For $n \geq 1$: $\mathsf{H}_{\omega^n}(m) \leq \mathsf{Ack}(2n, m)$.*

Proof. We recall the definition of the Ackermann function:

$$\begin{aligned} \mathsf{Ack}(0, m) &= m + 1 \\ \mathsf{Ack}(n+1, 0) &= \mathsf{Ack}(n, 1) \\ \mathsf{Ack}(n+1, m+1) &= \mathsf{Ack}(n, \mathsf{Ack}(n+1, m)) \end{aligned}$$

In the following we sometimes denote the Ackermann function as a unary function, indexed by its first argument: $\mathsf{Ack}(n, m) = \mathsf{Ack}_n(m)$. To prove the lemma, we proceed by induction on the lexicographic comparison of n and m. We only present the case, where n and m are greater than 0. As preparation note that $m + 1 \leq \mathsf{H}_{\omega^n}(m)$ holds for any n and $\mathsf{Ack}_n^2(m+1) \leq \mathsf{Ack}_{n+1}(m+1)$ holds for any n, m.

$$\begin{aligned} \mathsf{H}_{\omega^{n+1}}(m+1) &= \mathsf{H}_{\omega^n(m+2)}(m+2) && \\ &\leq \mathsf{H}_{\omega^n(m+2)+\omega^n}(m+1) && \text{Lemma 9(3,4)} \\ &= \mathsf{H}^2_{\omega^n} \mathsf{H}_{\omega^n(m+1)}(m+1) && \text{Lemma 9(4)} \\ &= \mathsf{H}^2_{\omega^n} \mathsf{H}_{\omega^{n+1}}(m) && \\ &\leq \mathsf{Ack}^2_{2n} \mathsf{Ack}_{2(n+1)}(m) && \text{IH} \\ &\leq \mathsf{Ack}_{2n+1} \mathsf{Ack}_{2(n+1)}(m) && \\ &= \mathsf{Ack}(2(n+1), m+1) \ . && \end{aligned}$$

□

Lemma 11. *Assume $u(m,n) \leq 2m+n$ and set $\ell = |p|$. For all $n \in \mathbb{N}$:*

$$\mathsf{Ah}_n(p) \leq \mathsf{H}_{\omega^2 \cdot o(p)}(u(\mathbf{max}(p),n)+1) < \mathsf{H}_{\omega^{4+\ell}}(\mathbf{max}(p)+n) \ . \tag{3}$$

Proof. To prove the first half of (3) , we make use of the following fact:

$$p >^{\mathrm{lex}} q \wedge n \geq \mathbf{max}(q) \Longrightarrow o(p) >_{(n)} o(q) \ . \tag{4}$$

To prove (4), one proceeds by induction on $>^{\mathrm{lex}}$ and uses that the embedding $o\colon \mathbb{N}^* \to \omega^\omega$ is essentially an order-isomorphism. We omit the details.

By definition, we have $\mathsf{Ah}_n(p) = \max(\{\mathsf{Ah}_n(q)+1 \mid p >^{\mathrm{lex}}_n q\})$. Hence it suffices to prove

$$p >^{\mathrm{lex}} q \wedge u(\mathbf{max}(p),n) \geq \mathbf{max}(q) \Longrightarrow \mathsf{Ah}_n(q) < \mathsf{H}_{\omega^2 \cdot o(p)}(u(\mathbf{max}(p),n)+1) \tag{5}$$

We fix p fulfilling the assumptions in (5); let $\alpha = o(p)$, $\beta = o(q)$, $v = u(\mathbf{max}(q),n)$. We use (4) to obtain $\alpha >_{(v)} \beta$. We proceed by induction on p.

Consider the case $\alpha_v = \beta$. As $p >^{\mathrm{lex}} q$, we can employ IH to conclude $\mathsf{Ah}_n(q) \leq \mathsf{H}_{\omega^2 \cdot o(q)}(u(\mathbf{max}(q),n)+1)$. It is not difficult to see that for any $p \in \mathbb{N}^*$ and $n \in \mathbb{N}$, $4\,\mathbf{max}(p)+2n+1 \leq \mathsf{H}_{\omega^2}(u(\mathbf{max}(p),n))$. In sum, we obtain:

$$\begin{aligned}
\mathsf{Ah}_n(q) &\leq \mathsf{H}_{\omega^2 \cdot o(q)}(u(\mathbf{max}(q),n)+1) \\
&\leq \mathsf{H}_{\omega^2 \cdot \alpha_v}(u(u(\mathbf{max}(p),n),n)+1) && \mathbf{max}(q) \leq u(\mathbf{max}(p),n) \\
&\leq \mathsf{H}_{\omega^2 \cdot \alpha_v}(4\,\mathbf{max}(p)+2n+1) && \text{Definition of } u \\
&\leq \mathsf{H}_{\omega^2 \cdot \alpha_v}\mathsf{H}_{\omega^2}(u(\mathbf{max}(p),n)) \\
&= \mathsf{H}_{\omega^2 \cdot (\alpha_v+1)}(u(\mathbf{max}(p),n)) && \text{Lemma 9(4)} \\
&< \mathsf{H}_{\omega^2 \cdot (\alpha_v+1)}(u(\mathbf{max}(p),n)+1) && \text{Lemma 9(3)} \\
&\leq \mathsf{H}_{\omega^2 \cdot \alpha}(u(\mathbf{max}(p),n)+1) && \text{Lemma 9(2)}
\end{aligned}$$

The application of Lemma 9(2) in the last step is feasible as by definition $\alpha >_{(v)} \alpha_v$. An application of Lemma 9(1) yields $\alpha_v + 1 \leq_{(v+1)} \alpha$. From which we deduce $\omega^2 \cdot (\alpha_v + 1) \leq_{(v+1)} \omega^2 \cdot \alpha$.

Secondly, consider the case $\alpha_v >_{(v)} \beta$. In this case the proof follows the pattern of the above proof, but an additional application of Lemma 9(4) is required. This completes the proof of(5).

To prove the second part of (3), we proceed as follows: The fact that $\omega^\ell > o(p)$ is immediate from the definitions. Induction on p reveals that even $\omega^\ell >_{(\mathbf{max}(p))} o(p)$ holds. Thus in conjunction with the first part of (3), we obtain:

$$\begin{aligned}
\mathsf{Ah}_n(p) &\leq \mathsf{H}_{\omega^2 \cdot o(p)}(u(\mathbf{max}(p),n)+1) \leq \mathsf{H}_{\omega^{2+\ell}}(u(\mathbf{max}(p),n)+1) \\
&\leq \mathsf{H}_{\omega^{4+\ell}}(\mathbf{max}(p)+n) \ .
\end{aligned}$$

The last step follows as $2\,\mathbf{max}(p)+n+1 \leq \mathsf{H}_{\omega^2}(\mathbf{max}(p)+n)$. □

As a consequence of Lemma 10 and 11, we obtain the following proposition.

Theorem 3. *For all $n \geq 1$: If $\ell = |p|$, then $\mathsf{Ah}_n(p) \leq \mathsf{Ack}(2\ell+8, \mathbf{max}(p)+n)$.*

7 Derivation height of TRSs over infinite signatures compatible with KBOs

Based on Theorem 2 and 3 we obtain that the derivation height of $t \in \mathcal{T}(\mathcal{F})$ is bounded in the Ackermann function.

Theorem 4. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS, compatible with KBO. Assume the set $M := \{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}\}$ is finite and the arities of the symbols in $\mathcal{F}$ are bounded; set $K := \max(M)$. Then $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{Ack}(\mathcal{O}(|\mathsf{tw}(t)|) + \mathbf{max}(\mathsf{tw}(t)) + K, 0)$.*

Proof. We set $u(n, m) = 2n + m$ and keep the polynomial u fixed for the remainder. Let $p = \mathsf{tw}(t)$ and $\ell = |p|$. Due to Theorem 2 we conclude that $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{Ah}_K(p)$. It is easy to see that $\mathsf{Ack}(n, m) \leq \mathsf{Ack}(n + m, 0)$. Using this fact and Theorem 3 we obtain: $\mathsf{Ah}_K(p) \leq \mathsf{Ack}(\mathcal{O}(\ell), \mathbf{max}(p) + K) \leq \mathsf{Ack}(\mathcal{O}(\ell) + \mathbf{max}(p) + K, 0)$. Thus the theorem follows. □

For fixed $t \in \mathcal{T}(\mathcal{F})$ we can bound the argument of the Ackermann function in the above theorem in terms of the size of t. We define

$$r_{\max} := \mathsf{mrk}(t) \qquad w_{\max} := \max(\{\mathrm{w}(u) \mid u \in \mathsf{FS}(t) \cup \mathsf{Var}(t)\} .$$

Lemma 12. *For $t \in \mathcal{T}(\mathcal{F})$, let $r_{\max}$, $w_{\max}$ be as above. Let $b := \max\{\mathsf{Ar}(\mathcal{F}), 3\} + 1$, and set $n := \mathrm{Size}(t)$. Then $\mathrm{w}(t) \leq w_{\max} \cdot n$, $\mathsf{sp}(t) \leq n$, $\mathsf{mrk}(t) \leq r_{\max}$. Hence $|\mathsf{tw}(t)| \leq b^{w_{\max}(n) \cdot n + 1}$ and $\mathbf{max}(\mathsf{tw}(t)) \leq w_{\max}(n) \cdot n + r_{\max}$.*

Proof. The proof proceeds by induction on t. □

Corollary 1. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS, compatible with a KBO $\succ_{\mathsf{kbo}}$. Assume the set $\{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}\}$ is finite and the arites of the symbols in $\mathcal{F}$ are bounded. Then for $t \in \mathcal{T}(\mathcal{F})$, there exists a constant c—depending on t, $(\mathcal{F}, \mathcal{R})$, and $\succ_{\mathsf{kbo}}$—such that $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{Ack}(c^n, 0)$.*

Proof. The corollary is a direct consequence of Theorem 4 and Lemma 12. □

Remark 1. Note that it is not straight-forward to apply Theorem 4 to classify the derivational complexity of $\mathcal{R}$, over infinite signature, compatible with KBO. This is only possible in the (unlikely) case that for every term t the maximal rank $\mathsf{mrk}(t)$ and the weight $\mathrm{w}(t)$ of t can be bounded uniformly, i.e. independent of the size of t.

We apply Corollary 1 to the motivating example introduced in Section 1.

Example 3. Recall the definition of $\mathcal{R}$ and $\mathcal{R}' := \mathcal{R}_{\mathrm{lab}} \cup \mathcal{D}ec$ from Example 1 and 2 respectively. Let $s \in \mathcal{T}(\mathcal{F}_{\mathrm{lab}})$ be fixed and set $n := \mathrm{Size}(s)$.

Clearly the arities of the symbols in $\mathcal{F}_{\mathrm{lab}}$ are bounded. In Example 2 we indicated that the set $M = \{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}'\}$ is finite. Hence, Corollary 1 is applicable to conclude the existence of $c \in \mathbb{N}$ with $\mathsf{dh}_{\mathcal{R}'}(s) \leq \mathsf{Ack}(c^n, 0)$. In order to bound the derivation height of $\mathcal{R}$, we employ Lemma 1 to observe that for all $t \in \mathcal{T}(\mathcal{F})$: $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{dh}_{\mathcal{R}'}(\mathsf{lab}_\alpha(t))$, for arbitrary α. As $\mathrm{Size}(t) = \mathrm{Size}(\mathsf{lab}_\alpha(t))$ the above calculation yields

$$\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{dh}_{\mathcal{R}'}(\mathsf{lab}_\alpha(t)) \leq \mathsf{Ack}(c^n, 0) \ .$$

Note that c depends only on t, $\mathcal{R}'$ and the KBO $\succ_{\mathsf{kbo}}$ employed.

The main motivation of this work was to provide an alternative proof of Lepper's result that the derivational complexity of any *finite* TRS, compatible with KBO, is bounded by the Ackermann function, see [11]. We recall the definition of the *derivational complexity*:

$$\mathsf{dc}_{\mathcal{R}}(n) := \max(\{\mathsf{dh}_{\mathcal{R}}(t) \mid \mathrm{Size}(t) \leq n\}) \ .$$

Corollary 2. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS, compatible with KBO, such that $\mathcal{F}$ is finite. Then $\mathsf{dh}_{\mathcal{R}}(n) \leq \mathsf{Ack}(2^{\mathcal{O}(n)}, 0)$.*

Proof. As $\mathcal{F}$ is finite, the $K = \max(\{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}'\})$ and $\mathsf{Ar}(\mathcal{F})$ are obviously well-defined. Theorem 4 yields that $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{Ack}(\mathcal{O}(|\mathsf{tw}(t)|) + \mathbf{max}(\mathsf{tw}(t)) + K, 0)$. Again due to the finiteness of $\mathcal{F}$, for any $t \in \mathcal{T}(\mathcal{F})$, $\mathsf{mrk}(t)$ and $\mathrm{w}(t)$ can be estimated independent of t. A similar argument calculation as in Lemma 12 thus yields $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{Ack}(2^{\mathcal{O}(\mathrm{Size}(t))}, 0)$. Hence the result follows. □

Remark 2. Note that if we compare the above corollary to Corollary 19 in [11], we see that Lepper could even show that $\mathsf{dc}_{\mathcal{R}}(n) \leq \mathsf{Ack}(\mathcal{O}(n), 0)$. On the other hand, as already remarked above, Lepper's result is not admissible if the signature is infinite.

In concluding, we want to stress that the method is also applicable to obtain bounds on the derivational height of non simply terminating TRSs, a feature only shared by Hofbauer's approach to utilise context-dependent interpretations, cf. [20].

Example 4. Consider the TRS consisting of the following rules:

$$f(x) \circ (y \circ z) \to x \circ (f^2(y) \circ z) \qquad a(a(x)) \to a(b(a(x)))$$
$$f(x) \circ (y \circ (z \circ w)) \to x \circ (z \circ (y \circ w))$$
$$f(x) \to x$$

Let us call this TRS $\mathcal{R}$ in the following. Due to the rule $a(a(x)) \to a(b(a(x)))$, $\mathcal{R}$ is not simply terminating. And due to the three rules, presented on the left, the derivational complexity of $\mathcal{R}$ cannot be bounded by a primitive recursive function, compare [8].

Termination can be shown by semantic labelling, where the natural numbers are used as semantics and as labels. The interpretations $a_{\mathbb{N}}(n) = n+1$, $b_{\mathbb{N}}(n) = \max(\{0, n-1\})$, $f_{\mathbb{N}}(n) = n$, and $m \circ_{\mathbb{N}} n = m+n$ give rise to a quasi-model. Using the labelling function $\ell_a(n) = n$, termination of $\mathcal{R}' := \mathcal{R}_{\mathrm{lab}} \cup \mathcal{D}ec$ can be shown by an instance $\succ_{\mathsf{kbo}}$ of KBO with weight function $(\mathrm{w}, 1)$: $\mathrm{w}(\circ) = \mathrm{w}(f) = 0$, w$(b) = 1$, and w$(a_n) = n$ and precedence: $f \succ \circ \succ \dots a_{n+1} \succ a_n \succ \cdots \succ a_0 \succ b$. The symbol f is special. Clearly the arities of the symbols in $\mathcal{F}_{\text{lab}}$ are bounded. Further, it is not difficult to see that the set $M = \{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}'\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rk}(l)) \mid l \to r \in \mathcal{R}'\}$ is finite and $K := \max(M) = 2$.

Proceeding as in Example 3, we see that for each $t \in \mathcal{T}(\mathcal{F})$, there exists a constant c (depending on t, $\mathcal{R}'$ and $\succ_{\mathsf{kbo}}$) such that $\mathsf{dh}_{\mathcal{R}}(t) \leq \mathsf{Ack}(c^n, 0)$.

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