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The Hydra Battle and Cichon's Principle

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Abstract In rewriting the Hydra battle refers to a term rewrite system $\mathcal{H}$ proposed by Dershowitz and Jouannaud. To date, $\mathcal{H}$ withstands any attempt to prove its termination automatically. This motivates our interest in term rewrite systems encoding the Hydra battle, as a careful study of such systems may prove useful in the design of automatic termination tools. Moreover it has been an open problem, whether any termination order compatible with $\mathcal{H}$ has to have the Howard-Bachmann ordinal as its order type, i.e., the proof theoretic ordinal of the theory of one inductive definition. We answer this question in the negative, by providing a reduction order compatible with $\mathcal{H}$, whose order type is at most ϵ_0, the proof theoretic ordinal of Peano arithmetic.

1 Introduction

Kirby and Paris have shown in [16] that the *Battle of Hercules and the Hydra* terminates and that this fact cannot be proven in Peano arithmetic. The latter is due to the rapid growth rate of the length of the battle. It is a worthwhile term rewriting exercise to define a term rewrite system (TRS for short) that faithfully describes this battle. A first such TRS $\mathcal{H}$ was presented in [10], where $\mathcal{H}$ mimics the definition of the Hydra battle in [16] to some extent.

The central theoretical motivation of this paper is the following question, posed by Cichon: *Must any termination order used for proving termination of the Battle of Hydra and Hercules-system have the Howard ordinal as its order type?* More precisely must any termination order used for proving termination of $\mathcal{H}$ have the Howard-Bachmann ordinal as its order type? Here a *termination order* is a well-founded order $\succ$ on terms, such that by showing that the

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rewrite relation is contained in $\succ$, i.e., $\to_{\mathcal{H}} \subseteq \succ$, we establish termination of $\mathcal{H}$, compare [8].

Incidentally this is open problem # 23 in the list of open problems in rewriting (*RTALooP* for short).[1] Note that the Howard-Bachmann ordinal is the proof theoretic ordinal of the theory of one inductive definitions, compare e.g. [23]. For related work, see for example [13, 26, 19, 5, 3, 12].

When slightly reformulated, the question can be directly answered negatively. Touzet indicated in [26] how the Battle of Hydra and Hercules can be formulated as a TRS $\mathcal{R}$ so that the order type of a termination order compatible with $\mathcal{R}$ equals ϵ_0, the proof theoretic ordinal of Peano arithmetic, or alternatively, the first fixpoint of the equation $\omega^\alpha = \alpha$. (Observe that the Howard-Bachmann ordinal easily dwarfs the ordinal ϵ_0.) However, in its strict sense—where we consider the TRS $\mathcal{H}$—the question remained open until now. It is worthy of note that the system $\mathcal{R}$ defined by Touzet is quite different from the TRS $\mathcal{H}$. In particular the former is totally terminating, while the latter is not even simply terminating.

In this paper we provide a negative answer, by defining a reduction order of order type at most ϵ_0 that is compatible with $\mathcal{H}$. (Clearly the existence of a reduction order of order type ϵ_0, implies the existence of termination order of order type ϵ_0.) Admittedly this is not a surprising result. The evidence that the answer to the problem is negative, is (almost) overwhelming. Apparently the question has its roots in a conjecture that is sometimes referred to as *Cichon's principle*. Cichon claimed in [8] that the (worst-case) complexity of a TRS compatible with a termination order of order type α is eventually dominated by a function from the *slow-growing hierarchy* along α. The correctness of this conjecture would imply a positive answer to the studied question. Although the conjecture holds for the lexicographic and multiset path orders (c.f. [14, 27] but also [4, 22]), it is incorrect in general, as has been shown by Touzet in [26]. Moreover, Touzet's result establishes that the principle already fails for the class of *simply terminating* TRSs, compare. problem # 81 in the RTALooP. Furthermore Lepper established that Cichon's principle fails for the oldest invented reduction order: the Knuth-Bendix order (c.f. [17, 21]).

This implies that the intuition behind the question is unfounded. The order type of a termination order compatible with a suitable encoding of the Hydra battle need not be higher than the proof-theoretic ordinal of Peano arithmetic. This suggests that the answer to the question is negative. Moreover, it even seems to imply that proving this should be easy, in particular based on the excellent work by Touzet and Lepper. Unfortunately, it turned out to be rather hard work.

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[1] See `http://rtaloop.mancoosi.univ-paris-diderot.fr/`.

Instead of studying the TRS $\mathcal{H}$ directly, we base our investigations on the following TRS proposed by Dershowitz in 2004.[2]

$$\begin{aligned}
1\colon&& h(e(x),y) &\rightarrow h(d(x,y),S(y))\\
2\colon&& d(g(g(0,x),y),S(z)) &\rightarrow g(e(x),d(g(g(0,x),y),z))\\
3\colon&& d(g(g(0,x),y),0) &\rightarrow e(y)\\
4\colon&& d(g(0,x),y) &\rightarrow e(x)\\
5\colon&& d(g(x,y),z) &\rightarrow g(d(x,z),e(y))\\
6\colon&& g(e(x),e(y)) &\rightarrow e(g(x,y))\ .
\end{aligned}$$

In the following this TRS is denoted as $\mathcal{D}$. (See [12] for an account on the differences between $\mathcal{H}$ and $\mathcal{D}$.) The TRSs $\mathcal{H}$ and $\mathcal{D}$ are known to be terminating and sketches of termination proofs by transfinite induction up-to ϵ_0 can be found e.g. in [10, 9] (but see also [12] for the limits of these sketches). Both systems are overlapping and not simply terminating. Furthermore observe that both systems are non-confluent. For example with respect to TRS $\mathcal{D}$ consider the following peak:

$$e(0) \leftarrow d(g(0,0),0) \rightarrow g(d(0,0),e(0))\ .$$

It is easily seen that this peak is not joinable. In the sequel, we first study the TRS $\mathcal{D}$, as this system faithfully encodes the Hydra battle, which simplifies the investigation. Only after the completion of this study, we indicate how the developed theory has to be adapted to deal with the original system $\mathcal{H}$, c.f. Section 6.

The motivation for this work is not purely theoretical, but also an attempt to understand why proving termination of the Hydra battle automatically seems so difficult. Both TRSs $\mathcal{H}$ and $\mathcal{D}$ are part of a collection of TRSs used in the international termination competition.[3] None of the termination provers that entered any of the termination competitions so far, can prove termination of these systems.[4]

The rest of this paper is organised as follows. In Sections 2 we present basic notions. In Section 3 we recall the definition of the *Battle of Hercules and the Hydra* and provide the starting points of the paper. In Section 4 we introduce a notation system for ordinal less than ϵ_0 that is of central importance for our termination proof. In Section 5 we prove the termination of $\mathcal{D}$. To this end, we introduce a well-founded, monotone algebra $(\mathcal{A}, \rhd)$ that is compatible with $\mathcal{D}$. And the induced reduction order $\rhd_{\mathcal{A}}$ has order type ϵ_0. In Section 6 we adapt this algebra to an algebra $(\mathcal{B}, \blacktriangleright)$ that is again well-founded, monotone,

[2] The TRS $\mathcal{D}$ was presented in the rewriting list (see `https://listes.ens-lyon.fr/wws/arc/rewriting`) on February 19, 2004. We swap the arguments of the symbols d and h and make use of the unary function symbol S instead of the original c.

[3] See `http://colo5-c703.uibk.ac.at:8080/termcomp`.

[4] The results for the Termination Competition in 2008 are to be found at `http://colo5-c703.uibk.ac.at:8080/termcomp`. The problem identification of $\mathcal{H}$ is `TRS/D33-33`, while the identification of $\mathcal{D}$ is `TRS/Zantema06-hydra`.

and compatible with $\mathcal{H}$. Subsequently we show that the induced reduction order $\blacktriangleright_{\mathcal{B}}$ has order type $\leqslant \epsilon_0$. Finally, we conclude with the discussion of related work in Section 7.

2 Preliminaries

2.1 Proper Orders and Ordinals

We assume very basic knowledge of set-theory and in particular ordinals, see [15]. In motivating this research, the Howard-Bachmann ordinal was mentioned, but in the remainder, ϵ_0 will be the largest ordinal to occur. We write $>$ to denote the standard order on ordinals. Recall that any ordinal $\alpha < \epsilon_0$, $\alpha \neq 0$ can be uniquely represented in *Cantor Normal Form* (*CNF* for short), i.e., it can be written as

$$\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \ ,$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_n$. For $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\alpha_{n+1}} + \cdots + \omega^{\alpha_{n+m}}$ define the *natural sum* $\alpha \oplus \beta$ as $\omega^{\alpha_{\pi(1)}} + \cdots + \omega^{\alpha_{\pi(n+m)}}$, where π denotes a permutation of the indices $\{1, \ldots, n+m\}$ such that $\alpha_{\pi(1)} \geqslant \alpha_{\pi(2)} \geqslant \cdots \geqslant \alpha_{\pi(n+m)}$ is guaranteed. We write $\alpha \cdot n$ as an abbreviation of $\alpha + \cdots + \alpha$ (n-times α). Further, we identify the natural numbers with the ordinals below ω. I.e. the ordinal $\{\varnothing, \{\varnothing\}\}$ is denoted as 2. We denote the set of limit ordinals by Lim.

A *proper order* $\succ$ is an irreflexive and transitive relation. The converse of $\succ$ is written as $\prec$. A *quasi-order* is a reflexive and transitive relation and a *partial order* is an anti-symmetric quasi-order. A proper order $\succ$ on a set A is *well-founded* (on A) if there exists no infinite descending sequence $a_1 \succ a_2 \succ \cdots$ of elements of A. A well-founded proper order is called a *well-founded order*. A proper order is called *linear* (or *total*) on A if for all $a, b \in A$, a different from b, a and b are comparable by $\succ$. A linear well-founded order is called a *well-order*.

To each well-founded order $\succ$ on a set A we can associate a (set-theoretic) ordinal, its *order type*. First we associate an ordinal to each element a of A by setting $\mathrm{otype}_{\succ}(a) := \sup\{\mathrm{otype}_{\succ}(b) + 1 \mid b \in A \text{ and } a \succ b\}$. Then the *order type* of $\succ$, denoted as $\mathrm{otype}(\succ)$, is defined as $\sup\{\mathrm{otype}_{\succ}(a) + 1 \mid a \in A\}$. For two proper orders $\succ$ and $\succ'$ on A and A', respectively, a mapping $o\colon A \to A'$ *embeds* $\succ$ into $\succ'$ if $\forall x, y \in A\ (x \succ y \Longrightarrow o(x) \succ' o(y))$. The proof of the next lemma can be found in [17].

Lemma 1 *If* $\succ$ *and* $\succ'$ *are well founded and if* $\succ$ *can be embedded into* $\succ'$*, then* $\mathrm{otype}(\succ) \leqslant \mathrm{otype}(\succ')$.

Two linear proper orders $(A, \succ)$ and $(B, \succ')$ are called *order-isomorphic* (or *equivalent*) if there exists a surjective mapping $o\colon A \to B$ such that $\forall x, y \in A\ (x \succ y \Longleftrightarrow o(x) \succ' o(y))$.

2.2 Nested Multisets

Ordinals below ϵ_0 are strongly related to *nested multisets* $(S)^{\sharp,*}$ over some set S, c.f. [11] or [25, Appendix A].

A *multiset* M over S is a function $M\colon S \to \mathbb{N}$ such that the set of elements of M is finite. For $s \in S$, $M(s)$ denotes the *multiplicity* of s in M. The *set of elements* of M is defined as $\{s \mid M(s) > 0\}$. The set of multisets over S is denoted as $S^\sharp$. We use the usual membership relation, to denote *membership* for multisets; i.e. $s \in M$ if $M(s) > 0$. For multisets M, N we define the *multiset union* $M + N$ of M and N by adding the two multiplicities: $(M + N)(s) := M(s) + N(s)$. Multiset union is associative and commutative.

For a proper order $\succ$, we define the *multiset extension* $\succ_\sharp$ of $\succ$ as follows: $\succ_\sharp$ is the smallest transitive relation that satisfies

$$\forall x \in M' \; s \succ x \Longrightarrow M + \{s\} \succ_\sharp M + M' \;,$$

for all $s \in S$, $M, M' \in S^\sharp$. Note that, if $\succ$ is well-founded, then $\succ_\sharp$ is well-founded, too. Moreover, if $\mathrm{otype}(\succ) = \alpha$, then $\mathrm{otype}(\succ_\sharp) = \omega^\alpha$. We set $\mathsf{I}_\sharp(S) := S \cup S^\sharp$ and define the set of *nested multisets* $(S)^{\sharp,*}$ over S by iterated application of the operator $\mathsf{I}_\sharp$:

$$(S)^{\sharp,*} := \bigcup_{n \geqslant 0} (\mathsf{I}_\sharp)^n(S) \;.$$

Note that ${\mathsf{I}_\sharp}^0(S) \subset {\mathsf{I}_\sharp}^1(S) \subset {\mathsf{I}_\sharp}^2(S) \ldots$ is an ascending sequence. For a proper order $\succ$ define the extension $\succ_{\mathsf{I}_\sharp}$ of $\succ_\sharp$ to $\mathsf{I}_\sharp(S)$ inductively:

- $\forall s, s' \in S$ $(s \succ s' \Longrightarrow s \succ_{\mathsf{I}_\sharp} s')$,
- $\forall M, M' \in (S^\sharp - S)$ $(M \succ_\sharp M' \Longrightarrow M \succ_{\mathsf{I}_\sharp} M')$,
- $\forall s \in S, \forall M \in (S^\sharp - S)$ $(M \succ_{\mathsf{I}_\sharp} s)$.

The results of the next two lemmas, have already been reported in [11].

Lemma 2 *Let $\succ$ be a well-founded order on S.*

1. *Then $\succ_{\mathsf{I}_\sharp}$ is a well-founded order on $\mathsf{I}_\sharp(S)$.*
2. *Let* $\mathrm{otype}(\succ) = \alpha$ *and α closed under addition. Then* $\mathrm{otype}(\succ_{\mathsf{I}_\sharp}) = \omega^\alpha$.

The given construction allows us to define the *nested multiset order* inductively. Let $\succ^0_{\mathsf{I}_\sharp} := \succ$; $\succ^{n+1}_{\mathsf{I}_\sharp} := (\succ^n_{\mathsf{I}_\sharp})_{\mathsf{I}_\sharp}$. It is easy to see that $\succ^n_{\mathsf{I}_\sharp}$ is a proper order on ${\mathsf{I}_\sharp}^n(S)$ for each n. Finally, we extend $\succ_{\mathsf{I}_\sharp}$ to the *nested multiset order* $\succ_{\sharp,*}$ on $(S)^{\sharp,*}$ by defining

$$\succ_{\sharp,*} := \bigcup_{n \geqslant 0} \succ^n_{\mathsf{I}_\sharp} \;.$$

We introduce the notion of ω-towers: $\omega_0 := 1$ and $\omega_{n+1} := \omega^{\omega_n}$.

Lemma 3 *Let $\succ$ be a well-founded order on S.*

1. *Then $\succ^n_{\mathsf{I}_\sharp}$ is a well-founded order on ${\mathsf{I}_\sharp}^n(S)$ and $\succ_{\sharp,*}$ is a well-founded order on $(S)^{\sharp,*}$.*

2. *Let* $\mathrm{otype}(\succ) = \alpha$ *and* α *closed under addition. Then* $\mathrm{otype}(\succ^n_{\mathsf{l}_\sharp}) = \omega^\alpha_n$.
3. *Let* $\mathrm{otype}(\succ) = \alpha$*,* α *closed under addition, and* $\alpha \leqslant \epsilon_0$*. Then we have* $\mathrm{otype}(\succ_{\sharp,*}) = \epsilon_0$.

Example 4 Consider $(\mathbb{N}, >)$, where $>$ denotes the usual order on $\mathbb{N}$. It is not difficult to see that $((\mathbb{N})^{\sharp,*}, >_{\sharp,*})$ is equivalent to $((\varnothing)^{\sharp,*}, \succ_{\sharp,*})$, where $\succ$ denotes the empty order. By the previous lemma, we obtain that $\mathrm{otype}(>_{\sharp,*}) = \mathrm{otype}(\succ_{\sharp,*}) = \epsilon_0$.

In the sequel, we write $(\mathsf{NMul}, >_{\sharp,*})$ instead of $((\varnothing)^{\sharp,*}, \succ_{\sharp,*})$.

2.3 Term Rewriting

Furthermore we assume familiarity with term rewriting. For further details see [2, 25]. Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a signature. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted as $\mathcal{T}(\mathcal{F}, \mathcal{V})$, while the set of ground terms is written as $\mathcal{T}(\mathcal{F})$. The set of variables occurring in a term t is denoted as $\mathcal{V}\mathsf{ar}(t)$. A term t is called *ground* or *closed* if $\mathcal{V}\mathsf{ar}(t) = \varnothing$.

A *term rewrite system* (*TRS* for short) $(\mathcal{F}, \mathcal{R})$ over $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a finite set of rewrite rules. If there is no need to indicate the signature $\mathcal{F}$, we simply write $\mathcal{R}$ to denote a TRS. A relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a *rewrite relation* if it is compatible with $\mathcal{F}$-operations and closed under substitutions. The smallest rewrite relation that contains $\mathcal{R}$ is denoted as $\to_{\mathcal{R}}$. The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to^+_{\mathcal{R}}$, and its transitive and reflexive closure by $\to^*_{\mathcal{R}}$. A TRS $\mathcal{R}$ is called *terminating* if there is no infinite sequence $(t_i : i \in \mathbb{N})$ of terms such that $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} t_m \to_{\mathcal{R}} \cdots$

A rewrite relation that is also a proper order is called *rewrite order*. A well-founded rewrite order is called *reduction order*. Let $\succ$ denote a proper order. A TRS $\mathcal{R}$ and a proper order $\succ$ are *compatible* if $\mathcal{R} \subseteq \succ$. We also say that $\mathcal{R}$ is compatible with $\succ$ or vice versa. A TRS $\mathcal{R}$ is terminating if and only if it is compatible with a reduction order $\succ$.

Let $\mathcal{F}$ be a signature. An $\mathcal{F}$*-algebra* (or simply *algebra*) $\mathcal{A}$ is a set A together with operations $f_{\mathcal{A}} \colon A^n \to A$ for each function symbol $f \in \mathcal{F}$ of arity n. The set A is called the *carrier* of $\mathcal{A}$. An $\mathcal{F}$-algebra $(\mathcal{A}, \succ)$ is called *monotone* if $\mathcal{A}$ is associated with a proper order $\succ$ and every algebra operation $f_{\mathcal{A}}$ is strictly monotone in all its arguments. A monotone algebra $(\mathcal{A}, \succ)$ is called *well-founded* if $\succ$ is well-founded. Let $(\mathcal{A}, \succ)$ denote a monotone algebra and let $\mathbf{a} \colon \mathcal{V} \to A$ denote an *assignment*. We write $[\mathbf{a}]_{\mathcal{A}}$ to denote the homomorphic extension of the assignment $\mathbf{a}$ and define a rewrite order $\succ_{\mathcal{A}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ in the usual way: $s \succ_{\mathcal{A}} t$ if $[\mathbf{a}]_{\mathcal{A}}(s) \succ [\mathbf{a}]_{\mathcal{A}}(t)$ for every assignment $\mathbf{a}$. Let $(\mathcal{A}, \succ)$ be a well-founded and monotone algebra (WMA for short). Then it is easy to see that $\succ_{\mathcal{A}}$ is a reduction order. We say the WMA $(\mathcal{A}, \succ)$ is compatible with a TRS $(\mathcal{F}, \mathcal{R})$, if $\succ_{\mathcal{A}}$ is compatible with $\mathcal{R}$. The following well-known theorem essentially traces back to [20]. In its modern form it can be found in [25, Chapter 6].

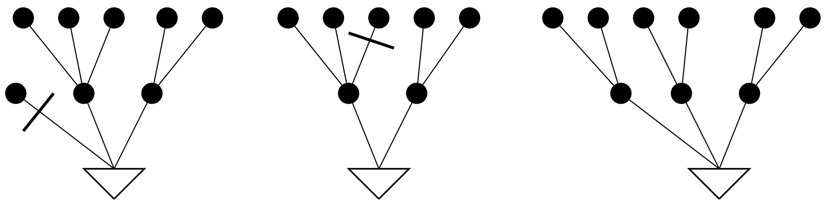
Theorem 5 *A TRS is terminating if and only if it is compatible with a well-founded, monotone algebra.*

3 The Hydra Battle

We recall the definition of the Hydra battle, see also [16, 26]. The beast is represented as a finite tree, where each leaf corresponds to a head of the Hydra. Hercules chops off heads of the Hydra, but the Hydra regrows according to the following rule: If the cut head has a pre-predecessor, then the branch issued from this node together with the remaining subtree is multiplied by the stage of the game. Otherwise the Hydra ignores the loss.

Let us consider a short example, more examples can be found in [16, 26, 12]. We write (H, n) to describe a single *configuration* in the game, where H denotes the Hydra and n the current stage of the game.

Example 6



$(\mathsf{H}_1, 1)$ $(\mathsf{H}_2, 2)$ $(\mathsf{H}_3, 3)$

In the first stage, Hercules chops off the leftmost head. As this head has no grandparent, Hydra shrinks. However, in Stage 2, Hercules chops off a head with a grandparent (the triangular root). Consequently, Hydra grows two replacement branches, as indicated.

The natural game-theoretic question is whether Hercules has a winning strategy. A *strategy* is a mapping determining which head Hercules chops off at each stage. In turns out that any strategy is a winning strategy.

Theorem 7 *Every strategy is a winning strategy.*

In proof, we follow Kirby and Paris and associate with each Hydra an ordinal strictly less than ϵ_0:

- To each leaf assign 0.
- To each other node v assign $\omega^{\alpha_1} \oplus \cdots \oplus \omega^{\alpha_n}$, if α_i are the ordinals assigned to the successors of v.

The ordinal representing the Hydra, is the ordinal assigned to the root.

Example 8 Consider the Hydras H_1–H_3, above. These have the representations:

$$\omega^3 \oplus \omega^2 \oplus 1\,, \quad \omega^3 \oplus \omega^2\ , \text{ and } \quad \omega^2 \cdot 3\,.$$

In the sequel, we often confuse the representation of a Hydra as finite tree and as ordinal. We fix a specific strategy S. Let (H, n) denote a configuration of the game. Then $(\mathsf{H})_n^{\mathsf{S}}$ denotes the resulting Hydra if S is applied to H at stage n. I.e. the next configuration is of form $((\mathsf{H})_n^{\mathsf{S}}, n+1)$. As Hydras are conceivable as ordinals, the next lemma follows easily, c.f. [16].

Lemma 9 *For any strategy* S*, Hydra* H*, and natural number* n*, we obtain that* $\mathsf{H} > (\mathsf{H})_n^{\mathsf{S}}$.

Then Theorem 7 follows from Lemma 9 together with the fact that $>$ is well-founded.

Remark 10 In Section 2 we indicated the connection between ordinals ($< \epsilon_0$) and nested multisets. It is a simple exercise to represent Hydras by elements of NMul and prove the variant of the above lemma where the ordinal comparison $>$ is replaced by $>_{\sharp,*}$. Consequently Theorem 7 is provable without any reference to ordinals.

In the remainder of this section, we formally define a specific strategy for the Hydra battle that has been called *standard* in [26]. Further, we show that $\mathcal{D}$ simulates the Hydra battle on the standard strategy. For $n \in \mathbb{N}$, we associate with every $\alpha \in \mathrm{CNF}$ an ordinal $\alpha_n \in \mathrm{CNF}$:

$$\alpha_n = \begin{cases} 0 & \text{if } \alpha = 0 \\ \beta & \text{if } \alpha = \beta + 1 \\ \beta + \omega^\gamma \cdot n & \text{if } \alpha = \beta + \omega^{\gamma+1} \\ \beta + \omega^{\gamma_n} & \text{if } \alpha = \beta + \omega^\gamma \text{ and } \gamma \in \mathsf{Lim} \ . \end{cases}$$

Then we can define the *standard Hydra battle* as follows.

Definition 11 A Hydra is an ordinal in CNF. The Hydra battle is a sequence of configurations. A *configuration* is a pair (α, n), where α denotes a Hydra and n the current step. Let (α, n) be a configuration, such that $\alpha \neq 0$. Then the next configuration in the standard strategy is defined as $(\alpha_n, n+1)$.

Remark 12 The sequence $(\alpha_n)_{n \in \mathbb{N}}$ is usually referred to as the *fundamental sequence* of α. A fundamental sequence fulfils the property that if α is a limit ordinal, i.e. $\alpha \in \mathsf{Lim}$, then the sequence is strictly increasing and its limit is α. For the connection between rewriting and fundamental sequences see e.g. [22].

We are going to show that the TRS $(\mathcal{F}, \mathcal{D})$ introduced in Section 1 faithfully represents the standard Hydra battle. In the sequel the signature $\mathcal{F}$ is fixed to the signature of the TRSs $\mathcal{D}$ and $\mathcal{H}$ respectively. Due to the definition of $\mathcal{D}$ this is particularly simple. We define a mapping $\mathcal{O}\colon \mathrm{CNF} \to \mathcal{T}(\mathcal{F})$:

$$\mathcal{O}(\alpha) := \begin{cases} 0 & \text{if } \alpha = 0 \\ g(\mathcal{O}(\gamma), 0) & \text{if } \alpha = \omega^\gamma \\ g(\mathcal{O}(\gamma), \mathcal{O}(\beta)) & \text{if } \alpha = \beta + \omega^\gamma \ . \end{cases}$$

Each configuration (α, n) of the game, is encoded by $h(e(\mathcal{O}(\alpha)), S^n(0))$.

Lemma 13 *Let $\alpha \in CNF$, $\alpha \neq 0$, $n \in \mathbb{N} - \{0\}$. Then $h(e(\mathcal{O}(\alpha)), S^n(0)) \to_{\mathcal{D}}^{+} h(e(\mathcal{O}(\alpha_n)), S^{n+1}(0))$.*

Proof Due to the presence of the rule $h(e(x), y) \to h(d(x, y), S(y))$ in $\mathcal{D}$ it suffices to verify that $d(\mathcal{O}(\alpha), S^n(0)) \to_{\mathcal{D}}^{+} e(\mathcal{O}(\alpha_n))$. This can be shown by transfinite induction on α. We restrict our attention to the case where $\alpha = \beta + \omega^{\gamma+1}$, see [12] for the full proof. By definition $\alpha_n = \beta + \omega^{\gamma} \cdot n$ and $t = \mathcal{O}(\alpha) = g(g(0, \mathcal{O}(\gamma)), \mathcal{O}(\beta))$, let $s = \mathcal{O}(\beta)$, $r = \mathcal{O}(\gamma)$. We obtain

$$\mathcal{O}(\alpha_n) = \underbrace{g(r, g(r, \cdots g(r, s) \cdots))}_{n \text{ occurrences of } r} ,$$

and the following rewrite sequence suffices:

$$\begin{aligned} d(g(g(0,r),s), S^n(0)) &\to_{\mathcal{D}}^{+} g(e(r), \cdots d(g(g(0,r),s),0) \cdots)) && \text{rule 2, } n \text{ times} \\ &\to_{\mathcal{D}} g(e(r), g(e(r), \cdots g(e(r), e(s)) \cdots)) && \text{rule 3} \\ &\to_{\mathcal{D}}^{+} e(g(r, g(r, \cdots g(r, s) \cdots))) && \text{rule 6, } n \text{ times .} \end{aligned}$$

□

Remark 14 In Remark 10 we indicated that Theorem 7 can be proven without the use of ordinals, by replacing the ordinal comparison $>$ by the proper order $>_{\sharp,*}$. The same result holds for Lemma 13. It is not difficult to see how to replace the definition of α_n by a corresponding definition acting on the set of nested multisets NMul. Hence, all results in this section can be proven without the use of ordinals.

From Theorem 7, we conclude that the standard strategy is a winning strategy for Hercules and that this fact can be proven by a well-founded order of order type ϵ_0. From Lemma 13, we know that $\mathcal{D}$ simulates the standard strategy. And the proof of this lemma clarifies the intended semantics of the symbols in $\mathcal{F}$. Thus the definition of a suitable termination order $\succ$ compatible with $\mathcal{D}$ (or $\mathcal{H}$) may appear to be a simple exercise. Unfortunately, the exercise turns out to be a bit more involved. Below we emphasise the major obstacles.

The proof of Lemma 13 suggests to interpret the symbol h as a pairing function that returns a configuration in the Hydra battle. However, in general the second argument of h can be an arbitrary term, for example the representation of a Hydra, not only (the representation of) a natural number. Thus Theorem 7 is not much help in a termination proof of $\mathcal{D}$ (or $\mathcal{H}$). A similar problem arises with the function symbol d: It is essential to interpret d so that it returns the n^{th} branch of the fundamental sequence $(\alpha_n)_{n\in\mathbb{N}}$ (see Remark 12), if the first argument of d represents α and the second n. However, what is the correct interpretation if the second argument is (the representation of) a Hydra?

This problem can be solved by the introduction of functions $\mathsf{C}_\alpha \colon \mathbb{N} \to \mathbb{N}$, indexed by (the representation of) a Hydra. These functions act as *collapsing functions*, by mapping Hydras to natural numbers so that the order relation on Hydras is preserved. (See [23] or alternatively [22, 21] for further reading on collapsing functions.)

On the other hand, our goal is not only to establish termination of the TRS $\mathcal{D}$ (or the TRS $\mathcal{H}$), but to define a termination order $\succ$ such that $\to_{\mathcal{D}} \subseteq \succ$ and $\mathrm{otype}(\succ) \leqslant \epsilon_0$. Note that to solve problem 23 in the RTALooP, any termination order with order type strictly less than the Howard-Bachmann ordinal, would suffice. But naturally, we strive for an optimal result. In order to define the order $\succ$ we go a step further and make sure that $\succ$ is actually a reduction order. Hence, instead of $\to_{\mathcal{D}} \subseteq \succ$, it suffices to verify $\mathcal{D} \subseteq \succ$. More precisely, we first define a WMA $(\mathcal{A}, \rhd)$ that is compatible with $\mathcal{D}$, such that $\mathrm{otype}(\rhd) = \epsilon_0$ and subsequently define a WMA $(\mathcal{B}, \blacktriangleright)$ compatible with $\mathcal{H}$, such that $\mathrm{otype}(\blacktriangleright) \leqslant \epsilon_0$.

Here a serious obstacle is the fact that neither the TRS $\mathcal{D}$, nor the TRS $\mathcal{H}$ are simply terminating. This implies that these TRSs are not totally terminating, c.f. [28]. Which in turn implies that there cannot be a total WMA compatible with either $\mathcal{D}$ or $\mathcal{H}$. In particular, we cannot hope to prove ϵ_0-termination of either $\mathcal{D}$ or $\mathcal{H}$. (Following [18], we call a TRS $\mathcal{R}$ α-terminating, if $\mathcal{R}$ is compatible with the monotone algebra $(\alpha, >)$, where $>$ denotes the standard order on ordinals.) Furthermore, we have to make sure that the algebra is *monotone*. As already observed by Touzet (see [26]) and Lepper (see [19]) the definition of *monotone* interpretation functions for TRSs simulating the Hydra battle is a non-trivial task.

We solve both this problems, by replacing the set-theoretic ordinals employed so far, by a suitable defined notation system for ordinals $< \epsilon_0$. This is the topic of the next section.

4 A Notation System for the TRS $\mathcal{D}$

In this section, we introduce an *ordinal notation system* for ordinals below ϵ_0. We follow an approach by Takeuti [24] (but see also [5] and [12]). We define $\mathsf{OT} := \mathcal{T}(\{\mathsf{g}, \mathsf{0}\})$, i.e., OT denotes the set of (ground) terms over the symbols g and $\mathsf{0}$, where the symbol $\mathsf{0}$ is a constant and the arity of g is 2. The elements of OT are called *ordinal terms* and are denoted by lower-case Greek letters. Sometimes, we drop the qualifier "term" and simply speak of *ordinals*. Any Hydra H becomes representable as an element of OT. Or, alternatively any $\alpha \in \mathsf{OT}$ is conceivable as an element of NMul, c.f. Section 2.2, such that $\mathsf{g}(\alpha, \beta)$ corresponds to the multiset $\beta + \{\alpha\}$. Note that in contrast to the connection between ordinals and Hydras, the correspondence between ordinals and nested multisets is not exact. This is due to the fact that ordinals are ordered, but nested multisets are not.

In the sequel the expression "ordinal" always refer to an element of OT. If we refer to set-theoretic ordinals, this will be explicitly mentioned. Clearly any element of OT different from $\mathsf{0}$ can be written as follows:

$$\mathsf{g}(\alpha_n, \mathsf{g}(\alpha_{n-1}, \ldots \mathsf{g}(\alpha_1, \mathsf{0}) \ldots)) \;, \tag{1}$$

where each of the $\alpha_1, \ldots, \alpha_n$ can also be written in form (1). To improve readability and clarify the connection to [24], we also denote terms of the form (1) as $\omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n}$.

Definition 15 We inductively define an equivalence $\sim$ and a proper order $\succ$ so that they satisfy the following clauses:

1. $\mathsf{0}$ is the minimal element of $\succ$.
2. For $\alpha \in \mathsf{OT}$ of form (1), assume α contains the consecutive terms ω^{α_i} and $\omega^{\alpha_{i+1}}$ with $\alpha_{i+1} \succ \alpha_i$. That is α has the form
$$\cdots + \omega^{\alpha_i} + \omega^{\alpha_{i+1}} + \cdots .$$
Let β be obtained by removing the expression "$\omega^{\alpha_i} +$" from α, so that β is of the form
$$\cdots + \omega^{\alpha_{i+1}} + \cdots .$$
Then $\alpha \sim \beta$.
3. Suppose $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$, $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, $\alpha_1 \succcurlyeq \alpha_2 \succcurlyeq \cdots \succcurlyeq \alpha_m$, and $\beta_1 \succcurlyeq \beta_2 \succcurlyeq \cdots \succcurlyeq \beta_n$, holds. ($\alpha \succcurlyeq \beta$ means $\alpha \succ \beta$ or $\alpha \sim \beta$.) Then $\alpha \succ \beta$ if either $\alpha_i \succ \beta_i$ for some $i \in \{1, \ldots, m\}$ and $\alpha_j \sim \beta_j$ for all $1 \leqslant j \leqslant i-1$ or $m > n$ and $\alpha_i \sim \beta_i$ holds for all $1 \leqslant i \leqslant m$.

Remark 16 A crucial idea of the notation system described, is the separation of the identity of ordinal terms (denoted by $=$) and the identity of their set-theoretic counterparts (denoted by $\sim$). We will see in the next section that this pedantry is essential.

We identify natural numbers with ordinals less than ω. For ordinal terms strictly less than ω the usual comparison of naturals coincides with Definition 15. First we abbreviate $\mathsf{g}(\mathsf{0}, \mathsf{0})$ as 1 and denote $1 + 1$ as 2, $1 + 1 + 1$ as 3, and so on. By definition, for any $\alpha \neq \mathsf{0} \in \mathsf{OT}$, there exists a unique $\beta \in \mathsf{OT}$ with $\alpha \sim \beta$ so that β can be written as

$$\omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n} \qquad \text{with } \beta_1 \succcurlyeq \cdots \succcurlyeq \beta_n , \tag{2}$$

where $\beta_1 \succcurlyeq \cdots \succcurlyeq \beta_n$ holds. If β is written in this way, we say that it is in *normal-form*. The set of all ordinals in normal-form together with $\mathsf{0}$ is denoted as NF. The unique normal-form of a given ordinal α is denoted as $\mathsf{NF}(\alpha)$. Any $\alpha \in \mathsf{NF}$ uniquely represent a set-theoretic ordinals less than ϵ_0 in CNF. The following lemma is immediate.

Lemma 17 *1. The relation $\succ$ is a linear proper order on* NF.
2. The relation $\succ$ is well-founded and $\mathrm{otype}(\succ) = \epsilon_0$.

We extend the well-founded, linear order $\succ$ on NF to a well-founded, proper order $\succ$ on OT. To simplify notation we denote the extended relation with the same symbol, no confusion will arise from this. For $\alpha, \beta \in \mathsf{OT}$ define: $\alpha \succ \beta$, if $\mathsf{NF}(\alpha) \succ \mathsf{NF}(\beta)$. It follows that $\succ$ is a proper order and that $\alpha \succcurlyeq \beta \succ \gamma \Longrightarrow \alpha \succ \gamma$; as well as $\alpha \succ \beta \succcurlyeq \gamma \Longrightarrow \alpha \succ \gamma$ holds. The next lemma is an easy consequence of the definitions.

Lemma 18 *Let* $\alpha, \beta, \gamma \in \mathsf{OT}$*.*

1. $\alpha \succ \beta \Longrightarrow \mathsf{g}(\alpha, \gamma) \succ \mathsf{g}(\beta, \gamma)$.
2. $\beta \succ \gamma \Longrightarrow \mathsf{g}(\alpha, \beta) \succcurlyeq \mathsf{g}(\alpha, \gamma)$.
3. $\mathsf{g}(\alpha, 0) \succ \alpha$ *and* $\mathsf{g}(0, \beta) \succ \beta$.

Remark 19 As expressed in Remarks 10, 14 the results of the previous section can be proven by replacing the membership relation $>$ on ordinals by $>_{\sharp,*}$ over the set of nested multisets NMul. However, employing nested multisets in the termination proof of $\mathcal{D}$ would be technically involved. By definition nested multisets allow the permutations of the occurrences of elements. This causes problems, if we want to interpret the function symbol d according to its intended semantics. The only solution is the introduction of ordered (nested) multisets, i.e., nested sequences. The latter are a roundabout way to denote ordinal terms.

5 A Termination Proof of the TRS $\mathcal{D}$

In this section we define a WMA $(\mathcal{A}, \rhd)$ that is compatible with $(\mathcal{F}, \mathcal{D})$. Then termination of $\mathcal{D}$ follows by Theorem 5. Before we can present this algebra we need some preparations, which are the subject of the following subsection.

5.1 Preparations

Based on $\succ$ and $\sim$, we define a proper order $\sqsupset$ and an equivalence relation $\equiv$ on OT. We write $\mathsf{N}(\alpha)$ to denote the number of occurrences of g in α., i.e., $\mathsf{N}(0) := 0$ and $\mathsf{N}(\mathsf{g}(\alpha_1, \ldots, \mathsf{g}(\alpha_n, 0) \ldots)) := n + \mathsf{N}(\alpha_1) + \cdots + \mathsf{N}(\alpha_n)$.

Definition 20 Let $\alpha, \beta \in \mathsf{OT}$; we define $\alpha \sqsupset \beta$, if either $\alpha \succ \beta$ and $\mathsf{N}(\alpha) \geqslant \mathsf{N}(\beta)$ or $\alpha \sim \beta$ and $\mathsf{N}(\alpha) > \mathsf{N}(\beta)$. On the other hand, we define $\alpha \equiv \beta$, if $\alpha \sim \beta$ and $\mathsf{N}(\alpha) = \mathsf{N}(\beta)$ holds. We define the quasi-order $\sqsupset^{\equiv}$: $\alpha \sqsupset^{\equiv} \beta$, if $\alpha \ (\sqsupset \cup \equiv)\ \beta$.

Example 21 Let us consider

$$\omega + \omega^2 = \mathsf{g}(\mathsf{g}(0, \mathsf{g}(0, 0)), \mathsf{g}(\mathsf{g}(0, 0), 0)) \ ,$$

with norm $\mathsf{N}(\omega + \omega^2) = 5$ and

$$\omega + 3 = \mathsf{g}(0, \mathsf{g}(0, \mathsf{g}(0, \mathsf{g}(\mathsf{g}(0, 0), 0)))) \ ,$$

where $\mathsf{N}(\omega + 3) = 5$. Hence $\omega + \omega^2 \sqsupset \omega + 3$. This follows as $\mathsf{NF}(\omega + \omega^2) = \omega^2 = \mathsf{g}(\mathsf{g}(0, \mathsf{g}(0, 0)), 0)$, $\mathsf{NF}(\omega + 3) = \omega + 3 = \mathsf{g}(0, \mathsf{g}(0, \mathsf{g}(0, \mathsf{g}(\mathsf{g}(0, 0), 0))))$, and $\omega + \omega^2 \succ \omega + 3$. On the other hand, we have for example $\omega^2 \not\sqsupset \omega + 3$ as $\mathsf{N}(\omega + 3) = 5 > 3 = \mathsf{N}(\omega^2)$.

The example shows that the relation $\sim$ is not compatible with the relation $\sqsupset$. Moreover, note that $\sqsupset^{\equiv}$ is not a partial order, as indicated in the next example.

Example 22 Consider the terms $1+\omega^\omega$ and $\omega^{1+\omega}$. Then we obtain: $1+\omega^\omega \sqsupset^\equiv \omega^{1+\omega}$ and $\omega^{1+\omega} \sqsupset^\equiv 1+\omega^\omega$, but $1+\omega^\omega \neq \omega^{1+\omega}$, while clearly $1+\omega^\omega \equiv \omega^{1+\omega}$.

Lemma 23 *The binary relation* $\sqsupset$ *is a well-founded order and* $\mathrm{otype}(\sqsupset) \leqslant \epsilon_0$. *Furthermore for all* $n, m \in \mathbb{N}$: $n \sqsupset m$ *if and only if* $n > m$.

Proof That $\sqsupset$ is a proper order is immediate from the definition and the observations that $\alpha \succcurlyeq \beta \succ \gamma$ implies $\alpha \succ \gamma$, as well as $\alpha \succ \beta \succcurlyeq \gamma$ yields $\alpha \succ \gamma$, which were made immediately before Lemma 18 above.

To verify that $\sqsupset$ is well-founded with $\mathrm{otype}(\sqsupset) \leqslant \epsilon_0$, it suffices to define an embedding $o\colon \mathsf{OT} \to \epsilon_0$: $o(\alpha) := \omega^{\mathsf{NF}(\alpha)} + \mathsf{N}(\alpha)$. By case-distinction on the definition of $\sqsupset$ one verifies that for all $\alpha, \beta \in \mathsf{OT}$, $\alpha \sqsupset \beta$ implies $o(\alpha) \succ o(\beta)$. Assume first $\alpha \succ \beta$ and $\mathsf{N}(\alpha) \geqslant \mathsf{N}(\beta)$. Then $\omega^{\mathsf{NF}(\alpha)} + \mathsf{N}(\alpha) > \omega^{\mathsf{NF}(\beta)} + \mathsf{N}(\beta)$ is immediate from the definition of the usual comparison $>$ of set-theoretic ordinals. Now assume $\alpha \sim \beta$ and $\mathsf{N}(\alpha) > \mathsf{N}(\beta)$. Then $\omega^{\mathsf{NF}(\alpha)} + \mathsf{N}(\alpha) > \omega^{\mathsf{NF}(\beta)} + \mathsf{N}(\beta)$ follows similarly.

The second half of the lemma is a direct result of the definition of $\sqsupset$ and the definition of N. $\square$

The next lemma follows easily from Lemma 18 and the definitions.

Lemma 24 *Let* $\alpha, \beta, \gamma \in \mathsf{OT}$.

1. $\alpha \sqsupset \beta \Longrightarrow \mathsf{g}(\alpha, \gamma) \sqsupset \mathsf{g}(\beta, \gamma)$.
2. $\beta \sqsupset \gamma \Longrightarrow \mathsf{g}(\alpha, \beta) \sqsupset^\equiv \mathsf{g}(\alpha, \gamma)$.
3. $\mathsf{g}(\alpha, 0) \sqsupset \alpha$ *and* $\mathsf{g}(0, \beta) \sqsupset \beta$.

Let $p\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ denote a fixed function, strictly monotone in each argument.

Definition 25 We define the set of *n-predecessors of α induced by p*. Let $\alpha \in \mathsf{OT}$, we set $\alpha[n] := \{\beta \mid \alpha \succ \beta \text{ and } p(\mathsf{N}(\alpha), n) \geqslant \mathsf{N}(\beta)\}$.

The notion of an n-predecessor stems from [7, Chapter 3]. However, in the definition, we follow the idea of norm-based fundamental sequences, compare [6].

Lemma 26 *Let* $\alpha \in \mathsf{OT}$ *and let* δ *denote a* $\sqsupset$*-maximal element of* $\alpha[n]$.

1. *For all n, the set $\alpha[n]$ is finite.*
2. *For each* $\beta \in \alpha[n]$: $\delta \sqsupset^\equiv \beta$.

Proof The first assertion of the lemma follows from the observation that only finitely many ordinals of a given norm can exist. For the second assertion, observe that it follows from the definition of δ that for all $\beta \in \alpha[n]$, either $\delta \sqsupset \beta$, $\beta \equiv \delta$, or β and δ are incomparable with respect to $\sqsupset$. We prove that the last case can never happen. We assume $\alpha \neq 0$, as otherwise the assertion follows trivially. Let $\beta \in \alpha[n]$ be arbitrary but fixed so that β, δ are incomparable.

The ordinals β and δ can only be incomparable if either of the following cases holds (i) $\delta \prec \beta$ and $\mathsf{N}(\beta) < \mathsf{N}(\delta)$, or (ii) $\delta \succ \beta$ and $\mathsf{N}(\beta) > \mathsf{N}(\delta)$. As the cases are dual, it suffices to consider the first one.

Without loss of generality we can assume that $\beta \succcurlyeq \omega$. Assume otherwise $\beta \in \mathbb{N}$, then $\delta \in \mathbb{N}$ and $\mathsf{N}(\beta) = \beta \succ \delta = \mathsf{N}(\delta)$, which contradicts the assumption $\mathsf{N}(\delta) > \mathsf{N}(\beta)$. We define an ordinal term β^* as follows: $\beta^* := (\mathsf{N}(\delta) - \mathsf{N}(\beta)) + \beta$. As $\beta \succcurlyeq \omega$, $\beta^* \sim \beta$ holds. Furthermore $\mathsf{N}(\beta^*) = \mathsf{N}(\delta) > \mathsf{N}(\beta)$, as $\mathsf{N}(\beta^*) = (\mathsf{N}(\delta) - \mathsf{N}(\beta)) + \mathsf{N}(\beta) = \mathsf{N}(\delta)$. Hence $\beta^* \sqsupset \beta$. We show that $\beta^* \in \alpha[n]$: $\alpha \succ \beta \sim \beta^*$ implies $\alpha \succ \beta^*$. And $p(\mathsf{N}(\alpha), n) \geqslant \mathsf{N}(\delta) = \mathsf{N}(\beta^*)$ implies $p(\mathsf{N}(\alpha), n) \geqslant \mathsf{N}(\beta^*)$. From this we derive a contradiction to the assumption that δ is a $\sqsupset$-maximal of $\alpha[n]$. As we have $\beta^* \succ \delta$, and $\mathsf{N}(\beta^*) \geqslant \mathsf{N}(\delta)$, hence $\beta^* \sqsupset \delta$, while $\beta^* \in \alpha[n]$. $\Box$

By the above lemma a $\sqsupset$-maximal element of $\alpha[n]$ is, up-to the equivalence $\equiv$, unique. In the following we fix (for each α and each n) an arbitrary $\sqsupset$-maximal element and denote it with $P_n(\alpha)$, such that if $\alpha \equiv \beta$, then $P_n(\alpha) \equiv P_n(\beta)$.

Lemma 27 *Let $\alpha \in \mathsf{OT}$ and suppose $\alpha \succcurlyeq \omega$. Then $\mathsf{N}(P_n(\alpha)) = p(\mathsf{N}(\alpha), n)$.*

Proof The proof follows the pattern of the proof of the previous lemma. $\Box$

The following lemma constitutes the main lemma of this subsection. Note that the first property fails for the standard definition of the n^{th} branch α_n of a fundamental sequence $(\alpha_n)_{n\in\mathbb{N}}$, c.f. Section 3, together with the usual order $>$ on (set-theoretic) ordinals. For this definition, we obtain $\omega > m$, but $\omega_n = n \not> m - 1 = (m)_n$ for any $m > n$.

Lemma 28 *Let $\alpha, \beta \in \mathsf{OT}$, $n \in \mathbb{N}$.*

1. *If $\alpha, \beta \neq 0$ and $\alpha \sqsupset \beta$, then $P_n(\alpha) \sqsupset P_n(\beta)$.*
2. *Suppose $m > n$. Then $P_m(\alpha) \sqsupset^{\equiv} P_n(\alpha)$.*

Proof We only show the first point, the arguments for the other points are similar, but simpler. Assume $\alpha \sqsupset \beta$. First we show the lemma for the special-case, where $\alpha \in \mathbb{N}$. This assumption implies $\beta \in \mathbb{N}$. Hence $P_n(\alpha) = \alpha - 1 \sqsupset \beta - 1 = P_n(\beta)$.

Consider the case $\alpha \succcurlyeq \omega$. We proceed by case-distinction on the definition of $\sqsupset$. Subcase $\alpha \succ \beta$ and $\mathsf{N}(\alpha) \geqslant \mathsf{N}(\beta)$: In particular $p(\mathsf{N}(\alpha), n) \geqslant \mathsf{N}(\beta)$. Thus $\beta \in \alpha[n]$. Utilising Lemma 26(2) we conclude $P_n(\alpha) \succcurlyeq \beta \succ P_n(\beta)$, which implies $P_n(\alpha) \succ P_n(\beta)$. By Lemma 27 we get: $\mathsf{N}(P_n(\alpha)) = p(\mathsf{N}(\alpha), n) \geqslant p(\mathsf{N}(\beta), n) \geqslant \mathsf{N}(P_n(\beta))$. In summary, we see $P_n(\alpha) \sqsupset P_n(\beta)$. Subcase $\alpha \sim \beta$ and $\mathsf{N}(\alpha) > \mathsf{N}(\beta)$: From the assumptions we conclude $P_n(\beta) \in \alpha[n]$, as $\alpha \sim \beta \succ P_n(\beta)$ and $p(\mathsf{N}(\alpha), n) > p(\mathsf{N}(\beta), n) \geqslant \mathsf{N}(P_n(\beta))$. Hence, Lemma 26 implies $P_n(\alpha) \sqsupset P_n(\beta)$ or $P_n(\alpha) \equiv P_n(\beta)$. If the former case holds, the lemma is established. Assume the latter. By definition of $\equiv$ we see that $\mathsf{N}(P_n(\alpha)) = \mathsf{N}(P_n(\beta))$. On the other hand, we have: $\mathsf{N}(P_n(\alpha)) = p(\mathsf{N}(\alpha), n) > p(\mathsf{N}(\beta), n) \geqslant \mathsf{N}(P_n(\beta))$. We derive a contradiction. $\Box$

We write $f^n(\cdot)$ to denote the n-fold application of the function f.

Definition 29 We define the *collapsing function* C_α (parametrised in p):

$$\mathsf{C}_\alpha(n) := \max(\{2^{n+1}\} \cup \{\mathsf{C}^2_\beta(n) \mid \alpha \succ \beta \wedge p(\mathsf{N}(\alpha), n) \geqslant \mathsf{N}(\beta)\}) .$$

Lemma 30 *Let* $\alpha, \beta \in \mathsf{OT}$*,* $n, m \in \mathbb{N}$*.*

1. $\mathsf{C}_\alpha(n) \geqslant 2^{n+1} > n+1$.
2. *If* $m > n$*, then* $\mathsf{C}_\alpha(m) > \mathsf{C}_\alpha(n)$*.*
3. *If* $\alpha \succ \beta$ *and* $p(\mathsf{N}(\alpha), n) \geqslant \mathsf{N}(\beta)$*, then* $\mathsf{C}_\alpha(n) > \mathsf{C}_\beta(n+1)$*.*
4. *If* $\alpha \sqsupset \beta$*, then* $\mathsf{C}_\alpha(n) > \mathsf{C}_\beta(n)$*.*
5. *If* $\alpha \equiv \beta$*, then* $\mathsf{C}_\alpha(n) = \mathsf{C}_\beta(n)$*.*
6. $\mathsf{C}_\alpha(n+m) \geqslant \mathsf{C}_\alpha(n) + m$.
7. $\mathsf{C}_\alpha(n+1) \geqslant 2 \cdot \mathsf{C}_\alpha(n)$.

Proof We only show point 4) as the other points follow similarly. We proceed by transfinite induction on α. The base case is trivial. For the step-case, we first consider the subcase, where $\alpha \succ \beta$ and $\mathsf{N}(\alpha) \geqslant \mathsf{N}(\beta)$. Then the property follows by application of point 2) and 3). On the other hand assume $\alpha \sim \beta$ and $\mathsf{N}(\alpha) > \mathsf{N}(\beta)$. This implies that $\alpha \succcurlyeq \omega$. It suffices to show that

$$\forall \gamma \ \left(\beta \succ \gamma \wedge p(\mathsf{N}(\beta), n) \geqslant \mathsf{N}(\gamma) \Longrightarrow \mathsf{C}_\alpha(n) > \mathsf{C}^2_\gamma(n)\right) .$$

Fix $\gamma \in \mathsf{OT}$ with $\beta \succ \gamma$ and $p(\mathsf{N}(\beta), n) \geqslant \mathsf{N}(\gamma)$. The assumptions imply $\gamma \in \alpha[n]$, as $\alpha \sim \beta \succ \gamma$ and $p(\mathsf{N}(\alpha), n) > p(\mathsf{N}(\beta), n) \geqslant \mathsf{N}(\gamma)$. Lemma 26 yields $P_n(\alpha) \sqsupset^{\equiv} \gamma$ which can be strengthened to $P_n(\alpha) \sqsupset \gamma$, as the assumption $\alpha \succcurlyeq \omega$ together with Lemma 27 yields $\mathsf{N}(P_n(\alpha)) = p(\mathsf{N}(\alpha), n) > \mathsf{N}(\gamma)$. Moreover as $\alpha \succ P_n(\alpha)$, induction hypothesis is applicable to conclude $\mathsf{C}_{P_n(\alpha)}(m) > \mathsf{C}_\gamma(m)$ for any m. We obtain:

$$\mathsf{C}_\alpha(n) \geqslant \mathsf{C}^2_{P_n(\alpha)}(n) > \mathsf{C}_{P_n(\alpha)}\mathsf{C}_\gamma(n) > \mathsf{C}^2_\gamma(n) .$$

Here we employ point 2 in the second inequality. □

5.2 A Well-founded, Monotone Algebra for the TRS $\mathcal{D}$

In this section we define the $\mathcal{F}$-algebra $(\mathcal{A}, \rhd)$ and provide a proof that $\mathcal{A}$ is well-founded and monotone. The carrier A of $\mathcal{A}$ is defined as the set

$$\{(\alpha, m, 1) \mid \alpha \in \mathsf{OT}, m \in \mathbb{N}\} \cup \{(0, m, 0) \mid m \in \mathbb{N}\} .$$

The triples are related as follows:

$$(\alpha, m, p) \rhd (\beta, n, q) \quad :\Longleftrightarrow \quad ((\alpha \sqsupset \beta \wedge m \geqslant n) \vee (\alpha \equiv \beta \wedge m > n)) \wedge (p \geqslant q) .$$

Lemma 31 *The binary relation* $\rhd$ *is a well-founded order and* $\operatorname{otype}(\rhd) \leqslant \epsilon_0$*.*

Proof The proof follows the same pattern as the proof of Lemma 23. □

We define the following operations as interpretations of the elements of $\mathcal{F}$.

$$
\begin{array}{lll}
d_{\mathcal{A}} & (\alpha, m, p), (\beta, n, q) \mapsto (P_n(\alpha), \mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0) + m + n), 1) & \alpha \neq \mathsf{0} \\
 & (0, m, p), (\beta, n, q) \mapsto (0, 2^{\mathsf{C}_\beta(0)+m+n}, 0) & \\
g_{\mathcal{A}} & (\alpha, m, 1), (\beta, n, q) \mapsto (\mathsf{g}(\alpha, \beta), \mathsf{C}_\beta(0) + m + n, 1) & \\
 & (\alpha, m, 0), (\beta, n, q) \mapsto (\mathsf{0}, \mathsf{C}_\beta(0) + m + n, 0) & \\
h_{\mathcal{A}} & (\alpha, m, 1), (\beta, n, q) \mapsto (\mathsf{0}, \mathsf{C}_\alpha(\mathsf{C}_\beta(0) + m + n), 1) & \\
 & (\alpha, m, 0), (\beta, n, q) \mapsto (\mathsf{0}, \mathsf{C}_\beta(0) + m + n, 0) & \\
e_{\mathcal{A}} & (\alpha, m, p) \mapsto (\alpha, m + 1, 1) & \\
S_{\mathcal{A}} & (\alpha, m, p) \mapsto (\alpha, m + 1, 1) & \\
0_{\mathcal{A}} & (\mathsf{0}, 0, 1)\ . &
\end{array}
$$

It is easy to see that these operations are well-defined; it remains to verify that the operations $d_{\mathcal{A}}$, $g_{\mathcal{A}}$, $h_{\mathcal{A}}$, $e_{\mathcal{A}}$, and $S_{\mathcal{A}}$ are strictly monotone in each argument.

Lemma 32 *For each $f_{\mathcal{A}} \in \{d_{\mathcal{A}}, g_{\mathcal{A}}, h_{\mathcal{A}}\}$ and $(\alpha, m, p), (\beta, n, q), (\gamma, k, r) \in A$ we have*

$$
\begin{aligned}
(\alpha, m, p) \rhd (\gamma, k, r) &\Longrightarrow f_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd f_{\mathcal{A}}((\gamma, k, r), (\beta, n, r))\ , \\
(\beta, n, q) \rhd (\gamma, k, r) &\Longrightarrow f_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd f_{\mathcal{A}}((\alpha, m, p), (\gamma, k, r))\ ,
\end{aligned}
$$

and, for each $f_{\mathcal{A}} \in \{e_{\mathcal{A}}, S_{\mathcal{A}}\}$, we have

$$
(\alpha, m, p) \rhd (\gamma, k, r) \Longrightarrow f_{\mathcal{A}}((\alpha, m, p)) \rhd f_{\mathcal{A}}((\gamma, k, r))\ .
$$

Proof We only consider the operations $d_{\mathcal{A}}$, $g_{\mathcal{A}}$, and $h_{\mathcal{A}}$, as the monotonicity of $e_{\mathcal{A}}$ and $S_{\mathcal{A}}$ is easily seen.

1. *Case $d_{\mathcal{A}}$*: Assume $(\alpha, m, p) \rhd (\gamma, k, r)$, we firstly show

$$
d_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd d_{\mathcal{A}}((\gamma, k, r), (\beta, n, q))\ . \tag{3}
$$

   We proceed by case-distinction on α, γ. *Subcase $\alpha, \gamma \neq \mathsf{0}$*: We have to show:

$$
(P_n(\alpha), \mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0) + m + n), 1) \rhd (P_n(\gamma), \mathsf{C}_{P_n(\gamma)}(\mathsf{C}_\beta(0) + k + n), 1)\ .
$$

   By assumption $\alpha \sqsupseteq^{\equiv} \gamma$, $m \geqslant k$ so that at least one of the inequalities is strict. Suppose $\alpha \sqsupset \gamma$, then Lemmata 28(1) and 30(2,4) yield $P_n(\alpha) \sqsupset P_n(\gamma)$, and $\mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0) + m + n) > \mathsf{C}_{P_n(\gamma)}(\mathsf{C}_\beta(0) + k + n)$. On the other hand suppose $m > k$, then Lemma 28(1) yields $P_n(\alpha) \sqsupseteq^{\equiv} P_n(\gamma)$ and due to Lemma 30(2), $\mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0)+m+n) > \mathsf{C}_{P_n(\gamma)}(\mathsf{C}_\beta(0)+k+n)$ follows. *Subcase* $\alpha \neq \mathsf{0}$, $\gamma = \mathsf{0}$: Then the right-hand side becomes

$$(\mathsf{0}, 2^{\mathsf{C}_\beta(0)+k+n}, 0) \; .$$

Then $P_n(\alpha) \sqsupset^{\equiv} 0$ and the second argument strictly decreases: $\mathsf{C}_\mathsf{0}(\mathsf{C}_\beta(0)+m+n) = 2^{\mathsf{C}_\beta(0)+m+n+1} > 2^{\mathsf{C}_\beta(0)+k+n}$; *Subcase* $\alpha = \gamma = \mathsf{0}$: We have to show:

$$(\mathsf{0}, 2^{\mathsf{C}_\beta(0)+m+n}, 0) \rhd (0, 2^{\mathsf{C}_\beta(0)+k+n}, 0) \; .$$

By the assumptions in this subcase, we have $\alpha \equiv \gamma$ and $m > k$. Hence the second argument decreases. In all considered cases the third component of the triple never increases, hence (3) follows. Now assume $(\beta, n, p) \rhd (\gamma, k, r)$, we show:

$$d_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd d_{\mathcal{A}}((\alpha, m, p), (\gamma, k, r)) \; . \tag{4}$$

Subcase $\alpha \neq \mathsf{0}$: We have to show:

$$(P_n(\alpha), \mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0)+m+n), 1) \rhd (P_k(\alpha), \mathsf{C}_{P_k(\alpha)}(\mathsf{C}_\gamma(0)+m+k), 1) \; .$$

Due to Lemma 28(2) $P_n(\alpha) \sqsupset^{\equiv} P_k(\alpha)$. By assumption $\beta \sqsupset^{\equiv} \gamma$, $n \geqslant k$, and at least one of the inequalities is strict. Suppose $\beta \sqsupset \gamma$. Then Lemma 30(4) yields $\mathsf{C}_\beta(0) > \mathsf{C}_\gamma(0)$. Hence by Lemma 30(2), $\mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0)+m+n) > \mathsf{C}_{P_k(\alpha)}(\mathsf{C}_\gamma(0)+m+k)$. If on the other hand $n > k$, then $\mathsf{C}_\beta(0) \geqslant \mathsf{C}_\gamma(0)$ holds and $\mathsf{C}_{P_n(\alpha)}(\mathsf{C}_\beta(0)+m+n) > \mathsf{C}_{P_k(\alpha)}(\mathsf{C}_\gamma(0)+m+k)$ follows by application of Lemma 30(2). *Subcase* $\alpha = \mathsf{0}$: We have to show:

$$(\mathsf{0}, 2^{\mathsf{C}_\beta(0)+m+n}, 0) \rhd (0, 2^{\mathsf{C}_\gamma(0)+m+k}, 0) \; ,$$

which either follows from Lemma 30(4), if $\beta \sqsupset \gamma$ or directly from $n > k$. In both sub cases the last argument remains equal, thus (4) follows.

2. *Case* $g_{\mathcal{A}}$: We assume $(\alpha, m, p) \rhd (\gamma, k, r)$ and show

$$g_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd g_{\mathcal{A}}((\gamma, k, r), (\beta, n, q)) \; . \tag{5}$$

Subcase $p = r = 1$: We have to show

$$(\mathsf{g}(\alpha, \beta), \mathsf{C}_\beta(0)+m+n, 1) \rhd (\mathsf{g}(\gamma, \beta), \mathsf{C}_\beta(0)+k+n, 1) \; .$$

By assumption $\alpha \sqsupset^{\equiv} \gamma$, $m \geqslant k$, and at least one of the inequalities is strict. Suppose $\alpha \sqsupset \gamma$. Lemma 24(1) shows that $\mathsf{g}(\alpha, \beta) \sqsupset \mathsf{g}(\gamma, \beta)$, while the second argument is non-increasing. On the other hand suppose $m > k$, then the second argument strictly decreases and $\mathsf{g}(\alpha, \beta) \sqsupset^{\equiv} \mathsf{g}(\gamma, \beta)$ holds. *Subcase* $p = 1$, $r = 0$: Then the right-hand side rewrites to

$$(\mathsf{0}, \mathsf{C}_\beta(0)+k+n, 0) \; .$$

By definition $\mathsf{g}(\alpha, \beta) \sqsupset 0$ and the second argument is non-increasing. *Subcase* $p = r = 0$: We have to show

$$(\mathsf{0}, \mathsf{C}_\beta(0)+m+n, 0) \rhd (\mathsf{0}, \mathsf{C}_\beta(0)+k+n, 0) \; ,$$

which follows trivially, if the assumptions imply $m > k$. To see that this always holds, note that $\alpha \sqsupset \gamma$ would imply $\alpha \neq 0$, and thus $(\alpha, m, 0) \in A$, for $\alpha \neq 0$, in contrast to the definition of the algebra $\mathcal{A}$. Finally, the last argument is non-increasing for all subcases, hence (5) follows. Assume $(\beta, n, p) \rhd (\gamma, k, q)$, we show

$$g_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd g_{\mathcal{A}}((\alpha, m, p), (\gamma, k, r)) \ . \tag{6}$$

Subcase $p = 1$: We have to show

$$(\mathsf{g}(\alpha, \beta), \mathsf{C}_{\beta}(0) + m + n, 1) \rhd (\mathsf{g}(\alpha, \gamma), \mathsf{C}_{\gamma}(0) + m + k, 1) \ .$$

Due to Lemmata 24(2) and 30(4,5), we conclude $\mathsf{g}(\alpha, \beta) \sqsupset^{\equiv} \mathsf{g}(\alpha, \gamma)$ and $\mathsf{C}_{\beta}(0) + m + n > \mathsf{C}_{\gamma}(0) + m + k$ from which (6) follows. *Subcase* $p = 0$: We have to show

$$(0, \mathsf{C}_{\beta}(0) + m + n, 0) \rhd (0, \mathsf{C}_{\gamma}(0) + m + k, 0) \ ,$$

which follows by Lemma 30(4) if $\beta \sqsupset \gamma$; directly if $\beta \equiv \gamma$. As the last argument of the triple remains equal in both subcases, we have established (6).

3. *Case* $h_{\mathcal{A}}$: We assume $(\alpha, m, p) \rhd (\gamma, k, q)$ and show

$$h_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd h_{\mathcal{A}}((\gamma, k, r), (\beta, n, q)) \ . \tag{7}$$

Subcase $p = r = 1$. We have to show

$$(0, \mathsf{C}_{\alpha}(\mathsf{C}_{\beta}(0) + m + n), 1) \rhd (0, \mathsf{C}_{\gamma}(\mathsf{C}_{\beta}(0) + k + n), 1)$$

By assumption $\alpha \sqsupset^{\equiv} \gamma$, $m \geqslant k$, so that at least one of the inequalities is strict. It suffices to show that $\mathsf{C}_{\alpha}(\mathsf{C}_{\beta}(0) + m + n) > \mathsf{C}_{\gamma}(\mathsf{C}_{\beta}(0) + k + n)$, which follows by Lemma 30. *Subcase* $p = 1$, $r = 0$. We have to show

$$(0, \mathsf{C}_{\alpha}(\mathsf{C}_{\beta}(0) + m + n), 1) \rhd (0, \mathsf{C}_{\beta}(0) + k + n, 0) \ ,$$

which follows by Lemma 30(1). *Subcase* $p = r = 0$. We have to show

$$(0, \mathsf{C}_{\beta}(0) + m + n), 0) \rhd (0, \mathsf{C}_{\beta}(0) + k + n), 0) \ ,$$

which follows trivially, as the assumptions imply $m > k$. As the third argument of the triple is non-increasing in all subcases, (7) follows. Finally assume $(\beta, n, q) \rhd (\gamma, k, r)$, we show

$$h_{\mathcal{A}}((\alpha, m, p), (\beta, n, q)) \rhd h_{\mathcal{A}}((\alpha, m, p), (\gamma, k, r)) \ . \tag{8}$$

Subcase $p = 1$. We have to show

$$(0, \mathsf{C}_{\alpha}(\mathsf{C}_{\beta}(0) + m + n), 1) \rhd (0, \mathsf{C}_{\alpha}(\mathsf{C}_{\gamma}(0) + m + k), 1) \ ,$$

which follows as Lemma 30(4) implies $\mathsf{C}_{\beta}(0) + m + n > \mathsf{C}_{\gamma}(0) + m + k$, and Lemma 30(2) yields that the second argument decreases. *Subcase* $p = 0$. We have to show

$$(0, \mathsf{C}_{\beta}(0) + m + n, 0) \rhd (0, \mathsf{C}_{\gamma}(0) + m + k, 0) \ ,$$

which follows as above. As the third argument of the triple remains equal in both subcases, (7) follows.

□

The next theorem is a direct consequence of Lemmata 31 and 32.

Theorem 33 *The $\mathcal{F}$-algebra $(\mathcal{A}, \rhd)$ is a WMA and* $\mathrm{otype}(\rhd) \leqslant \epsilon_0$.

5.3 Termination

We fix the parameter in the definition of the n-predecessors and the functions C_α:

$$p(m,n) := (m+1)\cdot(n+1) \ .$$

Theorem 34 *The WMA $(\mathcal{A}, \rhd)$ is compatible with the TRS $(\mathcal{F}, \mathcal{D})$.*

Proof Let $\rhd_\mathcal{A}$ denote the reduction order induced by the algebra $(\mathcal{A}, \rhd)$. Due to Theorem 5 it remains to verify that for each rule $l \to r \in \mathcal{D}$, $l \rhd_\mathcal{A} r$ holds. To this end, suppose $\mathbf{a}\colon \mathcal{V} \to A$ denotes an arbitrary, but fixed assignment. Then we must show for each rule $l \to r$: $[\mathbf{a}]_\mathcal{A}(l) \rhd [\mathbf{a}]_\mathcal{A}(r)$ holds. Let (α, m, p), (β, n, q), and (γ, k, r) denote the interpretations of the variables x, y, and z, respectively. In proof we only consider the rules 1, 2, 5, and 6. To show compatibility with the rules 3 and 4, similar, but simpler arguments suffice.

1. *Rule* 1: $h(e(x),y) \to h(d(x,y),S(y))$. *Subcase* $\alpha \neq 0$: Set $\delta := P_n(\alpha)$. We simplify the left-hand side and right-hand side of the interpretations of the rule:

$$\begin{aligned} &h_\mathcal{A}(e_\mathcal{A}((\alpha,m,p)),(\beta,n,q)) = h_\mathcal{A}((\alpha,m+1,1),(\beta,n,q)) \\ &= (0, \mathsf{C}_\alpha(\mathsf{C}_\beta(0)+m+n+1),1) \ , \end{aligned}$$

and

$$\begin{aligned} &h_\mathcal{A}(d_\mathcal{A}((\alpha,m,p),(\beta,n,q)),S_\mathcal{A}((\beta,n,q))) \\ &= h_\mathcal{A}(d_\mathcal{A}((\alpha,m,p),(\beta,n,q)),(\beta,n+1,1)) \\ &= h_\mathcal{A}((\delta,\mathsf{C}_\delta(\mathsf{C}_\beta(0)+m+n),1),(\beta,n+1,1)) \\ &= (0,\mathsf{C}_\delta(\mathsf{C}_\beta(0)+\mathsf{C}_\delta(\mathsf{C}_\beta(0)+m+n)+n+1),1) \ . \end{aligned}$$

Due to Lemma 30(1,2,7), we obtain:

$$\begin{aligned} \mathsf{C}_\alpha(\mathsf{C}_\beta(0)+m+n+1) &\geqslant \mathsf{C}_\delta(\mathsf{C}_\delta(\mathsf{C}_\beta(0)+m+n+1)) \\ \geqslant \mathsf{C}_\delta(2\mathsf{C}_\delta(\mathsf{C}_\beta(0)+m+n)) &= \mathsf{C}_\delta(\mathsf{C}_\delta(\mathsf{C}_\beta(0)+m+n)+\mathsf{C}_\delta(\mathsf{C}_\beta(0)+m+n)) \\ &> \mathsf{C}_\delta(\mathsf{C}_\beta(0)+n+1+\mathsf{C}_\delta(\mathsf{C}_\beta(0)+m+n)) \ . \end{aligned}$$

Subcase $\alpha = 0$: The right-hand side becomes

$$\begin{aligned} &h_\mathcal{A}(d_\mathcal{A}((\alpha,m,p),(\beta,n,q)),S_\mathcal{A}((\beta,n,q))) \\ &= h_\mathcal{A}((0,2^{\mathsf{C}_\beta(0)+m+n},0),(\beta,n+1,1)) \\ &= (0,\mathsf{C}_\beta(0)+2^{\mathsf{C}_\beta(0)+m+n}+n+1),0) \ . \end{aligned}$$

And, a few calculations reveal:

$$\begin{aligned}\mathsf{C}_0(\mathsf{C}_\beta(0)+m+n+1) &= 2^{\mathsf{C}_\beta(0)+m+n+2} > 2^{\mathsf{C}_\beta(0)+m+n} + 2^{\mathsf{C}_\beta(0)+m+n+1} \\ &> \mathsf{C}_\beta(0)+n+1+2^{\mathsf{C}_\beta(0)+m+n} \ .\end{aligned}$$

Thus rule 1 is compatible with $\mathcal{A}$, as in both subcases the first argument of the compared triples remains unchanged and the last argument of the triple is not increased.

2. *Rule* 2: $d(g(g(0,x),y),S(z)) \to g(e(x),d(g(g(0,x),y),z))$. For that we set $\delta := P_{k+1}(\mathsf{g}(\mathsf{g}(0,\alpha),\beta))$ and $\epsilon := P_k(\mathsf{g}(\mathsf{g}(0,\alpha),\beta))$. Define $\ell = \mathsf{C}_\beta(0) + \mathsf{C}_\alpha(0)+m+n$. The left-hand side of the rule is transformed as follows:

$$\begin{aligned}& d_\mathcal{A}(g_\mathcal{A}(g_\mathcal{A}(0_\mathcal{A},(\alpha,m,p)),(\beta,n,q)),S_\mathcal{A}((\gamma,k,r))) \\ &= d_\mathcal{A}(g_\mathcal{A}(g_\mathcal{A}((0,0,1),(\alpha,m,p)),(\beta,n,q)),(\gamma,k+1,1)) \\ &= d_\mathcal{A}(g_\mathcal{A}((\mathsf{g}(0,\alpha),\mathsf{C}_\alpha(0)+m,1),(\beta,n,q)),(\gamma,k+1,1)) \\ &= d_\mathcal{A}((\mathsf{g}(\mathsf{g}(0,\alpha),\beta),\ell,(\gamma,k+1,1)) \\ &= (\delta,\mathsf{C}_\delta(\mathsf{C}_\gamma(0)+\ell+k+1),1) \ ,\end{aligned}$$

while for the right-hand side we have

$$\begin{aligned}& g_\mathcal{A}(e_\mathcal{A}((\alpha,m,p)),d_\mathcal{A}(g_\mathcal{A}(g_\mathcal{A}(0_\mathcal{A},(\alpha,m,p)),(\beta,n,q)),(\gamma,k,r))) \\ &= g_\mathcal{A}((\alpha,m,1),d_\mathcal{A}((\mathsf{g}(\mathsf{g}(0,\alpha),\beta),\ell,1),(\gamma,k,q))) \\ &= g_\mathcal{A}((\alpha,m,1),(\epsilon,\mathsf{C}_\epsilon(\mathsf{C}_\gamma(0)+\ell+k),1)) \\ &= (\mathsf{g}(\alpha,\epsilon),\mathsf{C}_\epsilon(0)+m+\mathsf{C}_\epsilon(\mathsf{C}_\gamma(0)+\ell+k)),1) \ .\end{aligned}$$

Let $\epsilon = \mathsf{g}(\epsilon_1,\mathsf{g}(\epsilon_2,\dots,\mathsf{g}(\epsilon_e,\beta)\dots))$ such that $\epsilon_i \in \mathsf{OT}$ for all $1 \leqslant i \leqslant e$. As $\mathsf{g}(\mathsf{g}(0,\alpha),\beta) \succ \epsilon$, we obtain $\mathsf{g}(0,\alpha) \succ \epsilon_i$ for all $1 \leqslant i \leqslant e$. By Lemma 18, we additionally have $\mathsf{g}(0,\alpha) \succ \alpha$. Thus $\mathsf{g}(\mathsf{g}(0,\alpha),\beta) \succ \mathsf{g}(\alpha,\epsilon)$ follows. Now, a simple calculation shows

$$p(\mathsf{N}(\mathsf{g}(\mathsf{g}(0,\alpha),\beta)),k+1) > \mathsf{N}(\mathsf{g}(\alpha,\epsilon)) \ .$$

Lemma 26 yields $\delta \sqsupset^{\equiv} \mathsf{g}(\alpha,\epsilon)$ and Lemma 24 yields $\mathsf{g}(\alpha,\epsilon) \sqsupset \epsilon$. Hence, we obtain $\delta \sqsupset \epsilon$ and by Lemma 30(4,6,7):

$$\begin{aligned}\mathsf{C}_\delta(\mathsf{C}_\gamma(0)+\ell+k+1) &\geqslant \mathsf{C}_\delta(\mathsf{C}_\gamma(0)+\ell+k)+\mathsf{C}_\delta(\mathsf{C}_\gamma(0)+l+k) \\ &> \mathsf{C}_\epsilon(\mathsf{C}_\gamma(0)+\ell+k)+1+\mathsf{C}_\epsilon(\mathsf{C}_\gamma(0)+l+k) \\ &> \mathsf{C}_\epsilon(0)+m+1+\mathsf{C}_\epsilon(\mathsf{C}_\gamma(0)+l+k) \ .\end{aligned}$$

Hence the second argument strictly decreases, while the first and the third do not increase.

3. *Rule* 5: $d(g(x,y),z) \to g(d(x,z),e(y))$. *Subcase* $p=1$, $\alpha \neq 0$: Set $\delta := P_k(\mathsf{g}(\alpha,\beta))$ and $\epsilon := P_k(\alpha)$. The left-hand side becomes

$$\begin{aligned}& d_\mathcal{A}(g_\mathcal{A}((\alpha,m,1),(\beta,n,q)),(\gamma,k,r)) \\ &= d_\mathcal{A}((\mathsf{g}(\alpha,\beta),\mathsf{C}_\beta(0)+m+n,1),(\gamma,k,r)) \\ &= (\delta,\mathsf{C}_\delta(\mathsf{C}_\gamma(0)+\mathsf{C}_\beta(0)+m+n+k),1) \ ,\end{aligned}$$

while the right-hand side amounts to

$$\begin{aligned}&g_{\mathcal{A}}(d_{\mathcal{A}}((\alpha,m,1),(\gamma,k,r)),e_{\mathcal{A}}((\beta,n,q)))\\&=g_{\mathcal{A}}((\epsilon,\mathsf{C}_{\epsilon}(\mathsf{C}_{\gamma}(0)+m+k),1),(\beta,n+1,1))\\&=(\mathsf{g}(\epsilon,\beta),\mathsf{C}_{\beta}(0)+\mathsf{C}_{\epsilon}(\mathsf{C}_{\gamma}(0)+m+k)+n+1,1)\ .\end{aligned}$$

Due to $\alpha \neq 0$, we have $\alpha \succ P_k(\alpha) = \epsilon$. By Lemma 18 we see $\mathsf{g}(\alpha,\beta) \succ \mathsf{g}(\epsilon,\beta)$ and by definition of the function p and the norm-function N:

$$p(\mathsf{N}(\mathsf{g}(\alpha,\beta)),k) \geqslant (\mathsf{N}(\alpha)+1)(k+1)+\mathsf{N}(\beta)+1 \geqslant \mathsf{N}(\mathsf{g}(\epsilon,\beta))\ .$$

This implies that $\delta \sqsupset^{\equiv} \mathsf{g}(\alpha,\epsilon)$, yielding a weak decrease in the first argument. Moreover by Lemma 18 we obtain: $\mathsf{g}(\alpha,\beta) \succ \alpha$ and Lemma 26 implies $\delta \succcurlyeq \alpha \succ P_k(\alpha) = \epsilon$. As $\mathsf{g}(\alpha,\beta) \succcurlyeq \omega$, Lemma 27 becomes applicable to show: $\mathsf{N}(\delta) = p(\mathsf{N}(\mathsf{g}(\alpha,\beta)),k) > \mathsf{N}(\epsilon)$. Due to Lemma 30(3,6), we have:

$$\begin{aligned}\mathsf{C}_{\delta}(\mathsf{C}_{\gamma}(0)+\mathsf{C}_{\beta}(0)+m+n+k) &> \mathsf{C}_{\epsilon}(\mathsf{C}_{\gamma}(0)+\mathsf{C}_{\beta}(0)+m+n+k+1)\\&\geqslant \mathsf{C}_{\epsilon}(\mathsf{C}_{\gamma}(0)+m+k)+\mathsf{C}_{\beta}(0)+n+1\ .\end{aligned}$$

Subcase $p=1$, $\alpha=0$: This setting implies that we have to show

$$(\delta,\mathsf{C}_{\delta}(\mathsf{C}_{\gamma}(0)+\mathsf{C}_{\beta}(0)+m+n+k),1) \rhd (0,\mathsf{C}_{\beta}(0)+2^{\mathsf{C}_{\gamma}(0)+m+k}+n+1,0)$$

We have $\delta \sqsupset^{\equiv} 0$. Moreover for all χ, $\mathsf{C}_{\chi}(0) \geqslant 2$ holds; we proceed as follows:

$$\begin{aligned}\mathsf{C}_{\delta}(\mathsf{C}_{\gamma}(0)+\mathsf{C}_{\beta}(0)+m+n+k) &\geqslant 2^{\mathsf{C}_{\gamma}(0)+\mathsf{C}_{\beta}(0)+m+n+k+1}\\&> 2^{\mathsf{C}_{\gamma}(0)+\mathsf{C}_{\beta}(0)+m+n+k} \geqslant 2^{\mathsf{C}_{\gamma}(0)+m+k+1}+2^{\mathsf{C}_{\beta}(0)+n+1}\\&> 2^{\mathsf{C}_{\gamma}(0)+m+k}+\mathsf{C}_{\beta}(0)+n+1\ .\end{aligned}$$

Subcase $p=0$: By definition this implies $\alpha=0$ and we have to show

$$(0,2^{\mathsf{C}_{\gamma}(0)+\mathsf{C}_{\beta}(0)+m+n+k},0) \rhd (0,\mathsf{C}_{\beta}(0)+2^{\mathsf{C}_{\gamma}(0)+m+k}+n+1,0)\ ,$$

which follows as above. In all subcases the last argument is not increased.

4. *Rule* 6: $g(e(x),e(y)) \to e(g(x,y))$. *Subcase* $p=1$: Independent of p the left-hand side rewrites to

$$\begin{aligned}&g_{\mathcal{A}}(e_{\mathcal{A}}(\alpha,m,p),e_{\mathcal{A}}((\beta,n,q))) = g_{\mathcal{A}}((\alpha,m+1,1),(\beta,n+1,1))\\&=(\mathsf{g}(\alpha,\beta),\mathsf{C}_{\beta}(0)+m+n+2,1)\ ,\end{aligned}$$

while the right-hand side becomes

$$\begin{aligned}&e_{\mathcal{A}}(g_{\mathcal{A}}((\alpha,m,1),(\beta,n,q))) = e_{\mathcal{A}}((\mathsf{g}(\alpha,\beta),\mathsf{C}_{\beta}(0)+m+n,1))\\&=(\mathsf{g}(\alpha,\beta),\mathsf{C}_{\beta}(0)+m+n+1,1)\ .\end{aligned}$$

This subcase follows directly from the definitions. *Subcase* $p = 0$: We re-calculate the right-hand side:

$$e_{\mathcal{A}}(g_{\mathcal{A}}((\alpha, m, 0), (\beta, n, q))) = e_{\mathcal{A}}((0, \mathsf{C}_\beta(0) + m + n, 0))$$
$$= (0, \mathsf{C}_\beta(0) + m + n + 1, 1) \ .$$

Thus we observe that $\mathsf{g}(\alpha, \beta) \sqsupset^{\equiv} 0$, while the second argument decreases as above. Note that the last argument does not increase.

$\square$

Corollary 35 *The Hydra battle $\mathcal{D}$ is terminating and termination can be established by a reduction order of order type ϵ_0.*

Proof The termination of $(\mathcal{F}, \mathcal{D})$ is established by Theorem 34. We consider the reduction-order $\rhd_{\mathcal{A}}$ induced by $\mathcal{A}$. Let $\mathbf{0}\colon \mathcal{V} \to A$ denote the assignment that substitutes 0 for each variable. By definition of $\rhd_{\mathcal{A}}$, for all s, t: $s \rhd_{\mathcal{A}} t$ implies $[\mathbf{0}]_{\mathcal{A}}(s) \rhd [\mathbf{0}]_{\mathcal{A}}(t)$. Thus $\rhd_{\mathcal{A}}$ is embeddable into $\rhd$ and by Lemma 1 and Lemma 31, we see

$$\mathrm{otype}(\rhd_{\mathcal{A}}) \leqslant \mathrm{otype}(\rhd) \leqslant \epsilon_0 \ .$$

To establish equality, we assume to the converse, that $\mathrm{otype}(\rhd_{\mathcal{A}}) =: \alpha < \epsilon_0$. From this, the termination proof, and Lemma 13 we see that for all $\beta < \epsilon_0$ the length of the Hydra battle is majorised by C_α. It is not difficult to argue that C_α is provably total in Peano arithmetic. Hence termination of the Hydra battle would be provable within Peano arithmetic in direct contradiction to the Kirby and Paris's result, c.f. [16]. $\square$

6 A Termination Proof of the TRS $\mathcal{H}$

We recall the definition of the TRS $\mathcal{H}$ introduced in [10]. Again, we swap the arguments of the symbols d and h and make use of the unary function symbol S instead of the original c.

$$\begin{aligned}
7\colon && h(e(x), y) &\to h(d(x, y), S(y)) \\
8\colon && d(g(0, 0), y) &\to e(0) \\
9\colon && d(g(g(x, y), 0), S(z)) &\to g(d(g(x, y), S(z)), d(g(x, y), z)) \\
10\colon && d(g(x, y), z) &\to g(e(x), d(y, z)) \\
11\colon && g(e(x), e(y)) &\to e(g(x, y)) \ .
\end{aligned}$$

Note that the rules 7 and 11 are part of the above studied TRS $\mathcal{D}$. Moreover, rule 8 is a specialisation of the rule 4 in $\mathcal{D}$. Despite these similarities, an attempt to prove compatibility of $\mathcal{H}$ with $\mathcal{A}$ fails.

In order to yield termination of $\mathcal{H}$ we make use of a subtle variant of the ordinal notation system introduced in Section 4: Instead of conceiving the term $\mathsf{g}(\alpha, \beta)$ as the the ordinal $\beta + \omega^\alpha$ (or similarly as the nested multiset $\beta + \{\alpha\}$), we forget about the earlier notation system, and now conceive $\mathsf{g}(\alpha, \beta)$ as $\omega^\alpha + \beta$ (i.e., as $\{\alpha\} + \beta$). This change is essential, as the role of the function symbol $g \in \mathcal{F}$ with respect to the TRS $\mathcal{H}$ changes.

For this new interpretation, all results from Section 4 are preserved. Thus we can use the "same" carrier and almost the same function interpretations as in the $\mathcal{F}$-algebra $\mathcal{A}$ to define a new $\mathcal{F}$-algebra $\mathcal{B}$. The only exception is the definition of interpretation for g that we change as follows:

$$\begin{array}{ll} g_{\mathcal{B}} & (\alpha, m, p), (\beta, n, 1) \mapsto (\mathsf{g}(\alpha, \beta), \mathsf{C}_\beta(0) + m + n, 1) \\ & (\alpha, m, p), (\beta, n, 0) \mapsto (0, \mathsf{C}_\beta(0) + m + n, 0) \ . \end{array}$$

The resulting $\mathcal{F}$-algebra $\mathcal{B}$ follows to some extent the corresponding suggestions in [10]. However, note that the algebra $\mathcal{B}$ crucially rests on the use of additional concepts like collapsing functions or the carefully crafted notion of an n-predecessor, c.f. Definition 25 and Definition 29.

In the spirit of the relation $\rhd$, we define the relation $\blacktriangleright$ as: $(\alpha, m, p) \blacktriangleright (\beta, n, q)$ if either $\alpha \sqsupset \beta$ and $m \geqslant n$, or if $\alpha \equiv \beta$ and $m > n$ such that in both cases $p \geqslant q$. Further we define the parameter for the definition of n-predecessors and the functions C_α as follows:

$$p(m, n) := 2^m \cdot (n + 1) \ .$$

Following the pattern of the proofs of Lemmata 31 and 32 it is not difficult to see that $(\mathcal{B}, \blacktriangleright)$ is a WMA and $\mathrm{otype}(\blacktriangleright) \leqslant \epsilon_0$.

We arrive at the main theorem of this paper.

Theorem 36 *The WMA $(\mathcal{B}, \blacktriangleright)$ is compatible with the TRS $(\mathcal{F}, \mathcal{H})$.*

Proof As in the proof of Theorem 34, $\blacktriangleright_{\mathcal{B}}$ denotes the reduction order induced by the $\mathcal{F}$-algebra $(\mathcal{B}, \blacktriangleright)$. For each rule $l \to r \in \mathcal{H}$ we show $[\mathbf{a}]_{\mathcal{B}}(l) \rhd [\mathbf{a}]_{\mathcal{B}}(r)$ for an arbitrary but fixed assignment $\mathbf{a}$. We suppose the variables x, y, and z are interpreted as (α, m, p), (β, n, q), and (γ, k, r), respectively. The arguments are similar to those in the proof of Theorem 34, exemplary we consider rule 9: $d(g(g(x, y), 0), S(z)) \to g(d(g(x, y), S(z)), d(g(x, y), z))$

Subcase $q = 1$: Let $\delta := P_{k+1}(\mathsf{g}(\mathsf{g}(\alpha, \beta), 0))$, $\epsilon := P_{k+1}(\mathsf{g}(\alpha, \beta))$, $\zeta := P_k(\mathsf{g}(\alpha, \beta))$. The left-hand side becomes

$$\begin{aligned} & d_{\mathcal{B}}(g_{\mathcal{B}}(g_{\mathcal{B}}((\alpha, m, p), (\beta, n, 1)), 0_{\mathcal{B}}), S_{\mathcal{B}}((\gamma, k, r))) \\ & = d_{\mathcal{B}}(g_{\mathcal{B}}((\mathsf{g}(\alpha, \beta), \mathsf{C}_\alpha(0) + m + n, 1), (0, 0, 1)), (\gamma, k + 1, 1)) \\ & = (\delta, \mathsf{C}_\delta(\mathsf{C}_\gamma(0) + \mathsf{C}_{\mathsf{g}(\alpha,\beta)}(0) + \mathsf{C}_\alpha(0) + m + n + k + 1), 1) \ , \end{aligned}$$

while the right-hand side becomes:

$$\begin{aligned} & g_{\mathcal{B}}(d_{\mathcal{B}}(g_{\mathcal{B}}((\alpha, m, p), (\beta, n, 1)), S_{\mathcal{B}}((\gamma, k, r))), d_{\mathcal{B}}(g_{\mathcal{B}}((\alpha, m, p), (\beta, n, 1)), (\gamma, k, r))) \\ & = g_{\mathcal{B}}(d_{\mathcal{B}}((\mathsf{g}(\alpha, \beta), \mathsf{C}_\alpha(0) + m + n, 1), (\gamma, k + 1, 1)), \\ & \quad d_{\mathcal{B}}((\mathsf{g}(\alpha, \beta), \mathsf{C}_\alpha(0) + m + n, 1), (\gamma, k, 1))) \\ & = (\mathsf{g}(\epsilon, \zeta), \mathsf{C}_\epsilon(0) + \mathsf{C}_\epsilon(\mathsf{C}_\gamma(0) + \mathsf{C}_\alpha(0) + m + n + k + 1) + \\ & \quad + \mathsf{C}_\zeta(\mathsf{C}_\gamma(0) + \mathsf{C}_\alpha 0) + m + n + k), 1) \ . \end{aligned}$$

It is not difficult to see that $\mathsf{g}(\mathsf{g}(\alpha,\beta),\mathsf{0}) \succ \mathsf{g}(\epsilon,\zeta)$. Moreover we have

$$
\begin{aligned}
&p(\mathsf{N}(\mathsf{g}(\mathsf{g}(\alpha,\beta),\mathsf{0}),k+1) = 2^{\mathsf{N}(\mathsf{g}(\alpha,\beta))+1}(k+2) \\
&\geqslant \mathsf{N}(P_{k+1}(\mathsf{g}(\alpha,\beta))) + \mathsf{N}(P_k(\mathsf{g}(\alpha,\beta))) + 1 = \mathsf{N}(\mathsf{g}(\epsilon,\zeta)) \ .
\end{aligned}
$$

The analogues of Lemmata 26, 24 yields $\delta \sqsupset^{\equiv} \mathsf{g}(\epsilon,\zeta) \sqsupset \epsilon,\zeta$. From which we obtain $\delta \sqsupset \zeta$, and $\delta \sqsupset \epsilon$. The latter can even strengthened as $\delta \succ \epsilon$ and $\mathsf{N}(\delta) > \mathsf{N}(\epsilon)$ holds. Employing Lemma 30 we conclude:

$$
\begin{aligned}
&\mathsf{C}_\delta(\mathsf{C}_\gamma(0) + \mathsf{C}_{\mathsf{g}(\alpha,\beta)}(0) + \mathsf{C}_\alpha(0) + m + n + k + 1) \\
&\geqslant \mathsf{C}_\delta(\mathsf{C}_\gamma(0) + \mathsf{C}_\alpha(0) + m + n + k + 2) \geqslant 4C\delta(\mathsf{C}_\gamma(0) + \mathsf{C}_\alpha(0) + m + n + k) \\
&> \mathsf{C}_\epsilon(0) + \mathsf{C}_\epsilon(\mathsf{C}_\gamma(0) + \mathsf{C}_\alpha(0) + m + n + k + 1) + \\
&\quad + \mathsf{C}_\zeta(\mathsf{C}_\gamma(0) + \mathsf{C}_\alpha(0) + m + n + k) \ .
\end{aligned}
$$

Subcase $q = 0$: This implies $\beta = 0$; the left-hand side becomes

$$
\begin{aligned}
&d_{\mathcal{B}}(g_{\mathcal{B}}(g_{\mathcal{B}}((\alpha,m,p),(\beta,n,0)),0_{\mathcal{B}}),S_{\mathcal{B}}((\gamma,k,r))) \\
&= d_{\mathcal{B}}(g_{\mathcal{B}}((0,\mathsf{C}_\alpha(0)+m+n,0),(0,0,1)),(\gamma,k+1,1)) \\
&= (0, 2^{\mathsf{C}_\gamma(0)+\mathsf{C}_\alpha(0)+m+n+k+3}, 0) \ ,
\end{aligned}
$$

while the right-hand side becomes:

$$
\begin{aligned}
&g_{\mathcal{B}}(d_{\mathcal{B}}(g_{\mathcal{B}}((\alpha,m,a),(\beta,n,0)),S_{\mathcal{B}}((\gamma,k,r))),d_{\mathcal{B}}(g_{\mathcal{B}}((\alpha,m,a),(\beta,n,0)),(\gamma,k,r))) \\
&= g_{\mathcal{B}}(d_{\mathcal{B}}((0,\mathsf{C}_\alpha(0)+m+n,0),(\gamma,k+1,1)),d_{\mathcal{B}}(0,\mathsf{C}_\alpha(0)+m+n,0),(\gamma,k,1))) \\
&= (0, 2 + 2^{\mathsf{C}_\gamma(0)+\mathsf{C}_\alpha(0)+m+n+k+1} + 2^{\mathsf{C}_\gamma(0)+\mathsf{C}_\alpha(0)+m+n+k}, 0) \ .
\end{aligned}
$$

It is easy to see that the second argument of the triple decreases. In both subcases, the last argument is not increased, hence the case follows. □

7 Conclusion

A long standing challenge in the annual termination competition is the TRS `D33-33` introduced by Dershowitz and Jouannaud. To date this systems defies all automated termination attempts. In this paper we study this system (denoted as $\mathcal{H}$ above) together with its variant `Zantema06-hydra` (denoted $\mathcal{D}$ above) in the light of Cichon's principle.

We provide conceptually simple, but arguably technical termination proofs of these systems by showing compatibility with suitably chosen well-founded, monotone algebras. To the best of our knowledge this is the first complete termination proof for these systems given in the literature. (See [12] for a termination proof based on the dependency pair method [1] that rests on ideas fully developed here.) Based on this result, we can solve open problem # 23 in RTALooP in the negative.

As a result of our investigations, we are convinced that in order to prove termination of the TRSs $\mathcal{H}$ and $\mathcal{D}$ automatically, it will not suffice to extend (well-studied) polynomial interpretation by interpretations into the ordinals, but additional investigations into ordinal notation systems will be necessary. Both considerations will be subject of future work.

In concluding, we want to mention an alternative term rewriting characterisation (denoted $\mathcal{Q}$ below) of the Hydra battle provided by Buchholz, see [5].

$$\begin{aligned}
h(g(x,y),z) &\to h(d(g(x,y),z),g(z,0)) \\
d(0,z) &\to 0 \\
d(g(x,0),z) &\to x \\
d(g(x,g(y,0)),0) &\to x \\
d(g(x,g(y,0)),g(z,v)) &\to g(d(g(x,g(y,0)),z),y) \\
d(g(x,g(y,g(u,v))),z) &\to g(x,d(g(y,g(u,v)),z)) \ .
\end{aligned}$$

The crucial difference between the TRS $\mathcal{Q}$ and the above studied TRSs $\mathcal{H}$ and $\mathcal{D}$ is the fact that $\mathcal{Q}$ is confluent. Note that the TRS $\mathcal{Q}$ shows the possibility to encode the *Battle of Hercules and the Hydra* faithfully as a standard TRS without having to resort to types or specific strategies. Termination of $\mathcal{Q}$ can be easily proven by transfinite induction up-to ϵ_0. In proof, one uses an interpretation from $\mathcal{Q}$ into OT that is actually a model of $\mathcal{Q}$, compare [5]. Unfortunately this elegant proof cannot be adapted to either of the system $\mathcal{H}$ or $\mathcal{D}$.

The (relative) simplicity of the termination proof of TRS $\mathcal{Q}$ seems to indicate that (variants) of this TRS may be more accessible to automatic termination provers making use of interpretations into the ordinals. Such an extension may also be suitable to deal with Touzet's formalisation of the Hydra battle, c.f. [26].

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References

1. T. Arts and J. Giesl. Termination of term rewriting using dependency pairs. *Theor. Comput. Sci.*, 236:133–178, 2000.
2. F. Baader and T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1998.
3. L. Beklemishev. Representing worms as a term rewriting system. In *Mini-Workshop: Logic, Combinatorics and Independence Results*, pages 3093–3095. Mathematisches Forschungsinstitut Oberwolfach, Report No. 52/2006, 2006, 2006. Abstract.
4. W. Buchholz. Proof-theoretical analysis of termination proofs. *Ann. Pure Appl. Logic*, 75:57–65, 1995.
5. W. Buchholz. Another rewrite system for the standard Hydra battle. In *Mini-Workshop: Logic, Combinatorics and Independence Results*, pages 3099–3011. Mathematisches Forschungsinstitut Oberwolfach, Report No. 52/2006, 2006, 2006. Abstract.

6. W. Buchholz, E.-A. Cichon, and A. Weiermann. A uniform approach to fundamental sequences and hierarchies. *Mathematical Logic Quarterly*, 40:273–286, 1994.
7. S.-R. Buss, editor. *Handbook of Proof Theory*, volume 137. Elsevier Science, 1998.
8. E.-A. Cichon. Termination orderings and complexity characterisations. In P. Aczel, H. Simmons, and S.S. Wainer, editors, *Proof Theory*, pages 171–193, 1992.
9. N. Dershowitz. 33 examples of termination. In *French Spring School of Theoretical Computer Science Advanced Course on Term Rewriting, Font Romeux, France, May 1993*, volume 909, pages 16–26. LNCS, 1995.
10. N. Dershowitz and J.-P. Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, pages 245–319. Elsevier Science, 1990.
11. N. Dershowitz and Z. Manna. Proving termination with multiset orderings. *Communications of the ACM*, 22(8):465–476, 1979.
12. N. Dershowitz and G. Moser. The Hydra battle revisited. In *Rewriting, Computation and Proof*, volume 4600 of *LNCS*, pages 1–27. Springer Verlag, 2007. Essays Dedicated to Jean-Pierre Jouannaud on the Occasion of His 60th Birthday.
13. M. C. F. Ferreira. *Termination of Term Rewriting*. PhD thesis, University of Utrecht, November 1995 1995. Well-foundedness, Totality and Transformations.
14. D. Hofbauer. Termination proofs by multiset path orderings imply primitive recursive derivation lengths. *Theor. Comput. Sci.*, 105:129–140, 1992.
15. T. Jech. *Set Theory*. Springer Verlag, 2002.
16. L. Kirby and J. Paris. Accessible independence results for Peano arithmetic. *Bulletin London Mathematical Society*, 4:285–293, 1982.
17. I. Lepper. Derivation lengths and order types of Knuth-Bendix orders. *Theor. Comput. Sci.*, 269:433–450, 2001.
18. I. Lepper. *Simplification Orders in Term Rewriting*. PhD thesis, WWU Münster, 2002. `http://wwwmath.uni-muenster.de/logik/publ/diss/9.html`.
19. I. Lepper. Simply terminating rewrite systems with long derivations. *Arch. Math. Logic*, 43:1–18, 2004.
20. Z. Manna and S. Ness. On the termination of Markov algorithms. In *Proc. of the Third Hawaii International Conference on System Science*, pages 789–792, 1970.
21. G. Moser. Derivational complexity of Knuth Bendix orders revisited. In *Proceedings of the 13th International Conference on Logic for Programming Artificial Intelligence and Reasoning*, number 4246 in LNCS, pages 75–89. Springer Verlag, 2006.
22. G. Moser and A. Weiermann. Relating derivation lengths with the slow-growing hierarchy directly. In *Proceedings of the 14th International Conference on Rewriting Techniques and Applications*, number 2706 in LNCS, pages 296–310. Springer Verlag, 2003.
23. W. Pohlers. *Proof Theory: The First Step into Impredicativity*. Universitext. Springer Verlag, 2008.
24. G. Takeuti. *Proof Theory*. North-Holland, Amsterdam, 2nd edition, 1987.
25. TeReSe. *Term Rewriting Systems*, volume 55 of *Cambridge Tracks in Theoretical Computer Science*. Cambridge University Press, 2003.
26. H. Touzet. Encoding the Hydra battle as a rewrite system. In *Proceedings of the 23rd International Symposium on Mathematical Foundations of Computer Science*, LNCS 1450, pages 267–276. Springer Verlag, 1998.
27. A. Weiermann. Termination proofs for term rewriting systems with lexicographic path ordering imply multiply recursive derivation lengths. *Theor. Comput. Sci.*, 139: 355–362, 1995.
28. H. Zantema. Termination of term rewriting: interpretation and type elimination. *J. Symbolic Comput.*, 17(1):23–50, 1994.