# Complexity Analysis by Graph Rewriting* 

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#### Abstract

Recently, many techniques have been introduced that allow the (automated) classification of the runtime complexity of term rewrite systems (TRSs for short). In this paper we show that polynomial (innermost) runtime complexity of TRSs induces polytime computability of the functions defined. In this way we show a tight correspondence between the number of steps performed in a given rewrite system and the computational complexity of an implementation of rewriting. The result uses graph rewriting as a first step towards the implementation of term rewriting. In particular, we prove the adequacy of (innermost) graph rewriting for (innermost) term rewriting.


## 1 Introduction

We study techniques to analyse the complexity of programs automatically. Instead of studying programs of a particular programming language directly, we focus on the complexity analysis of term rewrite systems instead. The reason is that term rewriting is a very simple formalism underlying many programming languages. Recently, many techniques to automatically assess the runtime complexity of term rewrite systems (TRSs for short) have been introduced. For example in [13] we introduced the polynomial path order POP* and extensions of it. POP* is a restriction of the multiset path order 13 and whenever compatibility of a TRS $\mathcal{R}$ with $\mathrm{POP}^{*}$ can be shown then the runtime complexity of $\mathcal{R}$ is polynomially bounded. Here the runtime complexity of a TRS measures the maximal number of rewrite steps as a function in the size of the initial term.

We have successfully implemented this technique ${ }^{1}$ Thus we can automatically verify for a given $\operatorname{TRS} \mathcal{R}$ that it admits at most polynomial runtime complexity. This opens the way to automatically verify for a given (functional) program P that its runtime complexity is polynomial (in the input size). The only restrictions in the applicability of the result are that (i) the program $P$ is transformable into a term rewrite system $\mathcal{R}$ and (ii) a feasible (i.e., polynomial) runtime complexity with respect to $\mathcal{R}$ gives rise to a feasible runtime complexity of P . In short the transformation has to be non-termination and complexity preserving.

[^0]As an example of the technique consider the following rendering of insert sort as TRS $\mathcal{R}_{1}$ given in [11. Observe that for a list $x s$ build from constructors cs and nil, $\operatorname{sort}(x s, \operatorname{len}(x s))$ returns $x s$ sorted.

$$
\begin{array}{rlrl}
\text { if(true }, x, y) & \rightarrow x & 0<\mathrm{s}(x) & \rightarrow \text { true } \\
\text { if(false, } x, y) & \rightarrow y & \operatorname{ins}(0, x, y s) & \rightarrow \mathrm{cs}(x, y s) \\
\text { len }(\text { nil }) & \rightarrow 0 & \operatorname{ins}(\mathrm{~s}(n), x, \mathrm{cs}(y, y s)) & \rightarrow \operatorname{if}(x<y, \operatorname{cs}(x, \mathrm{cs}(y, y s)), \\
\text { len }(\mathrm{cs}(x, x s)) & \rightarrow \mathrm{s}(\operatorname{len}(x s)) & \operatorname{cs}(y, \operatorname{ins}(n, x, y s))) \\
x<0 & \rightarrow \text { false } & \operatorname{sort}(\text { nil }, 0) & \rightarrow \text { nil } \\
\mathrm{s}(x)<\mathrm{s}(y) & \rightarrow x<y & \operatorname{sort}(\mathrm{cs}(x, x s), \mathrm{s}(n)) & \rightarrow \operatorname{ins}(n, x, \operatorname{sort}(x s, n))
\end{array}
$$

It is not difficult to see that the innermost runtime complexity of the TRS $\mathcal{R}_{1}$ is polynomially bounded. Moreover this fact can be verified automatically by showing that $\mathcal{R}_{1} \subseteq>_{\text {pop* }}$ for a suitable instance $>_{\text {pop* }}$ of $\mathrm{POP}^{*}$, cf. [1].

Once we have established a bound on the (innermost) runtime complexity of a TRS, it is natural to direct our attention to the computational complexity of the functions defined by the TRS. In particular with respect to TRS $\mathcal{R}_{1}$ we also want to verify (automatically if possible) that insertion sort is polytime computable. Due to the restrictive definition of $>_{\text {pop* }}$ this is not difficult as the signature of $\mathcal{R}_{1}$ is simple. (See [2] for a proof.) Here a simple signature [11] essentially means that the size of any constructor term depends polynomially on its depth.

Unfortunately the restriction to a simple signature is rather restrictive. Consider the following TRS $\mathcal{R}_{2}$. (This is Example 8 in [9].)

$$
\begin{array}{ll}
\mathrm{D}(\mathrm{c}) \rightarrow 0 & \mathrm{D}(x+y) \rightarrow \mathrm{D}(x)+\mathrm{D}(y) \\
\mathrm{D}(\mathrm{t}) \rightarrow 1 & \mathrm{D}(x-y) \rightarrow \mathrm{D}(x)-\mathrm{D}(y) \\
\mathrm{D}(x \times y) \rightarrow y \times \mathrm{D}(x)+x \times \mathrm{D}(y) &
\end{array}
$$

Employing runtime complexity techniques developed in 10, we can automatically verify that the runtime complexity of $\mathcal{R}_{2}$ is polynomially bounded. However the signature of $\mathcal{R}_{2}$ is not simple and thus we cannot directly conclude polytime computability of the function computed. The main obstacle here is that due to the last rule, a single step may copy arbitrary large subterms.

In this paper we show that polynomial runtime complexity of TRSs induces polytime computability of the functions defined. The only restriction is that we consider an eager evaluation strategy, i.e., we base our study on an innermost rewriting strategy. We show the precise correspondence between the number of steps performed in a given rewrite system and the computational complexity of an implementation of rewriting.

In order to overcome the problem of copying we use graph rewriting-here copying is replaced by sharing-as a first step towards the implementation of term rewriting. We re-prove the adequacy of innermost graph rewriting for innermost term rewriting, compare for example 1213. A new proof becomes necessary as we need to control the resources needed in the simulation. This
pedantry is then used to establish the tight correspondence between the complexity of a given TRS $\mathcal{R}$ and the intrinsic computational complexity of the functions computed by $\mathcal{R}$.

The rest of the paper is organised as follows. In Section 2 we present basic notions and recall (briefly) the central concepts of graph rewriting. The adequacy theorem is provided in Section 3 and in Section 4 we show how innermost graph rewriting can be encoded efficiently. Finally we conclude in Section5, where we also relate our work to recent work. Due to space limitations, we omit some proofs in the presentation. Those are available in the technical report 4].

## 2 Preliminaries

We assume familiarity with term rewriting [513], but no familiarity with graph rewriting (see [13]) is assumed. The purpose of this section is to fix the term rewriting notions and introduce a formulation of graph rewriting that is sufficient for our purposes.

Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a finite signature. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}$. The arity of a function symbol $f$ is denoted as $\operatorname{ar}(f) . \mathcal{V} \operatorname{ar}(t)$ denotes the set of variables occurring in a term $t$. The size of a term $t$, i.e., the number of symbols appearing in $t$, is denoted as $|t|$. We write $\left.t\right|_{p}$ for the subterm of $t$ at position $p$.

A term rewrite system ( $T R S$ for short) $\mathcal{R}$ (over a signature $\mathcal{F}$ ) is a finite set of rewrite rules $l \rightarrow r$, such that $l \notin \mathcal{V}$ and $\mathcal{V} \operatorname{ar}(l) \supseteq \mathcal{V} \operatorname{ar}(r)$. The root symbols of left-hand sides of rewrite rules are called defined, while all other function symbols are called constructors. Constructor symbols are collected in $\mathcal{C} \subseteq \mathcal{F}$. The smallest rewrite relation, i.e., closed under contexts and substitutions, that contains $\mathcal{R}$ is denoted by $\rightarrow_{\mathcal{R}}$. We write $\rightarrow_{\mathcal{R}}^{*}$ for the transitive and reflexive closure of $\rightarrow_{\mathcal{R}}$. Let $s$ and $t$ be terms. If exactly $n$ steps are performed to contract $s$ to $t$ we write $s \rightarrow_{\mathcal{R}}^{n} t$. A term $s \in \mathcal{T}$ is called a normal form if there is no $t \in \mathcal{T}$ such that $s \rightarrow_{\mathcal{R}} t$. With $\operatorname{NF}(\mathcal{R})$ we denote the set of all normal forms of a term rewrite system $\mathcal{R}$. We write $s \rightarrow_{\mathcal{R}}^{!} t$ if $s \rightarrow_{\mathcal{R}}^{*} t$ and $t \in \operatorname{NF}(\mathcal{R})$. A term $t$ is called argument normalised (with respect to $\mathcal{R}$ ) if every proper subterm $\left.t\right|_{p} \in \operatorname{NF}(\mathcal{R})$.

Let $\square$ be a fresh constant. Terms over $\mathcal{F} \cup\{\square\}$ and $\mathcal{V}$ are called contexts. The empty context is denoted as $\square$. For a context $C$ with $n$ holes, we write $C\left[t_{1}, \ldots, t_{n}\right]$ for the term obtained by replacing the holes from left to right in $C$ with the terms $t_{1}, \ldots, t_{n}$. The innermost rewrite relation $\stackrel{i}{\rightarrow}_{\mathcal{R}}$ of a TRS $\mathcal{R}$ is defined on terms as follows: $s \stackrel{i}{\mathcal{T}}_{\mathcal{R}} t$ if there exists a rewrite rule $l \rightarrow r \in \mathcal{R}$, a context $C$, and a substitution $\sigma$ such that $s=C[l \sigma]$ and $t=C[r \sigma]$ for $l \sigma$ argument normalised.

A TRS is called confluent if for all $s, t_{1}, t_{2} \in \mathcal{T}$ with $s \rightarrow_{\mathcal{R}}^{*} t_{1}$ and $s \rightarrow_{\mathcal{R}}^{*} t_{2}$ there exists a term $t_{3}$ such that $t_{1} \rightarrow_{\mathcal{R}}^{*} t_{3}$ and $t_{2} \rightarrow_{\mathcal{R}}^{*} t_{3}$.

In the sequel we introduce the central concepts of term graph rewriting or graph rewriting for short. We concentrate on standard approaches in the context of rewriting, compare also [12]13].

A graph $G=\left(\mathrm{V}_{G}, \operatorname{succ}_{G}, \mathrm{~L}_{G}\right)$ over the set $\mathcal{L}$ is a structure such that $\mathrm{V}_{G}$ is a finite set, the nodes or vertexes, $\operatorname{succ}_{G}: \mathrm{V}_{G} \rightarrow \mathrm{~V}_{G}^{*}$ is a mapping that associates a node $n$ with an (ordered) sequence of nodes, called the successors of $n$. Finally $\mathrm{L}_{G}: \mathrm{V}_{G} \rightarrow \mathcal{L}$ is a mapping that associates each node $n$ with its label $\mathrm{L}_{G}(n)$. Often we drop the reference to the graph $G$ from $\mathrm{V}_{G}$, $\operatorname{succ}_{G}$, and $\mathrm{L}_{G}$, i.e., we write $G=(\mathrm{V}$, succ, L$)$. If possible, we will omit references to the constituents of the structure $G$; in particular we often write $n \in G$ instead of $n \in \mathrm{~V}$. Note that the sequence of successors of $n$ may be empty: $\operatorname{succ}(n)=[]$. Typically the set of labels $\mathcal{L}$ is clear from context and not explicitly mentioned.

Definition 1. Let $G=(\mathrm{V}$, succ, L$)$ be a graph and let $n \in \mathrm{~V}$. Consider $\operatorname{succ}(n)=$ $\left[n_{1}, \ldots, n_{k}\right]$. We call $n_{i}$ the $i$-th successor of $n$ (denoted as $n \stackrel{i}{\rightharpoonup} n_{i}$ ). If there exists $i$ such that $n \stackrel{i}{\rightharpoonup} m$, then we simply write $n \rightharpoonup m$. A node $m$ is called reachable from $n$ if $n \xrightarrow{*} m$, where $\xrightarrow{*}$ denotes the reflexive and transitive closure of $\rightarrow$. We write $\stackrel{\text { 土 }}{\rightharpoonup}$ for $\rightharpoonup \cdot \stackrel{*}{\rightharpoonup}$.

A graph $G$ is acyclic if $n \stackrel{\rightharpoonup}{\rightharpoonup} m$ implies $n \neq m$ and $G$ is rooted if there exists a unique node $n$ such that every other node in $G$ is reachable from $n$. The node $n$ is called the root $\operatorname{rt}(G)$ of $G$. The size of $G$, i.e. the number of nodes, is denoted as $|G|$. We write $G\lceil n$ for the subgraph of $G$ reachable from $n$.
Definition 2. $A$ term graph (with respect to $\mathcal{F}$ and $\mathcal{V}$ ) is an acyclic and rooted graph $G=(\mathrm{V}$, succ, $\mathbf{L})$ over $\mathcal{F} \cup \mathcal{V}$. Let $n \in \mathbf{V}$ and suppose $\mathrm{L}(n)=f \in \mathcal{F}$ such that $f$ is $k$-ary $(k \geqslant 0)$. Then $\operatorname{succ}(n)=\left[n_{1}, \ldots, n_{k}\right]$. On the other hand if $\mathrm{L}(n) \in \mathcal{V}$, then $\operatorname{succ}(n)=[]$. We demand that any variable node is shared, i.e., for $n \in \mathrm{~V}$ with $\mathrm{L}(n) \in \mathcal{V}$, if $\mathrm{L}(n)=\mathrm{L}(m)$ for $m \in \mathrm{~V}$ then $n=m$.

We set $\mathcal{V} \operatorname{ar}(G):=\{n \mid n \in G, \mathrm{~L}(n) \in \mathcal{V}\}$ to denote the set of variable nodes in $G$. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$. We keep $\mathcal{R}$ and $\mathcal{F}$ fixed for the remainder of this paper. We write $\mathcal{G}$ raph for the set of all term graphs with respect to $\mathcal{F}$ and $\mathcal{V}$.

Example 3. Consider the graph $G=(\{1,2\}$, succ, L) where succ $(1)=[2,2]$ and $\mathrm{L}(1)=\mathrm{f} \in \mathcal{F}$ and $\mathrm{L}(2)=x \in \mathcal{V}$. Then $G$ is a term graph. Intuitively $G$ represents the term $\mathrm{f}(x, x)$ such that the variable $x$ is shared.

We define the term $t:=\operatorname{term}(G)$ represented by $G$ as follows:

$$
t:= \begin{cases}x & \text { if } \mathrm{L}(\operatorname{rt}(G))=x \in \mathcal{V} \\ f\left(t_{1}, \ldots, t_{k}\right) & \text { if } \mathrm{L}(\operatorname{rt}(G))=f \in \mathcal{F} \text { and } \operatorname{succ}(\operatorname{rt}(G))=\left[n_{1}, \ldots, n_{k}\right] .\end{cases}
$$

Here $t_{i}:=\operatorname{term}\left(G \mid n_{i}\right)$ for $i=1, \ldots, k$. We write Term $(G)$ for the set of subterms of term $(G)$.

Let $G \in \mathcal{G}$ raph. A position in $G$ is a finite sequence of positive integers. The position of $\operatorname{rt}(G)$ is the empty sequence (denoted as $\varepsilon$ ). For positions $p$ and $q$ we write $p q$ for their concatenation. The set of positions $\mathcal{P o s}_{G}(n)$ of $n \in G$ is defined as $\mathcal{P}$ os $_{G}(n):=\{\varepsilon\}$ if $n=\operatorname{rt}(G)$ and $\mathcal{P o s}_{G}(n):=\left\{i_{1} \ldots i_{k} \mid \operatorname{rt}(G) \xrightarrow{i_{1}} \ldots \stackrel{i_{k}}{ } n\right\}$.

Note that for any node $n: \mathcal{P o s}_{G}(n) \neq \varnothing$. It is easy to see that for $p \in \mathcal{P}_{o s}(n)$, $\operatorname{term}(G \upharpoonright n)=\left.\operatorname{term}(G)\right|_{p}$.

We say that $n \in G$ is shared, if $n$ represents more than one subterm of term $(G)$. Note that $n$ is shared if the set of positions $\mathcal{P}_{o s}(n)$ of $n$ is not a singleton. If $\mathcal{P o s}_{G}(n)$ is a singleton, then $n$ is unshared.

Definition 4. A node $n$ is minimally shared if it is a variable node or unshared. We say $n$ is maximally shared if for all $m \in G$ such that $\operatorname{term}(G \upharpoonright m)=\operatorname{term}(G\lceil n)$ implies $n=m$. A term graph $G$ is normal form sharing (with respect to $\mathcal{R}$ ) if for all nodes $n \in G$, if $\operatorname{term}(G \upharpoonright n) \in \operatorname{NF}(\mathcal{R})$ then $n$ is maximally shared and minimally shared otherwise. Let $t$ be a term. We write $\triangle(t)(\diamond(t))$ for the set of minimally (normal form) sharing term graphs that represent $t$.

Example 5. Reconsider the TRS $\mathcal{R}_{2}$ from the introduction. Let $t=\mathrm{D}(x+x) \times$ $\mathrm{D}(x+x)$, represented by the term graphs $T_{\min }, T_{\mathrm{nf}}$ and $T_{\max }$ depicted as follows:


| $\times$ |  |
| :---: | :---: |
| ()$^{2}$ |  |
| D |  |
| $\vdots$ |  |
| + |  |
| $\left({ }_{x}\right)$ | $T_{\max }$ |

Then $T_{\min } \in \triangle(t)$ as only variable nodes are shared. Further $T_{\mathrm{nf}} \in \diamond(t)$ since the subterms $\{x+x, x\}=\operatorname{Term}\left(T_{\mathrm{nf}}\right) \cap \operatorname{NF}(\mathcal{R})$ are represented by unique nodes whereas all other nodes are unshared. $T_{\max }$ contains only maximally shared nodes.

Let $G$ and $H$ be two graphs, possibly sharing nodes. We write $G \cup H$ for their union. We say that $G$ and $H$ are properly sharing if $n \in G \cap H$ implies $\mathrm{L}_{G}(n)=\mathrm{L}_{H}(n)$ and $\operatorname{succ}_{G}(n)=\operatorname{succ}_{H}(n)$. For two properly sharing graphs $G$ and $H, G \cup H$ is again a graph (but possibly not a term graph).

Definition 6. Let $G$ be a term graph and $H$ a rooted graph over $\mathcal{F} \cup \mathcal{V}$. We denote by $G[H]_{n}$ the graph obtained by replacing the subgraph at node $n$ in $G$ by $H: G[H]_{n}:=(G[n \leftarrow \mathrm{rt}(H)] \cup H) \upharpoonright m$ where $m:=\operatorname{rt}(H)$ if $n=\mathrm{rt}(G)$ and $m:=\operatorname{rt}(G)$ otherwise. Here $G[n \leftarrow m]$ denotes the redirection of $n$ to $m$ in $G$ : Set $\mathrm{V}_{G^{\prime}}:=\left(\mathrm{V}_{G} \cup\{m\}\right) \backslash\{n\}$ and for all $p \in \mathrm{~V}_{G^{\prime}}$, $\operatorname{succ}_{G^{\prime}}(p):=r^{*}\left(\operatorname{succ}_{G}(p)\right)$. Here $r$ replaces $n$ by $m: r(n)=m$ and $r(p)=p$ for all $p \in \mathrm{~V}_{G^{\prime}}$. Further $r^{*}$ denotes the standard extension of $r$ to sequences. Finally, set $G[n \leftarrow m]:=G^{\prime}=$ $\left(\mathrm{V}_{G^{\prime}}, \operatorname{succ}_{G^{\prime}}, \mathrm{L}_{G}\right)$.

Observe that for properly sharing term graphs $G$ and $H$ such that $n \notin$ $H$ and $H$ acyclic, $G[H]_{n}$ is again a term graph. Let $G$ and $H$ be two term graphs. A morphism (denoted $m: G \rightarrow H$ ) is a function $m: \mathrm{V}_{G} \rightarrow \mathrm{~V}_{H}$ such that $m(\operatorname{rt}(G))=\operatorname{rt}(H)$, and for all $n$ with $\mathrm{L}_{G}(n) \in \mathcal{F}, \mathrm{L}_{G}(n)=\mathrm{L}_{H}(m(n))$ and $m^{*}\left(\operatorname{succ}_{G}(n)\right)=\operatorname{succ}_{H}(m(n))$. The next lemma follows directly from the definition.

Lemma 7. Suppose $m: G \rightarrow H$ and $m^{\prime}: G \rightarrow H$.

1) For any $n \in G$ we have $m: G \upharpoonright n \rightarrow H\lceil m(n)$.
2) For any $n \in G$ we have $m(n)=m^{\prime}(n)$.

We write $G \geqslant_{m} H$ (or $G \geqslant H$ for short) if there exists a surjective morphism $m: G \rightarrow H$ that preservers labels from $\mathcal{V}$. In this case we have for all $n \in \mathrm{~V}_{G}$ : $\mathrm{L}_{G}(n)=\mathrm{L}_{H}(m(n))$. When the graph morphism $m$ is non-injective we write $G>_{m} H$ (or $G>H$ for short). If $m$ is injective and surjective then $m$ is an isomorphism. Conclusively $G$ and $H$ are called isomorphic (denoted as $G \cong H$ ).

The next lemma follows by a simple inductive argument.
Lemma 8. For all term graph $G$ and $H, G \geqslant_{m} H$ implies term $(G)=\operatorname{term}(H)$.
Definition 9. Let $L, R$ be two properly sharing term graphs. Suppose $\operatorname{V} \operatorname{ar}(R) \subseteq$ $\mathcal{V} \operatorname{ar}(L), \mathrm{L}(\operatorname{rt}(L)) \in \mathcal{F}$ and $\operatorname{rt}(L) \notin R$. Then the graph $L \cup R$ is called a graph rewrite rule (rule for short). A rule is denoted as $L \rightarrow R$, where $L, R$ denotes the left-hand, right-hand side of $L \rightarrow R$, respectively. $A$ graph rewrite system (GRS) $\mathcal{G}$ is a set of graph rewrite rules.

Note that for a rule $L \rightarrow R$, the graphs $L$ and $R$ share at least all variable nodes. A rule $L^{\prime} \rightarrow R^{\prime}$ is called a renaming of $L \rightarrow R$ with respect to $S$ if $L^{\prime} \rightarrow R^{\prime} \cong L \rightarrow R$ and $\bigvee_{S} \cap \mathrm{~V}_{L^{\prime} \rightarrow R^{\prime}}=\varnothing$. Let $\mathcal{G}$ be a GRS, let $S \in \mathcal{G}$ raph and let $L \rightarrow R$ be a rule. A redex of $S$ with $L \rightarrow R$ is a node $u \in S$ such that there exists a renaming $L^{\prime} \rightarrow R^{\prime}$ of $(L \rightarrow R) \in \mathcal{G}$ with respect to $S$ such that $m: L^{\prime} \rightarrow S \upharpoonright u$ is a morphism and $T=S\left[m\left(R^{\prime}\right)\right]_{u}$. Here $m\left(R^{\prime}\right)$ denotes the structure obtained by replacing in $R^{\prime}$ every node $v \in L^{\prime} \cap R^{\prime}$ by $m(v) \in S$, where the labels of $m(v) \in m\left(R^{\prime}\right)$ are the labels of $m(v) \in S$. From $m: L^{\prime} \rightarrow S\lceil v$ we obtain that $T$ is a term graph.

Definition 10. We say $S$ rewrites to $T$ if there exists a rule $(L \rightarrow R) \in \mathcal{G}$ such that $n \in S$ is a redex with this rule. This is denoted as $S \Rightarrow_{\mathcal{G}, n, L \rightarrow R} T$. We set $S \Rightarrow_{\mathcal{G}} T$ if $S \Rightarrow_{\mathcal{G}, n, L \rightarrow R} T$ for some $n \in S$ and $(L \rightarrow R) \in \mathcal{G}$. The relation $\Rightarrow_{\mathcal{G}}$ is called the graph rewrite relation induced by $\mathcal{G}$.

We denote the set of normal forms with respect to $\Rightarrow_{\mathcal{G}}$ as $\operatorname{NF}(\mathcal{G})$. The innermost graph rewrite relation $\stackrel{i}{\mathcal{G}}_{\mathcal{G}}$ is the restriction of $\Rightarrow_{\mathcal{G}}$ where arguments need to be in normal form, i.e. $S \stackrel{i}{\mathcal{G}}_{\mathcal{G}, n, L \rightarrow R} T$ if $S \Rightarrow_{\mathcal{G}, n, L \rightarrow R} T$ and for all $m \in \operatorname{succ}_{S}(n), S \mid m \in \operatorname{NF}(\mathcal{G})$.

## 3 Adequacy of Graph Rewriting for Term Rewriting

In this section we show that graph rewriting is adequate for term rewriting if we restrict our attention to innermost (graph) rewriting. This holds without further restrictions on the studied $\operatorname{TRS} \mathcal{R}$. (However, we assume that $\mathcal{R}$ is finite.) The here presented adequacy theorem (see Theorem 19) is not essentially new. Related results can be found in the extensive literature, see for example [13]. In
particular in [12] the adequacy theorem is even generalised to full rewriting. As this approach involves copying, it is currently not clear whether it applies in our context. Still our treatment of innermost rewriting for unrestricted TRSs is new. (See 8 for strongly related work on orthogonal constructor TRSs.) Furthermore the detailed analysis given in this section is a necessary foundation for the precise characterisation of the implementation of graph rewriting, presented in Section 4.

Definition 11. The simulating graph rewrite system $\mathcal{G}(\mathcal{R})$ of $\mathcal{R}$ contains for each rule $l \rightarrow r \in \mathcal{R}$ some rule $L \rightarrow R$ such that $L \in \triangle(l)$ and $R \in \triangle(r)$ have only variable-nodes in common, i.e. $\mathrm{V}_{L} \cap \mathrm{~V}_{R}=\mathcal{V} \operatorname{ar}(R)$.

Below $L, R, S$ and $T$ denote term graphs. Suppose $m: L \rightarrow S$ is a morphism. Then $m$ induces a substitution $\sigma_{m}: \mathcal{V} \rightarrow \operatorname{Term}(S)$ : For any $n \in S$ such that $\mathrm{L}(n)=x \in \mathcal{V}$ we set $\sigma_{m}(x):=\operatorname{term}(S \backslash m(n))$.

Lemma 12. Let $L$ and $S$ be term graphs. Suppose $m: L \rightarrow S$ for some morphism $m$. Then for each node $n \in L$, $\operatorname{term}(L \backslash n) \sigma_{m}=\operatorname{term}(S \backslash m(n))$. In particular, term $(L) \sigma_{m}=\operatorname{term}(S)$.

Proof. We prove the lemma by induction on $l:=\operatorname{term}(L \mid n)$. We write $\sigma$ instead of $\sigma_{m}$. If $l \in \mathcal{V}$, then by definition $L \upharpoonright n$ consists of a single (variable-)node and $l \sigma=\operatorname{term}(S \upharpoonright m(n))$ follows by the definition of the induced substitution $\sigma$. If $l=f\left(l_{1}, \ldots, l_{k}\right)$, then we have $\operatorname{succ}_{L}(n)=\left[n_{1}, \ldots, n_{k}\right]$ for some $n_{1}, \ldots, n_{k} \in L$. As $m: L \rightarrow S$ holds, Lemma 7 yields $m: L\left\lceil n_{i} \rightarrow S\left\lceil m\left(n_{i}\right)\right.\right.$ for all $i=1, \ldots, k$. And induction hypothesis becomes applicable so that $l_{i} \sigma=\operatorname{term}\left(S\left\lceil m\left(n_{i}\right)\right)\right.$. Thus

$$
l \sigma=f\left(l_{1} \sigma, \ldots, l_{k} \sigma\right)=f\left(\operatorname { t e r m } \left(S\left\lceil m\left(n_{1}\right)\right), \ldots, \operatorname{term}\left(S\left\lceil m\left(n_{k}\right)\right)\right)\right.\right.
$$

By definition of $m, \mathrm{~L}_{S}(m(n))=\mathrm{L}_{L}(n)=f$ and $\operatorname{succ}_{S}(m(n))=m^{*}\left(\operatorname{succ}_{L}(n)\right)=$ $\left[m\left(n_{1}\right), \ldots, m\left(n_{k}\right)\right]$. Hence $f\left(\operatorname{term}\left(S \upharpoonright m\left(n_{1}\right)\right), \ldots, \operatorname{term}\left(S \upharpoonright m\left(n_{k}\right)\right)\right)=\operatorname{term}(S \upharpoonright$ $m(n))$.

Finally, in order to conclude term $(L) \sigma=\operatorname{term}(S)$, it suffices to observe that by definition of $m$ : $m(\operatorname{rt}(L))=\operatorname{rt}(S)$ holds.

Lemma 13. Let $L \rightarrow R$ be a graph rewrite rule and let $S$ be a term graph such that $S \cap R=\varnothing$ and $m: L \rightarrow S$ for some morphism $m$. Let $m^{\prime}$ denote the standard extension of $m$ to all nodes in $R$, i.e., $m^{\prime}(n):=m(n)$ if $n$ is in the domain of $m$ and $m^{\prime}(n):=n$ otherwise. Set $T:=\left(m^{\prime}(R) \cup S\right) \upharpoonright \operatorname{rt}\left(m^{\prime}(R)\right)$. Then for each $n \in R$, term $\left(R\lceil n) \sigma_{m}=\operatorname{term}\left(T\left\lceil m^{\prime}(n)\right)\right.\right.$. In particular, term $(R) \sigma_{m}=\operatorname{term}(T)$.

Proof. We write $\sigma$ instead of $\sigma_{m}$. Suppose $n \in R \cap L$. Then $R \upharpoonright n=L \upharpoonright n$ as $L$, $R$ are properly shared. By definition of the morphism $m: m(n) \in S$. Thus by definition of $T$ we have term $\left(T \upharpoonright m^{\prime}(n)\right)=\operatorname{term}(S \upharpoonright m(n))$. Moreover, employing Lemma 12, we have $\operatorname{term}(R \upharpoonright n) \sigma=\operatorname{term}(L \upharpoonright n) \sigma=\operatorname{term}(S \upharpoonright m(n))$. From this the assertion follows.

Thus suppose $n \in R \backslash L$. This subcase we prove by induction on $r=$ $\operatorname{term}(R \upharpoonright n)$. The base case $r \in \mathcal{V}$ follows as variables are shared in $L \rightarrow R$. For the inductive step, let $r=f\left(r_{1}, \ldots, r_{k}\right)$ with $\operatorname{succ}_{R}(n)=\left[n_{1}, \ldots, n_{k}\right]$. The
induction hypothesis yields $r_{i} \sigma=\operatorname{term}\left(T \upharpoonright m^{\prime}\left(n_{i}\right)\right)$ for $i=1, \ldots, k$. By definition of $m^{\prime}: m^{\prime}(n)=n \in m^{\prime}(R) \subseteq T$. Hence $\operatorname{succ}_{T}\left(m^{\prime}(n)\right)=\operatorname{succ}_{m^{\prime}(R)}(n)=$ $m^{\prime *}\left(\operatorname{succ}_{R}(n)\right)=\left[m^{\prime}\left(n_{1}\right), \ldots, m^{\prime}\left(n_{k}\right)\right]$. Moreover $\mathrm{L}_{T}(n)=\mathrm{L}_{m^{\prime}(R)}(n)=f$ by definition. We conclude $r \sigma=f\left(r_{1} \sigma, \ldots, r_{k} \sigma\right)=f\left(\operatorname{term}\left(T \upharpoonright m^{\prime}\left(n_{1}\right)\right), \ldots\right.$, term $(T \upharpoonright$ $\left.\left.m^{\prime}\left(n_{k}\right)\right)\right)=\operatorname{term}\left(T \upharpoonright m^{\prime}(n)\right)$. This concludes the inductive argument.

Finally, in order to conclude term $(R) \sigma_{m}=\operatorname{term}(T)$ observe that $m(\operatorname{rt}(T))=$ $\mathrm{rt}(R)$ holds.

Below we also write $\square$ for the unique (up-to isomorphism) graph representing the constant $\square$.

Lemma 14. Let $S$ and $T$ be two properly sharing term graphs, let $n \in S \backslash T$. Then $\operatorname{term}\left(S[T]_{n}\right)=C[\operatorname{term}(T), \ldots, \operatorname{term}(T)]$ where $C=\operatorname{term}\left(S[\square]_{n}\right)$ and the number of occurrences of the term term $(T)$ equals $\left|\mathcal{P o s}_{S}(n)\right|$.

Proof. We proceed by induction on the size of $S$. In the base case $S$ consists of a single node $n$. Hence the context $C$ is empty and the lemma follows trivially. For the induction step we can assume without loss of generality that $n \neq \operatorname{rt}(S)$.

We assume $\mathrm{L}_{S}(\operatorname{rt}(S))=f \in \mathcal{F}$ and $\operatorname{succ}_{S}(\operatorname{rt}(S))=\left[m_{1}, \ldots, m_{k}\right]$. For all $i$ $(1 \leqslant i \leqslant k)$ such that $m_{i}=n$ set $C_{i}=\square$ and for all $i$ such that $m_{i} \neq n$ but $\left(S[T]_{n}\right) \upharpoonright m_{i}=\left(S \backslash m_{i}\right)[T]_{n}$ we set $C_{i}=\operatorname{term}\left(\left(S \backslash m_{i}\right)[\square]_{n}\right)$. In the latter sub-case induction hypothesis is applicable to conclude

$$
\operatorname{term}\left(\left(S[T]_{n}\right)\left\lceil m_{i}\right)=C_{i}[\operatorname{term}(T), \ldots, \operatorname{term}(T)]\right.
$$

Finally we set $C:=f\left(C_{1}, \ldots, C_{k}\right)$ and obtain $C=\operatorname{term}\left(S[\square]_{n}\right)$. In sum we have $\operatorname{term}\left(S[T]_{n}\right)=f\left(\left(S[T]_{n}\right) \upharpoonright m_{1}, \ldots,\left(S[T]_{n}\right) \upharpoonright m_{k}\right)=C[\operatorname{term}(T)]$, where $\overline{\operatorname{term}(T)}$ denotes the sequences of terms term $(T)$ of the required length $\left|\mathcal{P} \operatorname{os}_{S}(n)\right|$.

Lemma 15. Let $\mathcal{R}$ be a TRS, $l$ be a term and let $\sigma: \mathcal{V} \rightarrow \operatorname{NF}(\mathcal{R})$ be a substitution such that $s=l \sigma$ is argument normalised with respect to $\mathcal{R}$. If $L \in \triangle(l)$ and $S \in \diamond(s)$, then there exists a morphism $m: L \rightarrow S$ such that for all variable nodes $n \in \mathcal{V} \operatorname{ar}(L)$ with $\mathrm{L}_{L}(n)=x, x \sigma=\operatorname{term}(S \mid m(n))$.

Proof. We prove the lemma by induction on $l$. It suffices to consider the induction step. Let $l=f\left(l_{1}, \ldots, l_{k}\right)$ and $s=f\left(l_{1} \sigma, \ldots, l_{k} \sigma\right)$. Let $\operatorname{succ}_{L}(\operatorname{rt}(L))=$ $\left[p_{1}, \ldots, p_{k}\right]$, let $\operatorname{succ}_{S}(\operatorname{rt}(S))=\left[q_{1}, \ldots, q_{k}\right]$, and let $i \in\{1, \ldots, k\}$. By induction hypothesis there exist morphisms $m_{i}: L\left\lceil p_{i} \rightarrow S \upharpoonright q_{i}\right.$ of the required form. Either $m_{i}(n)=m_{j}(n)$ or $n \notin\left(\operatorname{dom}\left(m_{i}\right) \cap \operatorname{dom}\left(m_{j}\right)\right)$.

Otherwise suppose $n \in\left(\operatorname{dom}\left(m_{i}\right) \cap \operatorname{dom}\left(m_{j}\right)\right)$. We show $m_{i}(n)=m_{j}(n)$. Since $L \in \triangle(l)$, only variable nodes are shared, hence $n$ needs to be a variable node. Suppose $\mathrm{L}_{L}(n)=x$. Then $\operatorname{term}\left(S \upharpoonright m_{i}(n)\right)=x \sigma=\operatorname{term}\left(S \upharpoonright m_{j}(n)\right)$. As $S \in \diamond(s)$ and $x \sigma \in \operatorname{NF}(\mathcal{R}), m_{i}(n)=m_{j}(n)$ has to hold.

Define a function $m: \mathrm{V}_{L} \rightarrow \mathrm{~V}_{S}$ as follows. Set $m(\mathrm{rt}(L))=\mathrm{rt}(S)$ and for $p \neq \operatorname{rt}(L)$ define $m(p)=m_{i}(p)$ if $p \in \operatorname{dom}\left(m_{i}\right)$. By the above observation, $m$ is well-defined and a morphism.

Lemma 16. Let $t$ be a term and let $T$ be a normal form sharing term graph such that $t=\operatorname{term}(T)$. Then $t \in \operatorname{NF}(\mathcal{R})$ if and only if $T \in \operatorname{NF}(\mathcal{G}(\mathcal{R}))$.

Let $T$ be a term graph; a node $n \in T$ is an $\mathcal{R}$-redex if $\operatorname{term}(T \upharpoonright n) \notin \operatorname{NF}(\mathcal{R})$. We call $T$ redex-unsharing if none of its $\mathcal{R}$-redexes are shared.
Lemma 17. Suppose $T$ is redex-unsharing. Then $T$ is normal form sharing if and only if it is not possible to find distinct nodes $p, q \in T$ such that (i) term $(T \uparrow$ $p) \in \operatorname{NF}(\mathcal{R})$, (ii) $\mathrm{L}(p)=\mathrm{L}(q)$ and (iii) $\operatorname{succ}(p)=\operatorname{succ}(q)$.

Proof. It suffices to consider the direction from right to left as the other is trivial. Thus suppose $T$ is not normal form shared. We show that there exist nodes $p, q$ fulfilling the properties stated.

We pick some node $p \in T$ with term $(T\lceil p) \in \operatorname{NF}(\mathcal{R})$ and there exists a distinct node $q \in T$ with term $(T \upharpoonright p)=\operatorname{term}(T \upharpoonright q)$. For that we assume that $p$ is - -minimal in the sense that there is no node $p^{\prime}$ with $p \stackrel{ \pm}{\rightharpoonup} p^{\prime}$ such that $p^{\prime}$ would fulfil the above properties. The node $p$ exists as $T$ is not normal form shared. By definition property (i) holds and property (ii) follows from term $(T\lceil p)=\operatorname{term}(T \upharpoonright q)$. To show property (iii) we assume $\operatorname{succ}(p)=\left[p_{1}, \ldots, p_{l}\right]$ and $\operatorname{succ}(q)=\left[q_{1}, \ldots, q_{l}\right]$ where for at least for one $i \in\{1, \ldots, l\}, p_{i} \neq q_{i}$. But this contradicts the minimality of $p$. We conclude $\operatorname{succ}(p)=\operatorname{succ}(q)$ as desired.

The next lemma follows from the above using a simple inductive argument.
Lemma 18. Let $S$ be redex-unsharing. Then for all $T \in \diamond(\operatorname{term}(S)), S \geqslant T$.
Theorem 19 (Adequacy). Let $\mathcal{R}$ be a TRS and let $\mathcal{G}(\mathcal{R})$ denote the simulating GRS. Suppose $s$ is a term and $S \in \diamond(s)$. Then $s \stackrel{i}{\rightarrow}_{\mathcal{R}} t$ if and only if $S \stackrel{i}{G}_{\mathcal{G}(\mathcal{R})}$ $\cdot \geqslant T$, where $T \in \diamond(t)$.

Proof. First we consider the direction from right to left. Suppose $S{\underset{\mathcal{G}}{ }{ }_{\mathcal{R}), p} T}$ such that $(L \rightarrow R) \in \mathcal{G}(\mathcal{R}), p \in S, m: L \rightarrow S\left\lceil p\right.$ and $T=S[m(R)]_{p}$. Moreover, $S \upharpoonright n \in \operatorname{NF}(\mathcal{G})$ for all $n \in \operatorname{succ}(p)$. Define the context $C=\operatorname{term}\left(S[\square]_{p}\right)$, we write $\sigma$ for the induced substitution $\sigma_{m}$ and prove

$$
\operatorname{term}(S)=C[\overline{\operatorname{term}(L) \sigma}] \stackrel{i}{\mathcal{T}}_{\mathcal{R}} C[\overline{\operatorname{term}(R) \sigma}]=\operatorname{term}(T) .
$$

Observe that $S=S[S \backslash p]_{p}$. By Lemma 12 , term $(S \upharpoonright p)=\operatorname{term}(L) \sigma$. Due to Lemma 14 we see

$$
\operatorname{term}(S)=\operatorname{term}\left(S[S \backslash p]_{p}\right)=C[\overline{\operatorname{term}(S \backslash p)}]=C[\overline{\operatorname{term}(L) \sigma}]
$$

By definition $T=S[m(R)]_{p}=S[(m(R) \cup S) \mid r t(m(R))]_{p}$. Due to Lemma 13 we have $\operatorname{term}(T)=C[\overline{\operatorname{term}(R) \sigma}]$. In order to show $\operatorname{term}(S) \stackrel{i}{\mathcal{R}}_{\mathcal{R}}$ term $(T)$ it suffices to verify that term $(L) \sigma=\operatorname{term}(S \backslash p)$ is argument normalised with respect to $\mathcal{R}$. Note that $l \sigma \notin \mathrm{NF}(\mathcal{R})$ and $S$ is a normal form sharing graph. Hence the context $C$ contains only one hole. Let $u$ be a proper subterm of term $(S \upharpoonright p)$. As for all $n \in \operatorname{succ}_{S}(p), S\lceil n \in \operatorname{NF}(\mathcal{G})$, and $S\lceil p$ is argument normalised, $u \in \operatorname{NF}(\mathcal{R})$ follows from Lemma 16. Hence term $(l) \sigma$ is argument normalised.

Finally, we prove the direction from left to right. Suppose $s=C[l \sigma] \stackrel{i}{\rightarrow}_{\mathcal{R}}$ $C[r \sigma]=t$ with $l \rightarrow r \in \mathcal{R}$ and $l \sigma$ argument normalised. Let $p \in S$ be the
node corresponding to $l \sigma$. As $l \sigma$ is not a normal form, $p$ is unique. Clearly $S \upharpoonright p \in \diamond(l \sigma)$. Let $(L \rightarrow R) \in \mathcal{G}(\mathcal{R})$. According to Lemma 15 , there exists a morphism $m: L \rightarrow S \backslash p$ such that $x \sigma=\operatorname{term}\left(S\lceil m(n))\right.$ where $L_{S}(n)=x \in \mathcal{V}$. As


It remains to prove that $T^{\prime}:=S[m(R)]_{p} \geqslant T \in \diamond(t)$. We claim $\operatorname{term}\left(T^{\prime}\right)=$ $C[r \sigma]=t$. For this, let $U:=(m(R) \cup S) \mid r t(m(R))$ and observe $T^{\prime}=S[m(R)]_{p}=$ $S[U]_{p}$. Due to Lemma 13 term $(U)=r \sigma$. Moreover, it is easy to see that $C=$ $\operatorname{term}\left(S[\square]_{p}\right)$. Hence by Lemma 14 , term $\left(S[U]_{p}\right)=C[\operatorname{term}(U)]$ and the claim follows. Let $T \in \diamond(t)$. We want to apply Lemma 18 to conclude $T^{\prime} \geqslant T$. For that we have to prove that $T^{\prime}$ is redex-unshared. Suppose there exists $q \in T^{\prime}$ such that $u:=\operatorname{term}\left(T^{\prime}\lceil q)\right.$ is reducible with respect to $\mathcal{R}$. We show that $\mathcal{P o s}_{T^{\prime}}(q)$ is a singleton set.

We distinguish two cases: Either $q$ is reachable from $\operatorname{rt}(m(R))$ in $T^{\prime}$ or it is not reachable from $\operatorname{rt}(m(R))$. If $q$ is reachable, then we can even conclude that $q \in m(R)$. For this, suppose there exists a variable node $n \in R$ such that $\mathrm{rt}(m(R)) \stackrel{\rightharpoonup}{\rightharpoonup} m(n) \stackrel{*}{\rightharpoonup} q$. Then term $\left(T^{\prime} \upharpoonright m(n)\right)=x \sigma$ for some variable $x \in \operatorname{V} \operatorname{ar}(r)$. But $x \sigma \in \mathrm{NF}(\mathcal{R})$ as $l \sigma$ is argument normalised. This contradicts the assumption that term $\left(T^{\prime} \upharpoonright q\right)$ is reducible. Hence suppose $q \in m(R)$. By Definition 9 in $L, R$ only variables are shared and in $S$ only normal forms are shared. Hence $q$ can only be shared in $m(R)$ if it represents a normal form, contrary to our assumption. Hence it is unshared. Finally consider the case that $q$ is not reachable from $\operatorname{rt}(m(R))$ in $T^{\prime}$. This implies that $q \in S$ and thus $q$ is unshared in $T^{\prime}$ if $q$ is unshared in $S$. Suppose $u=\operatorname{term}(S \upharpoonright q)$, i.e., the term represented by node $q$ is unchanged by the (graph) rewrite step. Then $q$ is unshared as by assumption $u$ is reducible and $S \in \diamond(s)$. Otherwise if $u \neq \operatorname{term}(S \backslash q)$, then $q \xrightarrow{*} p$ for the redex $p \in S$. Hence term $(S \upharpoonright q)$ is reducible and again not shared in $S$.

Sometimes it is convenient to combine graph rewrite and collapse steps into
 notion we can rephrase the conclusion of the theorem as: $s{ }^{\mathrm{i}}{ }_{\mathcal{R}} t$ if and only if $S \stackrel{\Rightarrow}{\mathcal{G}} T$, whenever the conditions of the theorem are fulfilled.

## 4 Complexity Considerations

We now prove a polynomial relationship between the number of rewrite steps admitted by $\mathcal{R}$ and the computational complexity of the functions defined. We give semantics to $\mathcal{R}$ in the most natural way. Let $\mathcal{V}$ al, the set of values, denote the set of terms generated from constructors $\mathcal{C}$ and variables $\mathcal{V}$. Further, suppose $\mathcal{R}$ is a confluent and terminating TRS. An $n$-ary partial function $f: \mathcal{V} \mathrm{al}^{n} \rightarrow \mathcal{V}$ al is computable by $\mathcal{R}$ if there exists a defined function symbol $f \in \mathcal{F}$ such that for all $s_{1}, \ldots, s_{n}, t \in \mathcal{V}$ al:

$$
\mathrm{f}\left(s_{1}, \ldots, s_{n}\right) \rightarrow{ }_{\mathcal{R}} t \Longleftrightarrow f\left(s_{1}, \ldots, s_{n}\right)=t
$$

where $f$ is defined. Note that this is well defined as $\mathcal{R}$ is confluent and terminating. We say that $\mathcal{R}$ computes $f$, if the function $f: \mathcal{V}$ al ${ }^{n} \rightarrow \mathcal{V}$ al is defined by
the above equation. We define the derivation length of a term $s$ with respect to a terminating TRS $\mathcal{R}$ and rewrite relation $\rightarrow \mathrm{dl}(s, \rightarrow)=\max \left\{n \mid \exists t s \rightarrow^{n} t\right\}$. The (innermost) runtime complexity (with respect to $\mathcal{R}$ ) is defined as follows:

$$
\operatorname{rc}_{\mathcal{R}}^{\mathrm{i}}(n)=\max \left\{\mathrm{dl}\left(s, \xrightarrow[\rightarrow]{\mathrm{i}}_{\mathcal{R}}\right) \mid s=f\left(s_{1}, \ldots, s_{n}\right), s_{1}, \ldots, s_{n} \in \mathcal{V} \text { al and }|s| \leqslant n\right\} .
$$

Suppose $\mathcal{R}$ admits polynomial runtime complexity, i.e. $\mathrm{rc}_{\mathcal{R}}{ }^{\mathrm{i}}$ is bounded polynomially. To conclude polytime computability of a function $f$ computed by $\mathcal{R}$, we implement $f$ using graph rewriting. More precisely, to evaluate $s:=\mathrm{f}\left(s_{1}, \ldots, s_{n}\right)$ for values $s_{1}, \ldots, s_{n}$, we normalise the graph $S$ corresponding to $s$ under ${\underset{\mathcal{G}}{(\mathcal{R})}}$. This is admissible by the Adequacy Theorem (Theorem 19). At most polynomially many reduction steps (in $|s|$ ) need to be performed. Below we show that each such intermediate step can be performed in time polynomial in the size of $s$.

Lemma 20. Let $\mathcal{G}$ denote a $G R S$, set $c:=\max \{|R| \mid(L \rightarrow R) \in \mathcal{G}\}$. If $S \underset{\mathcal{G}^{\prime}}{ } T$ then $|T| \leqslant|S|+c$.

Hence, sizes of intermediate graphs are polynomially related to the size of $S$ and $s$. (Note that $|S| \leqslant|s|$ ). From this we derive that polynomial innermost runtime complexity induces polytime computability.

In the following, we fix the set of nodes $\mathrm{V}:=\mathbb{N}$. We represent $S \in \mathcal{G}$ raph as a pair $\langle\mathrm{rt}(S)$, spec $\rangle$ where spec is a sequence containing for each node $n \in S$ the triple $\left\langle n, \mathrm{~L}_{S}(n), \operatorname{succ}_{S}(n)\right\rangle$. Following [6], we call such triples node specifications. We say that a term graph $S$ is normalised if $\vee_{S}=\{1, \ldots,|S|\}$. Representing normalised term graphs $S$ requires space $\mathrm{O}(\log (|S|) *|S|)$ on any reasonable model of computation: Each node $n \in S$ requires space at most $\log (|S|)$. Consequently, each node specification $\left\langle n, \mathrm{~L}_{S}(n), \operatorname{succ}_{S}(n)\right\rangle$ is representable in space $\mathrm{O}(\log (|S|))$. Here we employ that for a fixed signature $\mathcal{F}$, arities are bounded by some constant, moreover each label $\mathrm{L}_{S}(n)$ requires just constant space. As we have to store at most $|S|$ specifications (and the root node), the assertion follows.

We define $\|S\|:=\mathrm{O}(\log (|S|) *|S|)$. Below we also employ the notation $\|\cdot\|$ for the space required to represent different data-types like nodes or lists. In the following, we tacitly assume that term graphs are normalised. This is justified as normalisation introduces negligible overhead as can be seen in the next lemma.

Lemma 21. Let $S$ be a term graph. The function that maps $S$ to an isomorphic and normalised term graph is computable in time $\mathrm{O}\left(\|S\|^{2}\right)$.

Proof. We traverse over $S$ and replace each encountered node $n \in S$ by the node $m(n)$, where $m$ is a graph morphism normalising $S$ to be constructed. To obtain $m$, we start the overall procedure using a counter $c:=1$ and initialise $m$ as the morphism that is undefined everywhere. At each call $m(n)$ we check whether $m$ is defined on $n$. If so, we return $m(n)$, otherwise we set $m(n):=c, c:=c+1$ and return $m(n)$ afterwards. When the procedure stops, $m$ will be an injective graph morphism, and $m(S)$ a normalised term graph isomorphic to $S$.

While traversing $S$ in time $\mathrm{O}(\|S\|)$, we replace the root node, and for each node specification encountered we replace at most a constant number of nodes. I.e., in total $\mathrm{O}(|S|)$ calls $m(n)$ have to be performed. We can represent $m$ in space $\|m\|=\mathrm{O}(\|S\|)$. From this we obtain that $m(n)$ is computable in time $\mathrm{O}(\|S\|)$. Overall, the procedure finished after at most $\mathrm{O}\left(\|S\|^{2}\right)$ steps.

The implementation of a graph rewrite step $S \Rightarrow_{\mathcal{G}} T$ is developed stepwise in the following. The function match: $(\mathcal{G r a p h} \times \mathcal{G} r a p h \times V) \rightarrow(\mathrm{V} \rightarrow$ V) $\cup\{\perp\}$ computes morphisms between term graphs. Here $\perp$ is a designator indicating that no morphism can be found. More precisely, for $m$ : $L \rightarrow S \upharpoonright n$ we set $\operatorname{match}(L, S, n):=m$; otherwise $\operatorname{match}(L, S, n):=\perp$. The definition of match is well-formed as $m: L \rightarrow S$ is unique, cf. Lemma 7. Second, we use the function apply: $(\mathcal{G r a p h} \times \mathrm{V} \times \mathcal{G}$ raph $\times(\mathrm{V} \rightarrow \mathrm{V})) \rightarrow \mathcal{G}$ raph defined by $\operatorname{apply}(S, n, R, m):=S[m(R)]_{n}$. Here we suppose $m: L \rightarrow S \upharpoonright n$ for some graph rewrite rule $L \rightarrow R$. Let Rule denote the set of (normalised) graph rewrite rules over labels $\mathcal{F} \cup \mathcal{V}$. A step $S \Rightarrow_{\mathcal{G}, n, L \rightarrow R} T$ is thus computed by the function step : $(\mathcal{G}$ raph $\times \mathrm{V} \times$ Rule $) \rightarrow \mathcal{G}$ raph $\cup\{\perp\}$ given by

$$
\begin{aligned}
& \operatorname{step}(S, n, L \rightarrow R):=\operatorname{apply}\left(S, n, R^{\prime}, \operatorname{match}\left(L^{\prime}, S, n\right)\right) \\
& \quad \text { where } L^{\prime} \rightarrow R^{\prime}=\operatorname{rename}(S, L \rightarrow R) .
\end{aligned}
$$

Here we suppose $\operatorname{step}(S, n, L \rightarrow R)=\perp$ if $\operatorname{match}\left(L^{\prime}, S, n\right)=\perp$. The function rename: $(\mathcal{G}$ raph $\times$ Rule $) \rightarrow$ Rule is defined such that rename $(S, L \rightarrow R)$ is a renaming of the rule $L \rightarrow R$ with respect to the term graph $S$.

We give bounds on the computational complexity of match, rename and apply. Here we essentially translate the definition into programs and prove that those programs operate under the desired bounds, c.f. [4. We implement match $(L, S, n)$ by recursion on the term representation of $L$, which explains why match $(L, S, n)$ operates in time exponential in $|L|$. For $\mathcal{R}$ fixed, $2^{\mathrm{O}(\|L\|)}$ is constant and thus harmless for our concerns.
Lemma 22. Let $S$ be a term graph and $L \rightarrow R$ a graph rewrite rule. Then

1) match $(L, S, n)$ is computable in time $2^{\mathrm{O}(\|L\|)} *\|S\|^{2}$, and
2) rename $(S, L \rightarrow R)$ is computable in time $\mathrm{O}(\|S\|+\|L \rightarrow R\|)$, and
3) apply $(S, n, R, m)$ is computable in time $\mathrm{O}\left((\|S\|+\|L \rightarrow R\|)^{3}\right)$.

Lemma 23. Let $S$ be a term graph, $L \rightarrow R$ a graph rewrite rule and $n \in L$ be a node. Then $\operatorname{step}(S, n, L \rightarrow R)$ is computable in time $\mathrm{O}\left(2^{\mathrm{O}(\|L \rightarrow R\|)} *\|S\|^{3}\right)$.

Proof. Employing $\|L\|<\|L \rightarrow R\|$, the lemma follows from Lemma 22 ,
Remark 24. Note that for $\mathcal{G}$ fixed and $(L \rightarrow R) \in \mathcal{G}$, the bound $\mathrm{O}\left(2^{\mathrm{O}(\|L \rightarrow R\|)} *\right.$ $\left.\|S\|^{3}\right)$ reduces to $\mathrm{O}\left(\|S\|^{3}\right)$.

Let $S$ and $T$ be term graphs such that $S \stackrel{\underset{\mathcal{G}}{ }}{ } T$. To show that $T$ is efficiently computable from $S$, it remains to verify that collapsing to normal form sharing graphs is feasible. We introduce the function share : $\mathcal{G}$ raph $\rightarrow \mathcal{G}$ raph that maps (redex-unsharing) term graphs $S$ to their normal form sharing counterparts. To be more precise, $S \geqslant \operatorname{share}(S)$ with share $(S)$ normal form sharing.

Lemma 25. Let $S$ be redex-unsharing and let $p \in S$ be some $\rightarrow$-minimal node such that there exists a distinct node $q \in S$ with (i) $\mathrm{L}(p)=\mathrm{L}(q)$ and (ii) $\operatorname{succ}(p)=$ $\operatorname{succ}(q)$. Then $S\lceil p$ is normal form sharing.

Proof. For a proof by contradiction, suppose $S \backslash p$ is not normal form sharing. By Lemma 17, there exist distinct nodes $p^{\prime}, q^{\prime} \in S \backslash p$ such that labels and successor function coincide in $S\left\lceil p\right.$ for $p^{\prime}$ and $q^{\prime}$. By definition $p \stackrel{*}{\rightharpoonup} p^{\prime}$ and $p \stackrel{*}{\rightharpoonup} q^{\prime}$ such that in at least one case the path considered in $S \backslash p$ has non-zero length. Suppose we have $p \stackrel{+}{\rightharpoonup} p^{\prime}$. Then the node $p^{\prime}$ is a counter example to minimality of $p$.

Lemma 26. Let $S$ be redex-unsharing. share $(S)$ is computable in time $\mathrm{O}\left(\|S\|^{4}\right)$.
Proof. We compute share $(S)$ by computing term graphs $S_{1}, \ldots, S_{n}$ with $S=$ $S_{1}>\cdots>S_{n}=S^{\prime}$ such that for all $i=1, \ldots, n$, the graph $S_{i}$ is redexunsharing. Lemma 18 guarantees that the above sequence is well-defined. Since $S_{i}>S_{i+1}$ implies $\left|S_{i}\right|>\left|S_{i+1}\right|$, the number of iterations is bound by $|S|$.

It suffices to show one iteration. We show that the computation of $S_{i+1}$ from $S_{i}$, can be performed in time $\mathrm{O}\left(\|S\|^{3}\right)$. Including normalisation (operating in time $\mathrm{O}\left(\|S\|^{2}\right)$, cf. Lemma 21), we obtain that share $(S)$ is computable in time $|S| * \mathrm{O}\left(\|S\|^{3}\right)+\mathrm{O}\left(\|S\|^{2}\right) \leqslant \mathrm{O}\left(\|S\|^{4}\right)$.

Suppose we have constructed the redex-unsharing graph $S_{i}$. To obtain $S_{i+1}$ from $S_{i}$, we search for two distinct nodes $p$ and $q$ such that $\operatorname{succ}_{S_{i}}(p)=\operatorname{succ}_{S_{i}}(q)$, $\mathrm{L}_{S_{i}}(p)=\mathrm{L}_{S_{i}}(q)$ and $\operatorname{term}\left(S_{i} \upharpoonright p\right) \in \operatorname{NF}(\mathcal{R})$. By Lemma 17, nodes $p$ and $q$ exist if and only if $S_{i}$ is normal form shared. If $p$ and $q$ exist we obtain $S_{i+1}$ by identifying nodes $p$ and $q$ in $S_{i}$. Otherwise $S_{i}$ is normal form shared, we set $S_{i+1}:=S_{i}$ and the procedure stops.

Set $R:=\left\{p \mid \exists q . p \neq q \wedge \operatorname{succ}_{S_{i}}(p)=\operatorname{succ}_{S_{i}}(q) \wedge \mathrm{L}_{S_{i}}(p)=\mathrm{L}_{S_{i}}(q)\right\}$. Thus, reformulating the above approach, we search for $p \in R$ with term $\left(S_{i} \upharpoonright p\right) \in \operatorname{NF}(\mathcal{R})$. To this extend, let $R_{\text {min }} \subseteq R$ be the restriction of $R$ to - -minimal nodes; $R_{\text {min }}:=\left\{p \mid p \in R \wedge \neg \exists p^{\prime} \in R . p^{\prime} \stackrel{+}{\rightharpoonup} p\right\}$. Clearly $p^{\prime} \stackrel{\rightharpoonup}{\rightharpoonup} p$ and term $\left(S_{i} \upharpoonright p^{\prime}\right) \in \operatorname{NF}(\mathcal{R})$ implies term $\left(S_{i} \upharpoonright p\right) \in \operatorname{NF}(\mathcal{R})$. It is thus sufficient to search for some $p \in R_{\text {min }}$ with term $\left(S_{i} \upharpoonright p\right) \in \operatorname{NF}(\mathcal{R})$, or equivalently $S_{i} \upharpoonright p \in \operatorname{NF}(\mathcal{G})$. The latter observation results from Lemma 16, since by Lemma 25 the subgraph $S_{i} \upharpoonright p$ is normal form sharing due to the definition of $R_{\text {min }}$.

Suppose $p \in R_{\text {min }}$ with term $\left(S_{i} \upharpoonright p\right) \in \operatorname{NF}(\mathcal{R})$. By definition of $R_{\text {min }}$ there exists a distinct node $q \in S_{i}$ such that term $\left(S_{i} \upharpoonright p\right)=\operatorname{term}\left(S_{i} \upharpoonright q\right)$. To obtain $S_{i+1}$ from $S_{i}$, we identify $p$ and $q$, i.e. we set $S_{i+1}:=m\left(S_{i}\right)$ where the graph morphism $m: S_{i} \rightarrow S_{i+1}$ is defined by $m(q)=p$ and $m(n)=n$ for $n \in \mathrm{~V}_{S_{i}} \backslash\{q\}$. Then $S_{i+1}$ is redex-unsharing, moreover $S_{i}>_{m} S_{i+1}$ as desired.

One easily verifies that $R$ is computable in time $\mathrm{O}\left(\left\|S_{i}\right\|^{2}\right) \leqslant \mathrm{O}\left(\|S\|^{2}\right)$. The cardinality of $R$ is bound by $\left|S_{i}\right| \leqslant|S|$, as $S$ is normalised we see that $\|R\| \leqslant$ $\mathrm{O}(\|S\|)$. Using a quadratic reachability-algorithm to check $p^{\prime} \stackrel{\rightharpoonup}{\rightharpoonup} p, R_{\text {min }}$ is computable from $R$ in time $\|R\| * \mathrm{O}\left(\left\|S_{i}\right\|^{2}\right) \leqslant \mathrm{O}\left(\|S\|^{3}\right)$. As $R_{\text {min }} \subseteq R$ we have to check $S_{i} \mid p \in \operatorname{NF}(\mathcal{G})$ at most $\left|S_{i}\right| \leqslant|S|$ times. For this, we check match $\left(L, S_{i}, p\right)=\perp$ for all rules $(L \rightarrow R) \in \mathcal{G}(\mathcal{R})$. The latter can be done in time $\mathrm{O}\left(\left\|S_{i}\right\|^{2}\right) \leqslant \mathrm{O}\left(\|S\|^{2}\right)$ (cf. Lemma 22). We conclude that $p \in R_{\text {min }}$ with term $\left(S_{i} \upharpoonright p\right) \in \mathrm{NF}(\mathcal{R})$ can be
found in time $\mathrm{O}\left(\|S\|^{2}\right)+\mathrm{O}\left(\|S\|^{3}\right)+|S| * \mathrm{O}\left(\|S\|^{2}\right) \leqslant \mathrm{O}\left(\|S\|^{3}\right)$. Searching the corresponding node $q \in S_{i}$ and applying the morphism $m$ can be done in time $\mathrm{O}(\|S\|)$. Overall, the runtime for computing $S_{i+1}$ from $S_{i}$ is bound by O $\left(\|S\|^{3}\right)$ as desired.

Theorem 27. Let $\mathcal{R}$ be a confluent and terminating TRS, moreover suppose $\mathrm{rc}_{\mathcal{R}}^{\mathrm{i}}(n)=\mathrm{O}\left(n^{k}\right)$ for all $n \in \mathbb{N}$ and some $k \in \mathbb{N}$. The functions computed by $\mathcal{R}$ are computable in time $\mathrm{O}\left(n^{5 *(k+1)}\right)$.

Proof. To compute the functions defined by $\mathcal{R}$, we encode terms as graphs and perform graph rewriting using the simulating GRS $\mathcal{G}:=\mathcal{G}(\mathcal{R})$. Let $f$ be a function computed by $\mathcal{R}$, let $\mathrm{f} \in \mathcal{F}$ be the associated function symbol. Fix values $s_{1}, \ldots, s_{n}$ such that $f\left(s_{1}, \ldots, s_{n}\right)=t$ is defined. Let $s=\mathrm{f}\left(s_{1}, \ldots, s_{n}\right)$ and $S \in \diamond(s)$. By definition and confluence of $\mathcal{R}$, we obtain $s \xrightarrow{i}!$ $f\left(s_{1}, \ldots, s_{n}\right)$ is computable in time $|s|^{5 *(k+1)}$. For this, consider some derivation

$$
S=T_{0} \stackrel{\underset{\mathcal{G}}{\Rightarrow}}{\mathcal{G}} \ldots \stackrel{\stackrel{\rightharpoonup}{\Rightarrow}}{\mathcal{G}}, T_{l}=T
$$

with $T \in \operatorname{NF}(\mathcal{G})$. By Theorem 19 we conclude term $(T)=t$.
We first analyse a single step $T_{i}{\underset{\mathcal{G}}{ }}_{{\underset{\mathcal{G}}{ }} T_{i+1} \text { from the derivation ( } \dagger \text { ). In order }}$ to compute $T_{i+1}$ from $T_{i}$, we identify the corresponding redex in $T_{i}$. Define $\operatorname{redex}(n):=T_{i} \upharpoonright n \notin \operatorname{NF}(\mathcal{G}) \wedge \forall n_{j} \in \operatorname{succ}_{T_{i}}(n) . T_{i} \upharpoonright n_{j} \in \operatorname{NF}(\mathcal{G})$. Then the node $n \in T_{i}$ is an innermost redex if and only if redex ( $n$ ) holds. In order to check $T_{i} \upharpoonright q \in$ $\operatorname{NF}(\mathcal{G})$ for node $q \in T_{i}$, as before we apply the function match a constant number of times. Due to Lemma 22 we see that redex $(n)$ is computable in time $\mathrm{O}\left(\left\|T_{i}\right\|^{2}\right)$ overall. We find the desired redex in $T_{i}$ in time $\mathrm{O}\left(\left\|T_{i}\right\|^{3}\right)$ : we traverse $T_{i}$ and return the first $n \in T_{i}$ encountered satisfying redex $(n)$. Without loss of generality, we can assume $n$ is the redex in $T_{i}{\underset{\mathcal{G}}{\mathcal{G}}} T_{i+1}$ (otherwise, we adapt the derivation (†) appropriately). Let $T_{i+1}^{\prime}=\operatorname{step}(s, n, L \rightarrow R)$ for the first applicable rule $(L \rightarrow R) \in \mathcal{G}$. By Lemma 23 , since we have to check only a constant number of rules $L \rightarrow R, T_{i+1}^{\prime}$ is computable from $T_{i}$ in time $\mathrm{O}\left(\left\|T_{i}\right\|^{3}\right)$. As observed in the proof of Theorem 19 the graph $T_{i+1}^{\prime}$ is redex-unsharing. Using Lemma 26, we finally obtain $T_{i+1}=\operatorname{share}\left(T_{i+1}^{\prime}\right)$ in time $\mathrm{O}\left(\left\|T_{i+1}^{\prime}\right\|^{4}\right)$. Summing all up, employing $\left\|T_{i+1}^{\prime}\right\| \leqslant\left\|T_{i}\right\|+c$ for some fixed $c \in \mathbb{N}$ (cf. Lemma 20 ), $T_{i+1}$ is computable from $T_{i}$ in time $\mathrm{O}\left(\left\|T_{i}\right\|^{4}\right)$.

We return to the derivation $\dagger$. Note $s \in \mathcal{B}$ and further $T_{i} \Rightarrow_{\mathcal{G}} T_{i+1}$ implies $\operatorname{term}\left(T_{i}\right) \stackrel{i}{\rightarrow}_{\mathcal{R}} \operatorname{term}\left(T_{i+1}\right)$. Thus we conclude $l=\mathrm{O}\left(|s|^{k}\right)$ for some $k \in \mathbb{N}$. Let $j \in\{0, \ldots, l-1\}$. Lemma 20 implies $\left|T_{j}\right| \leqslant|S|+j * c \leqslant \mathrm{O}\left(|s|^{k}\right)$ for some fixed $c \in \mathbb{N}$. Here we employ that $|S| \leqslant|s|$. Recall $\left\|T_{j}\right\|=\mathrm{O}\left(\log \left(\left|T_{j}\right|\right) *\left|T_{j}\right|\right)$. Thus $\left\|T_{j}\right\| \leqslant \mathrm{O}\left(\log \left(|s|^{k}\right) *|s|^{k}\right) \leqslant \mathrm{O}\left(|s|^{k+1}\right)$. And so each step $T_{j}{\underset{\mathcal{G}}{ }}^{\mathcal{Z}_{j+1}}$ can be performed in time $\mathrm{O}\left(\left\|T_{j}\right\|^{4}\right) \leqslant \mathrm{O}\left(|s|^{4 *(k+1)}\right)$ using the above observation. In total, we obtain that $S \in \diamond(s)$ can be $\stackrel{\rightharpoonup}{\Rightarrow}_{\mathcal{G}^{\text {-normalised }} \text { in time } \mathrm{O}\left(|s|^{k}\right) * \mathrm{O}\left(|s|^{4 *(k+1)}\right) \leqslant ~}^{\text {- }}$ $\mathrm{O}\left(|s|^{5 *(k+1)}\right)$. We conclude the theorem.

## 5 Conclusion

Recently, many techniques have been introduced that allow the (automated) classification of the runtime complexity of TRS. In this paper we show that polynomial innermost runtime complexity of TRSs induces polytime computability of the functions defined. As a side-result we present a simulation between (innermost) term rewriting and (innermost) graph rewriting.

The latter result is related to implicit computational complexity and in particular to a recent result by Dal Lago and Martini. In 8 Dal Lago and Martini establish that orthogonal constructor TRSs and lambda calculus with weak call-by-value reduction simulate each other with linear overhead. The proof of this [8] (compare also [7) provides a variant of Theorem 19 for the restricted case that the TRS in question is constructor and orthogonal. By augmenting the innermost graph rewrite relation by collapse steps, our result prevail also in the general case.

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    ${ }^{1}$ The here mentioned polynomial path orders are one complexity technique implemented in the Tyrolean Complexity Tool, a complexity tool to analyse the runtime complexity of TRSs. The program is open-source and freely available at http://cl-informatik.uibk.ac.at/software/tct/.

