Foundations of Software Technology and Theoretical Computer Science (Bangalore) 2008.
Editors: R. Hariharan, M. Mukund, V. Vinay; pp 304-315

# Complexity Analysis of Term Rewriting Based on Matrix and Context Dependent Interpretations*

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ABSTRACT. For a given (terminating) term rewriting system one can often estimate its *derivational complexity* indirectly by looking at the proof method that established termination. In this spirit we investigate two instances of the interpretation method: *matrix interpretations* and *context dependent interpretations*. We introduce a subclass of matrix interpretations, denoted as *triangular matrix interpretations*, which induce polynomial derivational complexity and establish tight correspondence results between a subclass of context dependent interpretations and restricted triangular matrix interpretations. The thus obtained new results are easy to implement and considerably extend the analytic power of existing results. We provide ample numerical data for assessing the viability of the method.

## 1 Introduction

Term rewriting is a conceptually simple but Turing-complete model of computation. The foundation of rewriting is equational logic and term rewrite systems are conceivable as sets of directed equations. This orientation of equations naturally gives rise to computations, where a term is rewritten by successively replacing subterms by equal terms until no further reduction is possible. Such a sequence of rewrite steps is also called a *derivation*. In order to assess the complexity of a (terminating) term rewrite system (TRS for short) it is natural to look at the maximal length of derivations, as suggested by Hofbauer and Lautemann in [10]. More precisely, the *derivational complexity function* with respect to a (terminating) TRS $\mathcal{R}$ relates the length of a longest derivation sequence to the size of the initial term. Observe that the derivational complexity function is conceivable as a measure of proof complexity. Suppose an equational theory is representable as a convergent (i.e. a confluent and terminating) TRS, then rewriting to normal form induces an effective procedure to decide whether two terms are equal over a given equational theory. Thus the derivational complexity with respect to a convergent TRS essentially amounts to the *proof complexity* of this proof of identity.

For a given terminating TRS one can often estimate its derivational complexity indirectly by looking at the proof method that established termination. For example *polynomial*

*This research is partly supported by FWF (Austrian Science Fund) project P20133.

*interpretations* induce double-exponential derivational complexity (see [10], but also compare [8, 19, 9, 14, 7, 5, 12] for the derivational complexity induced by other termination techniques). The following example illustrates the situation.

*Example 1 ([9]).* Consider the following TRS $\mathcal{R}$ over the signature $\mathcal{F} = \{\circ, \mathsf{c}\}$.

$$(x \circ y) \circ z \to x \circ (y \circ z) \ .$$

It is easy to see that the polynomial interpretation $\mathcal{A}$ on the carrier $\mathbb{N} - \{0\}$ given through the interpretation functions $\circ_{\mathcal{A}}(n, m) = 2n + m$ and $\mathsf{c}_{\mathcal{A}} = 1$, is compatible with $\mathcal{R}$. Now, consider the (ground) terms $(t_n)_{n \in \mathbb{N}}$, defined as $t_0 := \mathsf{c}$ and $t_{n+1} = t_n \circ \mathsf{c}$. Note that the evaluation $[t_n]_{\mathcal{A}}$ of $t_n$ with respect to the algebra $\mathcal{A}$ is exponential in $n$. Hence the maximal length of a derivation starting from $t_n$ is (at most) exponential in $n$.

However the upper bound given in Example 1 is not optimal: The derivation length can be easily seen to be bounded quadratically in $n$. This overestimation is typical for polynomial interpretations. Hofbauer introduced *context dependent interpretations* as a remedy, cf. [9]. These interpretations extend traditional interpretations by introducing an additional parameter. The parameter changes in the course of evaluating a term, which makes the interpretation dependent on the context. With respect to Example 1 an interpretation can be found that estimates the derivation length optimally, compare [9]. Recently the first and second author introduced a technique to *automatically* search for context dependent interpretations, cf. [15]. This was achieved by delineating two subclasses of context dependent interpretations that made automation possible. However, up to now, we couldn't handle the TRS in Example 1 automatically.

In this paper we introduce an (easily automatable) technique to overcome this obstacle. We restrict *matrix interpretations* for terms (see [5], but compare also [11]) in such a way that we only employ matrices of particular simple form in the interpretation functions. Such interpretations (called *triangular matrix interpretations*) induce at most polynomial derivational complexity, where the degree of the polynomial depends on the dimension of the matrix.

Moreover, we identify a subclass of context dependent interpretations and a subclass of (two-dimensional) matrix interpretations which correspond to each other with respect to orientability: For any context dependent interpretation $\mathcal{C}$ from this class that is compatible with a TRS $\mathcal{R}$ there exists a matrix interpretation $\mathcal{A}$ compatible with $\mathcal{R}$ and vice versa. This theoretical result is interesting in its own right as it links two different termination techniques, which were previously conceived as incomparable.

The obtained new techniques are easy to implement and considerably extend the analytic power of existing results. We provide ample numerical data for assessing the viability of the method. In particular, we want to emphasise that Example 1 can be handled fully automatically and the resulting estimation on the derivational complexity is optimal.

The remainder of this paper is organised as follows. In the next section we recall basic notions. Section 3 introduces triangular matrix interpretations, while in Section 4 we recall context dependent interpretations and state the correspondence result mentioned above. In Section 5 we provide the experimental data for our implementation. Finally in Section 6 we conclude and mention possible future work.

## 2 Preliminaries

We assume familiarity with term rewriting [2, 18] but briefly review basic concepts and notations. Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a signature. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$. $\mathcal{V}\mathsf{ar}(t)$ denotes the set of variables occurring in a term $t$ and the *size* $|t|$ of a term is defined as the number of symbols in $t$, i.e., for example the size of the term $\mathsf{f}(\mathsf{a},x)$ is 3. The *depth* $\mathsf{dp}(t)$ of a term $t$ is defined as follows: (i) $\mathsf{dp}(t) := 0$, if $t$ is a variable or a constant and (ii) $\mathsf{dp}(f(t_1,\ldots,t_n)) := 1 + \max\{\mathsf{dp}(t_i) \mid 1 \leqslant i \leqslant n\}$.

A *term rewrite system* (*TRS* for short) $\mathcal{R}$ over $\mathcal{T}(\mathcal{F},\mathcal{V})$ is a *finite* set of rewrite rules $l \to r$, such that $l \notin \mathcal{V}$ and $\mathcal{V}\mathsf{ar}(l) \supseteq \mathcal{V}\mathsf{ar}(r)$. A relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ is a *rewrite relation* if it is compatible with $\mathcal{F}$-operations and closed under substitutions. The smallest rewrite relation that contains $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$. The transitive and reflexive closure of $\to_{\mathcal{R}}$ is denoted by $\to^*_{\mathcal{R}}$. We simply write $\to$ for $\to_{\mathcal{R}}$ if $\mathcal{R}$ is clear from context. A term $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ is called a *normal form* if there is no $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$ such that $s \to t$.

A TRS is called *confluent* if for all $s, t_1, t_2 \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with $s \to^* t_1$ and $s \to^* t_2$ there exists a term $t_3$ such that $t_1 \to^* t_3$ and $t_2 \to^* t_3$. We call a TRS *terminating* if no infinite rewrite sequence exists. Let $s$ and $t$ be terms. If exactly $n$ steps are performed to rewrite $s$ to $t$ we write $s \to^n t$. The *derivation length* of a terminating term $t$ with respect to a TRS $\mathcal{R}$ is defined as: $\mathsf{dl}(s,\to_{\mathcal{R}}) = \max\{n \mid \exists t\ s \to^n_{\mathcal{R}} t\}$. The *derivational complexity* (with respect to $\mathcal{R}$) is defined as follows:

$$\mathsf{dc}_{\mathcal{R}}(n) = \max\{\mathsf{dl}(t,\to_{\mathcal{R}}) \mid |t| \leqslant n\} .$$

We sometimes say the derivational complexity of $\mathcal{R}$ is *linear*, *quadratic*, or *polynomial* if $\mathsf{dc}_{\mathcal{R}}(n)$ is bounded linearly, quadratically, or polynomially in $n$, respectively.

A *proper order* is a transitive and irreflexive relation. A proper order $\succ$ is *well-founded* if there is no infinite decreasing sequence $t_1 \succ t_2 \succ t_3 \cdots$. A well-founded proper order that is also a rewrite relation is called a *reduction order*. We say a reduction order $\succ$ and a TRS $\mathcal{R}$ are *compatible* if $\mathcal{R} \subseteq \succ$. It is well-known that a TRS is terminating if and only if there exists a compatible reduction order. An $\mathcal{F}$-*algebra* $\mathcal{A}$ consists of a carrier set $A$ and a collection of interpretations $f_{\mathcal{A}}$ for each function symbol in $\mathcal{F}$. A *well-founded* and *monotone* algebra (*WMA* for short) is a pair $(\mathcal{A},\succ)$, where $\mathcal{A}$ is an algebra and $\succ$ is a well-founded proper order on $A$ such that every $f_{\mathcal{A}}$ is monotone in all arguments. An *assignment* $\alpha\colon \mathcal{V} \to A$ is a function mapping variables to elements in the carrier. Let $[\alpha]_{\mathcal{A}}(\cdot)$ denote the usual evaluation function associated with $\mathcal{A}$. A WMA naturally induces a proper order $\succ_{\mathcal{A}}$ on terms: $s \succ_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) \succ [\alpha]_{\mathcal{A}}(t)$ for all assignments $\alpha\colon \mathcal{V} \to A$.

## 3 Matrix Interpretations That Induce Polynomial Derivational Complexity

In this section we introduce a specific form of matrix interpretations, called *triangular matrix interpretations*, which induce a polynomial upper bound on the derivational complexity. This contrasts with general matrix interpretations, which yield an exponential upper bound, cf. [5]. Hence the introduced restriction defines a strict subclass of those TRSs that admit a matrix interpretation.

We start by recalling the concept of *matrix interpretations* (see [5] but compare also [11]). Let $\mathcal{F}$ denote a signature. We fix a dimension $d \in \mathbb{N}$ and use the set $\mathbb{N}^d$ as the carrier of an algebra $\mathcal{A}$, together with the following extension of the natural order $>$ on $\mathbb{N}$:

$$(x_1, x_2, \ldots, x_d) > (y_1, y_2, \ldots, y_d) :\Longleftrightarrow x_1 > y_1 \wedge x_2 \geqslant y_2 \wedge \ldots \wedge x_d \geqslant y_d \ .$$

For each $n$-ary function symbol $f$, we choose as an interpretation a linear function of the following shape:

$$f_{\mathcal{A}} : (\mathbb{N}^d)^n \to \mathbb{N}^d : (\vec{v}_1, \ldots, \vec{v}_n) \mapsto F_1\vec{v}_1 + \ldots + F_n\vec{v}_n + \vec{f} \ ,$$

where $\vec{v}_1, \ldots, \vec{v}_n$ are (column) vectors of variables, $F_1, \ldots, F_n$ are matrices (each of size $d \times d$), and $\vec{f}$ is a vector over $\mathbb{N}$. Moreover, for any $i$ ($1 \leqslant i \leqslant n$) the top left entry $(F_i)_{1,1}$ is positive. It is easy to see that the algebra forms a well-founded monotone algebra.

The following lemma states how compatibility of a matrix interpretation $\mathcal{A}$ with a given rewrite system can be easily verified (compare [5, Lemma 4]).

**LEMMA 2.** *Let $\mathcal{A}$ be a matrix interpretation and let $\mathcal{R}$ be a TRS. Let $l \to r \in \mathcal{R}$, let $k$ denote the number of variables in $l$ (and $r$) and let $\alpha$ be an assignment. Then there exist matrices $L_1, \ldots, L_k, R_1, \ldots, R_k$ and vectors $\vec{l}, \vec{r}$ such that $[\alpha]_{\mathcal{A}}(l) = \sum_{i=1}^{k} L_i\vec{x}_i + \vec{l}$ and $[\alpha]_{\mathcal{A}}(r) = \sum_{i=1}^{k} R_i\vec{x}_i + \vec{r}$. Moreover $l >_{\mathcal{A}} r$ if and only if $\vec{l} > \vec{r}$ and $L_i \geqslant R_i$ for all $1 \leqslant i \leqslant k$. (Here $\geqslant$ refers to the point-wise extension of the standard order on natural numbers to matrices.)*

We are now going to restrict the shape of the matrices, in order to obtain better bounds on derivational complexities.

**DEFINITION 3.** *An* upper triangular matrix *is a matrix $M$ in $\mathbb{N}^{d \times d}$ such that for all $d \geqslant i > j \geqslant 1$, we have $M_{i,j} = 0$, and for all $d \geqslant i \geqslant 1$, we have $M_{i,i} \leqslant 1$.*

We say that a TRS $\mathcal{R}$ admits a *triangular matrix interpretation* (*TMI* for short) if $\mathcal{R}$ is compatible with a matrix interpretation $\mathcal{A}$ and all matrices employed in $\mathcal{A}$ are of upper triangular form.

*Example 4 (continued from Example 1).* We define a triangular matrix interpretation $\mathcal{A}$, as follows:

$$\circ_{\mathcal{A}}(\vec{x}, \vec{y}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is easy to see that $\mathcal{A}$ is compatible with $\mathcal{R}$, i.e., $[\alpha]_{\mathcal{A}}((x \circ y) \circ z) > [\alpha]_{\mathcal{A}}(x \circ (y \circ z))$ holds for all assignments $\alpha$.

**LEMMA 5.** *Let $M$ be an upper triangular matrix in $\mathbb{N}^{d \times d}$ and $n \in \mathbb{N}$. Then all entries of $M^n$ are polynomially bounded in $n$. More precisely, if $i > j$ then $(M^n)_{i,j} = 0$, otherwise $(M^n)_{i,j} \leqslant (j-i)!(an)^{j-i}$, where $a = \max\{M_{i,j} \mid 1 \leqslant i, j \leqslant d\}$.*

PROOF. The case $i > j$ is easy to see. In the other case, we have $j \geqslant i$. Then the lemma follows by a straightforward induction on $j - i$. ∎

Due to Lemma 5, for any finite subset $M \subseteq \mathbb{N}^{d \times d}$ of upper triangular matrices, there is a polynomial $p$ of degree $d - 1$ such that for each sequence $M_1 \in M, \ldots, M_n \in M$, and for each $i, j$, it holds that $(M_1 \cdot \ldots \cdot M_n)_{i,j} \leqslant p(n)$. Such products occur when computing values of matrix interpretations on (ground) terms: For example, let $\mathcal{A}$ denote a matrix interpretation, $\alpha$ an arbitrary assignment and $t = \mathsf{f}(\mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{c})$. Then

$$[\alpha]_{\mathcal{A}}(t) = F_1 G_1 \vec{a} + F_1 G_2 \vec{b} + F_1 \vec{g} + F_2 \vec{c} + \vec{f} \,.$$

Clearly the length of each product is at most the depth of the term, which is smaller or equal to its size. Hence the entries in each product are polynomially bounded (with degree $d-1$) in the size of $t$. The number of products equals the number of subterms of $t$, which is exactly the size of $t$. Therefore, the entries in $[\alpha]_{\mathcal{A}}(t)$ are bounded by a polynomial of degree $d$ in the size of $t$. This observation leads us directly to the main result of this section.

**THEOREM 6.** *If a TRS $\mathcal{R}$ admits a triangular matrix interpretation $\mathcal{A}$ of dimension $d$, then the derivational complexity of $\mathcal{R}$ is bounded by a polynomial of degree $d$.*

PROOF. Any $k$-step derivation $s \to_{\mathcal{R}}^{k} t$ implies $[\alpha]_{\mathcal{A}}(s) >^{k} [\alpha]_{\mathcal{A}}(t)$, referring to the $k$-th iterate of the relation $>$ on $\mathbb{N}^d$ defined earlier. This implies $[\alpha]_{\mathcal{A}}(s)_1 \geq k + [\alpha]_{\mathcal{A}}(t)_1 \geq k$. So the top entry in $[\alpha]_{\mathcal{A}}(s)$ bounds the length of any derivation starting at $s$. In conjunction with the above observation, this suffices to prove the theorem. ∎

*Example 7.* It is easy to see that the derivational complexity of the following TRS $\mathcal{R}_1 = \{\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(x)), \mathsf{c}(\mathsf{a}(x)) \to \mathsf{b}(\mathsf{c}(x)), \mathsf{c}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{c}(x))\}$ is (at least) cubic. The following TMI $\mathcal{A}$ is compatible with $\mathcal{R}_1$.

$$\mathsf{a}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathsf{c}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Applying Theorem 6 we conclude that the derivational complexity function with respect to $\mathcal{R}_1$ is cubic.

Despite Example 7 the criterion is in general not complete. There exist TRSs of polynomial derivational complexity that do not admit a compatible TMI. One such example can be found in [5]: $\mathcal{R}_2 = \{\mathsf{f}(\mathsf{a}, \mathsf{b}) \to \mathsf{f}(\mathsf{b}, \mathsf{b}), \mathsf{f}(\mathsf{b}, \mathsf{a}) \to \mathsf{f}(\mathsf{a}, \mathsf{a})\}$. Then $\mathcal{R}_2$ has linear derivational complexity, but in fact no compatible matrix interpretation can exist, cf. [5]. Even if there is a compatible matrix interpretation and the complexity of the system is polynomial, it might be lacking a triangular interpretation. Consider the following TRS: $\mathcal{R}_3 = \{\mathsf{a}(\mathsf{a}(x)) \to \mathsf{b}(\mathsf{c}(x)), \mathsf{b}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{c}(x)), \mathsf{c}(\mathsf{c}(x)) \to \mathsf{a}(\mathsf{b}(x))\}$, a straightforward adaption of the string rewrite system `z086` introduced by Zantema as TRS. We conjecture that $\mathcal{R}_3$ admits at most polynomial derivational complexity and is not compatible with a triangular matrix interpretation. (This is related to the open problem number 105 in the RTA list of open problems, see `http://rtaloop.pps.jussieu.fr/`.)

We conclude this section by considering the following example, that can be handled automatically by TMIs, but not with any other known method. (See Section 5 for further details about the implementation.)

*Example 8.* Consider the TRS $\mathcal{R}_4$ with the following rewrite rules, which is example 4.30 from [17]:

$$\begin{aligned} \mathsf{f}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{g}(\mathsf{nil}) &\to \mathsf{nil} \\ \mathsf{f}(\mathsf{nil} \circ y) &\to \mathsf{nil} \circ \mathsf{f}(y) & \mathsf{g}(x \circ \mathsf{nil}) &\to \mathsf{g}(x) \circ \mathsf{nil} \\ \mathsf{f}((x \circ y) \circ z) &\to \mathsf{f}(x \circ (y \circ z)) & \mathsf{g}(x \circ (y \circ z)) &\to \mathsf{g}((x \circ y) \circ z) \end{aligned}$$

It is not difficult to check that the following TMI $\mathcal{A}$ of dimension 4 is compatible with $\mathcal{R}_4$.

$$\circ_{\mathcal{A}}(\vec{x}, \vec{y}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathsf{nil}_{\mathcal{A}} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathsf{f}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathsf{g}_{\mathcal{A}}(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Due to Theorem 6 we conclude that $\mathsf{dc}_{\mathcal{R}_4}$ can be asymptotically bounded by a polynomial of degree 4. Notice that this bound is not optimal, as it is easy to see that the derivational complexity is quadratic.

## 4 Context Dependent Interpretations and Matrices

In this section, we show a tight correspondence between (triangular) matrix interpretations as introduced in Section 3 and context dependent interpretations, see [9]. More precisely we define a subclass of context dependent interpretations such that any such interpretation $\mathcal{C}$ gives rise to a restricted TMI $\mathcal{A}$ and vice versa. Moreover $\mathcal{C}$ is compatible with a TRS $\mathcal{R}$ if and only if $\mathcal{A}$ is compatible with $\mathcal{R}$.

We recall context dependent interpretations. For that we follow the presentation in [15] in a simplified form. See [9, 15] for motivating examples and intuitions behind the definitions. A *context dependent $\mathcal{F}$-algebra* (*CDA* for short) $\mathcal{C}$ is a family of $\mathcal{F}$-algebras over the reals. A CDA $\mathcal{C}$ associates to each function symbol $f \in \mathcal{F}$ of arity $n$, a collection of $n+1$ mappings $f_{\mathcal{C}} \colon \mathbb{R}^+ \times (\mathbb{R}_0^+)^n \to \mathbb{R}_0^+$ and $f_{\mathcal{C}}^i \colon \mathbb{R}^+ \to \mathbb{R}^+$ for all $1 \leqslant i \leqslant n$. As usual $f_{\mathcal{C}}$ is called the *interpretation function*, while the mappings $f_{\mathcal{C}}^i$ are called the *parameter functions*. In addition $\mathcal{C}$ is equipped with a set $\{>_\Delta \mid \Delta \in \mathbb{R}^+\}$ of strict orders, where we define: $z >_\Delta z'$ if and only if $z - z' \geqslant \Delta$. Let $\mathcal{C}$ be a CDA and let a $\Delta$-*assignment* denote a mapping $\alpha : \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$. We define a mapping $[\alpha, \Delta]_{\mathcal{C}}$ from the set of terms into the set $\mathbb{R}_0^+$ of non-negative reals:

$$[\alpha, \Delta]_{\mathcal{C}}(t) := \begin{cases} \alpha(\Delta, t) & \text{if } t \in \mathcal{V} \\ f_{\mathcal{C}}(\Delta, [\alpha, f_{\mathcal{C}}^1(\Delta)]_{\mathcal{C}}(t_1), \ldots, [\alpha, f_{\mathcal{C}}^n(\Delta)]_{\mathcal{C}}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \,. \end{cases}$$

We fix some notational conventions: Due to the special role of the additional variable $\Delta$, we often write $f_{\mathcal{C}}[\Delta](z_1, \ldots, z_n)$ instead of $f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n)$. If $t$ is a ground term, we sometimes write $[\Delta]_{\mathcal{C}}(t)$ instead of $[\alpha, \Delta]_{\mathcal{C}}(t)$. We say that a CDA $\mathcal{C}$ is $\Delta$-*monotone* if for all

$\Delta \in \mathbb{R}^+$ and for all $a_1, \ldots, a_n, b \in \mathbb{R}_0^+$ with $a_i >_{f_\mathcal{C}^i(\Delta)} b$ for some $i \in \{1, \ldots, n\}$, we have $f_\mathcal{C}[\Delta](a_1, \ldots, a_i, \ldots, a_n) >_\Delta f_\mathcal{C}[\Delta](a_1, \ldots, b, \ldots, a_n)$. A CDA $\mathcal{C}$ and a TRS $\mathcal{R}$ are *compatible* if for every rewrite rule $l \to r \in \mathcal{R}$, every $\Delta \in \mathbb{R}^+$, and any $\Delta$-assignment $\alpha$: $[\alpha, \Delta](l) >_\Delta [\alpha, \Delta](r)$ holds.

**DEFINITION 9.** *A* $\Delta^2$-interpretation *is a CDA* $\mathcal{C}$ *with interpretation functions and parameter functions of the following form:*

$$f_\mathcal{C}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_{(f,i)} z_i + \sum_{i=1}^{n} b_{(f,i)} z_i \Delta + g_f + h_f \Delta \tag{†}$$

$$f_\mathcal{C}^i(\Delta) = \frac{c_{(f,i)} + d_{(f,i)}\Delta}{a_{(f,i)} + b_{(f,i)}\Delta}, \tag{‡}$$

*where* $a_{(f,i)} > 0$ *or* $b_{(f,i)} > 0$ *(for each* $f \in \mathcal{F}$, $1 \leqslant i \leqslant n$*) and the occurring coefficients are natural numbers.*

The following lemma is a direct consequence of the definitions.

**LEMMA 10.** *Let* $\mathcal{C}$ *denote a* $\Delta^2$*-interpretation. If for all* $f \in \mathcal{F}$, $1 \leqslant i \leqslant n$ *in* (‡)*, we have* $d_{(f,i)} \geqslant 1$*, then* $\mathcal{C}$ *is* $\Delta$*-monotone.*

In [15] two (strict) subclasses of $\Delta^2$-interpretations were studied: $\Delta$-linear interpretations and $\Delta$-restricted interpretations. A $\Delta$-*linear interpretation* is a $\Delta^2$-interpretation, where for the parameter functions as presented in (‡) we have $c_{(f,i)} = 0$ and $d_{(f,i)} = 1$ for all $f \in \mathcal{F}$, $1 \leqslant i \leqslant n$, and a $\Delta$-*restricted interpretation* is a $\Delta$-linear interpretation with the additional requirement that $a_{(f,i)} \in \{0, 1\}$.

*Example 11 (continued from Example 1).* Consider the following $\Delta$-linear interpretation $\mathcal{C}$:

$$\circ_\mathcal{C}[\Delta](x, y) = (1 + \Delta)x + y + 1 \qquad \circ_\mathcal{C}^1(\Delta) = \frac{\Delta}{1 + \Delta} \qquad \circ_\mathcal{C}^2(\Delta) = \Delta$$

For all ground terms $r, s, t$, we have $[\Delta]_\mathcal{C}((r \circ s) \circ t) - [\Delta]_\mathcal{C}(r \circ (s \circ t)) \geqslant \Delta$. This is shown in [9, Lemma 3] by an inductive argument. However, this argument is not well-suited for automation: The implementation described in [15] doesn't find this interpretation.

*Example 12 (continued from Example 11).* The $\Delta$-linear interpretation $\mathcal{C}$ is also a $\Delta$-restricted interpretation. Due to [15, Theorem 29] we conclude quadratic derivational complexity for $\mathcal{R}$. Note that this upper bound is optimal.

It seems worthy of note, that the matrix interpretation $\mathcal{A}$ employed in Example 4 is obtained fully automatically, while the context dependent interpretation $\mathcal{C}$ is obtained by hand. On the other hand $\mathcal{A}$ and $\mathcal{C}$ use exactly the same coefficients. We exploit this observation below.

**DEFINITION 13.** *Let* $\mathcal{C}$ *be a CDA and let* $\mathcal{A}$ *be a matrix interpretation over two dimensions. We say the* $\Delta$*-assignment* $\alpha$: $\mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ *and the assignment* $\alpha'$: $\mathcal{V} \to \mathbb{N}^2$ *are* corresponding *if for all variables* $x$ *and all* $\Delta \in \mathbb{R}^+$: $\alpha(\Delta, x) = a + b\Delta$ *if and only if* $\alpha'(x) = \begin{pmatrix} b \\ a \end{pmatrix}$.

We arrive at the main lemma of this section, whose technical, but not difficult proof has been omitted due to space restrictions.

**LEMMA 14.** *Let $\mathcal{C}$ denote a $\Delta^2$-interpretation:*

$$f_{\mathcal{C}}(\Delta, z_1, \dots, z_n) = \sum_{i=1}^{n} a_{(f,i)} z_i + \sum_{i=1}^{n} b_{(f,i)} z_i \Delta + g_f + h_f \Delta$$
$$f_{\mathcal{C}}^i(\Delta) = \frac{c_{(f,i)} + d_{(f,i)}\Delta}{a_{(f,i)} + b_{(f,i)}\Delta} ,$$

*and let $\mathcal{A}$ denote a matrix interpretation of the following form:*

$$f_{\mathcal{A}}(x_1, \dots, x_n) = (\sum_{i=1}^{n} \begin{pmatrix} d_{(f,i)} & b_{(f,i)} \\ c_{(f,i)} & a_{(f,i)} \end{pmatrix} \cdot x_i) + \begin{pmatrix} h_f \\ g_f \end{pmatrix} ,$$

*where $d_{(f,i)} \geqslant 1$, and either $a_{(f,i)} > 0$ or $b_{(f,i)} > 0$ for all $f \in \mathcal{F}$ and $1 \leqslant i \leqslant n$. Then for any term $t$*

$$[\alpha, \Delta]_{\mathcal{C}}(t) = s_1 + s_2\Delta \quad \Longleftrightarrow \quad [\alpha']_{\mathcal{A}}(t) = \begin{pmatrix} s_2 \\ s_1 \end{pmatrix} ,$$

*whenever $\alpha$ and $\alpha'$ are corresponding.*

We say a matrix interpretation $\mathcal{A}$ *corresponds* to a $\Delta^2$-interpretation $\mathcal{C}$, if $\mathcal{A}$ and $\mathcal{C}$ are defined as in Lemma 14.

*Example 15.* Consider the triangular matrix interpretation $\mathcal{A}$ introduced in Example 4 and the $\Delta$-restricted interpretation $\mathcal{C}$ from Example 11. Then it is easy to see that $\mathcal{A}$ and $\mathcal{C}$ are corresponding.

**THEOREM 16.** *Let $\mathcal{R}$ be a TRS and let $\mathcal{C}$ be a $\Delta^2$-interpretation such that $\mathcal{R}$ is compatible with $\mathcal{C}$. Then there exists a corresponding matrix interpretation $\mathcal{A}$ (of dimension 2) compatible with $\mathcal{R}$.*

PROOF. Let $\alpha \colon \mathcal{V} \to \mathbb{N}^2$ be arbitrary, but fixed. To prove the theorem, it suffices to verify that for any rule $l \to r \in \mathcal{R}$: $[\alpha]_{\mathcal{A}}(l) > [\alpha]_{\mathcal{A}}(r)$ holds, where $\mathcal{A}$ is the matrix interpretation constructed in Lemma 14. To apply Lemma 14, we choose a $\Delta$-assignment $\alpha' \colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$ that corresponds to $\alpha$. For every $l \to r \in \mathcal{R}$, for every $\Delta \in \mathbb{R}^+$, and every $\alpha' \colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$, we have $\big([\alpha', \Delta]_{\mathcal{C}}(l) - [\alpha', \Delta]_{\mathcal{C}}(r)\big) = a + b\Delta$. Here we make use of the fact that for any term $t$: $[\alpha', \Delta]_{\mathcal{C}}(t) = c + d\Delta$. This follows from an inductive argument employing the assumed form of the assignment $\alpha'$. Moreover, as $a + b\Delta \geqslant \Delta$, we conclude $a \geqslant 0$ and $b \geqslant 1$. Thus an application of Lemma 14 yields

$$\big([\alpha]_{\mathcal{A}}(l) - [\alpha]_{\mathcal{A}}(r)\big) = \begin{pmatrix} b \\ a \end{pmatrix} \geqslant \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

from which compatibility with $\mathcal{A}$ directly follows. ∎

An easy consequence of Theorem 16 in conjunction with Theorem 6 is that $\Delta^2$-interpretations induce at most exponential derivational complexity. In particular, we obtain the following corollary. (A direct proof of this result, i.e., a proof that argues only about context dependent interpretations, can be found in [15].)

**COROLLARY 17.** *Let $\mathcal{R}$ be a TRS and let $\mathcal{C}$ denote a $\Delta$-linear interpretation compatible with $\mathcal{R}$. Then $\mathcal{R}$ is terminating and $\mathsf{dc}_{\mathcal{R}}(n) = 2^{\mathsf{O}(n)}$. Moreover, if $\mathcal{C}$ is a $\Delta$-restricted interpretation, then $\mathsf{dc}_{\mathcal{R}}(n) = \mathsf{O}(n^2)$. Observe that the bounds are tight, i.e., we can find TRSs $\mathcal{R}$ that fulfill these requirements, such that $\mathsf{dc}_{\mathcal{R}}$ is an exponential or quadratic function, respectively.*

Theorem 16 raises the question, whether the other direction may hold for $\Delta^2$-interpretations: Given a matrix interpretation, compatible with $\mathcal{R}$, does there exist a $\Delta^2$-interpretation that is compatible with $\mathcal{R}$? Recall that a CDA $\mathcal{C}$ is compatible with a TRS $\mathcal{R}$, if for every $l \to r \in \mathcal{R}$, for every $\Delta \in \mathbb{R}^+$, and every $\alpha\colon \mathbb{R}^+ \times \mathcal{V} \to \mathbb{R}_0^+$, we have $[\alpha, \Delta]_{\mathcal{C}}(l) - [\alpha, \Delta]_{\mathcal{C}}(r) \geqslant \Delta$. The next example shows that this definition of compatibility is too general in this context.

*Example 18 (continued from Example 4).* The $\Delta$-restricted interpretation $\mathcal{C}$ is not compatible with $\mathcal{R}$ as defined above, i.e. we do not have $[\alpha, \Delta]_{\mathcal{C}}((x \circ y) \circ z) - [\alpha, \Delta]_{\mathcal{C}}(x \circ (y \circ z)) \geqslant \Delta$ for arbitrary assignments $\alpha$. To construct a counter-example we set $\alpha(\Delta, x) := \Delta^2$, and $\alpha(\Delta, u)$ is arbitrary for $u \neq x$. Following the proof of Lemma 3 in [9], we obtain

$$[\alpha, \Delta]_{\mathcal{C}}((x \circ y) \circ z) - [\alpha, \Delta]_{\mathcal{C}}(x \circ (y \circ z)) = (1 + 2\Delta)[\alpha, \frac{\Delta}{1+2\Delta}]_{\mathcal{C}}(x) +$$
$$+\Delta - (1+\Delta)[\alpha, \frac{\Delta}{1+\Delta}]_{\mathcal{C}}(x) = \Delta + \frac{\Delta^2}{1+2\Delta} - \frac{\Delta^2}{1+\Delta} \not\geqslant \Delta\ .$$

This violates the compatibility condition.

Example 18 motivates the next definition.

**DEFINITION 19.** *We say a $\Delta$-assignment $\alpha\colon \Delta \times \mathcal{V} \to \mathbb{R}_0^+$ is* linear*, if there exist natural numbers $a$ and $b$, such that $\alpha(\Delta, x) = a + b\Delta$.*

*Example 20 (continued from Example 18).* For any linear $\Delta$-assignment $\alpha$, we have $[\alpha, \Delta]_{\mathcal{C}}((x \circ y) \circ z) - [\alpha, \Delta]_{\mathcal{C}}(x \circ (y \circ z)) \geqslant \Delta$. This can be seen by just applying a linear $\Delta$-assignment of the following parametric form: $\alpha(\Delta, x) = x_1 + x_2\Delta$, $\alpha(\Delta, y) = y_1 + y_2\Delta$, and $\alpha(\Delta, z) = z_1 + z_2\Delta$.

**LEMMA 21.** *Let $\sigma$ be a ground substitution and let $\mathcal{C}$ be a $\Delta^2$-interpretation. Then there exists a linear $\Delta$-assignment $\alpha$ such that $[\Delta]_{\mathcal{C}}(t\sigma) = [\alpha, \Delta]_{\mathcal{C}}(t)$ for all $\Delta \in \mathbb{R}^+$ and terms $t$.*

PROOF. We set $\alpha(\Delta, x) = [\Delta]_{\mathcal{C}}(x\sigma)$ for any $x \in \mathsf{dom}(\sigma)$ and $\alpha(\Delta, x) = 0$ otherwise. The fact that $[\Delta]_{\mathcal{C}}(x\sigma)$ has a linear shape can be shown by an easy induction on $x\sigma$. ∎

By now, the following main result of this section, is an easy consequence of Lemma 14, Theorem 16 and Lemma 21.

**THEOREM 22.** *Let $\mathcal{A}$ be a monotone matrix interpretation of dimension 2 such that no zero column occurs in any matrix, let $\mathcal{C}$ be the corresponding $\Delta^2$-interpretation and let $\mathcal{R}$ be a TRS. Then $\mathcal{A}$ is compatible with $\mathcal{R}$ if and only if for all linear $\Delta$-assignments $\alpha$, all $\Delta \in \mathbb{R}^+$ and all rules $l \to r \in \mathcal{R}$, we have $[\alpha, \Delta]_{\mathcal{C}}(l) >_{\Delta} [\alpha, \Delta]_{\mathcal{C}}(r)$.*

Note that the restriction on zero columns expressed in the assumptions of the theorem appears to be negligible, if we consider the automation of the introduced techniques. This is the subject of the next section.

Table 1: Termination Methods as Complexity Analysers

| | BOUNDS | CDI | TMI | | | | TMI+BOUNDS |
|---|---|---|---|---|---|---|---|
| dimension | | | 2 | 3 | 4 | 5 | 3 |
| # successes | 125 | 85 | 143 | 158 | 154 | 156 | 216 |
| avg. success time | 0.68 | 3.84 | 0.19 | 1.33 | 0.56 | 2.39 | 8.65 |
| # timeouts | 328 | 272 | 66 | 224 | 237 | 244 | 237 |

# 5 Experiments

We have implemented the methods described in this paper, and tested their viability to analyse polynomial derivational complexity on version 4.0 of the Termination Problem Data Base (TPDB for short), which is used in the annual RTA termination competition. (Available at `http://www.lri.fr/~marche/tpdb/`, but we also included the secret systems from the competition 2007.) This database contains a total of 1381 TRSs, 957 of which are known to be terminating. Arguably, the TPDB is an imperfect choice as it has been designed to test the strength of termination provers in rewriting, not as a testbed to analyse feasible bounds on the derivational complexity of TRSs. Examples such as TRS encodings of the Ackermann function and the Hydra Battle reinforce this point. On the other hand, the TPDB is the only relatively large collection of TRSs that is publicly available.

We briefly sketch our implementation: Similar to [3], we build a set of Diophantine constraints which express all necessary restrictions on the matrix interpretation. Then, we put a finite upper bound on the variables in the constraints and encode these constraints as a problem of propositional logic (see [6] but also [5]). We give the final SAT problem to MiniSAT [4] and use a satisfying assignment to construct a suitable matrix interpretation, where we use the fact that all matrix products are upper triangular, so values for entries below the main diagonal can be ignored.

In order to compare $\Delta$-restricted interpretations (referred to by CDI) and triangular matrix interpretations (TMI) to other results, we compared them to the implementation of the match-bound technique (BOUNDS for short) as in [13]: Linear TRSs are tested for match-boundedness, non-linear, but non-duplicating TRSs are tested for match-raise-boundedness. It is not difficult to see that this technique implies *linear* derivational complexity. Last, we tested the union of the two strongest methods (BOUNDS and TMI for dimension 3) by using half of the time on TMI and the rest on BOUNDS. For both CDI and TMI, we restricted all coefficients to at most 15 (allowing us to use at most 4 bits to encode each coefficient).

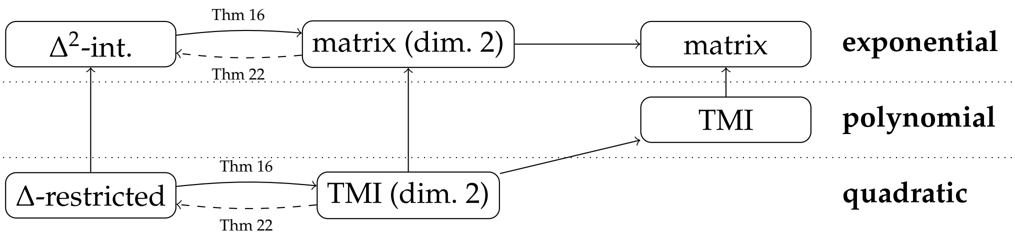
All tests were executed single-threaded on a server equipped with 8 AMD Opteron™ 2.8 GHz dual core processors with 64GB of RAM. We used a timeout of 60 seconds for each TRS. The results of the tests are shown in Table 1 (see
`http://cl-informatik.uibk.ac.at/users/aschnabl/experiments/08msw/`
for the full experimental evidence). The times given in the table are seconds. (In examples for which the according method was neither successful nor had a timeout, the proof attempt was given up before the timeout.)

As we can see, triangular matrix interpretations are clearly the most powerful method for proving polynomial derivational complexity of rewriting on our testbed. As suggested by our results in Section 4, the systems that can be handled by Δ-restricted interpretations are a strict subset of the problems solved by TMI with matrices of dimension 2. We want to note that out of the latter 143 systems, 140 can still be handled with the restriction on zero columns in Theorem 22. In total, TMIs of dimensions 2 to 5 are successful in 162 instances.

It is worthy of note that "full" matrix interpretations (for dimensions 2 to 5 and coefficients at most 15) can handle (only) 222 TRSs. Hence a clear majority of those systems that can in principle shown to be terminating with matrix interpretations have polynomial derivational complexity.

## 6 Conclusion

In this paper we studied the complexity of rewrite systems $\mathcal{R}$ as expressed by the derivational complexity function $\mathsf{dc}_{\mathcal{R}}$. The following diagram provides a condensed view of the studied classes of matrix and context dependent interpretations, where the right column gives the induced derivational complexity. (The arrows depict set inclusions and the dashed arrows refer to the additional restriction on the zero columns.)




We emphasise the pictured correspondence result: Consider $\Delta^2$-interpretations and triangular matrix interpretations of dimension 2, where no zero columns occur, then these interpretations are *equivalent* for orientability. This correspondence sheds light on the expressivity of matrix interpretations and (significantly) extends the class of rewrite systems whose compatibility with context dependent interpretations can be shown automatically. As witnessed by Example 1, it is sometimes possible to automatically obtain a context-dependent interpretation via the correspondence result, where the direct approach fails. The mentioned techniques have been implemented and the experimental data clearly shows that triangular matrix interpretations extend the power of previously known methods to automatically analyse polynomial derivational complexity. In particular, the TMI method is the only known automatic method to prove polynomial complexity for Example 1, 7, and 8.

In concluding, we note that matrix interpretations for termination of string rewriting systems are also known as $\mathbb{N}$-rational series in the theory of weighted (tree) automata, and they have been investigated in connection with the growth of DT0L systems (see [16]). It remains to connect this knowledge to the present application. Other directions for research are concerned with extending the presented *direct* termination techniques with transformation techniques like the *dependency pair method* [1] or with *multi-step termination proofs* as invoked in [5]. Although these extensions are necessary to increase the power of the studied techniques further, already simple examples show the challenging nature of this endeavour, compare [5].

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