Monotonicity Criteria for Polynomial Interpretations over the Naturals*

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Abstract. Polynomial interpretations are a useful technique for proving termination of term rewrite systems. In an automated setting, termination tools are concerned with parametric polynomials whose coefficients (i.e., the parameters) are initially unknown and have to be instantiated suitably such that the resulting concrete polynomials satisfy certain conditions. We focus on monotonicity and well-definedness, the two main conditions that are independent of the respective term rewrite system considered, and provide constraints on the abstract coefficients for linear, quadratic and cubic parametric polynomials such that monotonicity and well-definedness of the resulting concrete polynomials are guaranteed whenever the constraints are satisfied. Our approach subsumes the absolute positiveness approach, which is currently used in many termination tools. In particular, it allows for negative numbers in certain coefficients. We also give an example of a term rewrite system whose termination proof relies on the use of negative coefficients, thus showing that our approach is more powerful.

1 Introduction

Polynomial interpretations are a simple yet useful technique for proving termination of term rewrite systems (TRSs). They come in various flavors. While originally conceived by Lankford [10] for establishing direct termination proofs, polynomial interpretations are nowadays often used in the context of the dependency pair (DP) framework [1,6,7]. Moreover, the classical approach of Lankford, who only considered polynomial algebras over the natural numbers, was extended by several authors to polynomial algebras over the real numbers [3,11].

This paper is concerned with automatically proving termination of term rewrite systems by means of polynomial interpretations over the natural numbers. In the classical approach, we associate with every n-ary function symbol f a polynomial P_f in n indeterminates with integer coefficients, which induces a mapping or *interpretation* from terms to integer numbers in the obvious way. In order to conclude termination of a given TRS, three conditions have to be satisfied. First, every polynomial must be *well-defined*, i.e., it must induce a well-defined polynomial function $f_{\mathbb{N}} \colon \mathbb{N}^n \to \mathbb{N}$ over the natural numbers. In addition, the interpretation functions $f_{\mathbb{N}}$ are required to be *strictly monotone* in all arguments. Finally, one has to show *compatibility* of the interpretation with

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the given TRS. More precisely, for every rewrite rule $l \to r$ the polynomial P_l associated with the left-hand side must be greater than P_r, the corresponding polynomial of the right-hand side, i.e., $P_l > P_r$, for all values (in $\mathbb{N}$) of the indeterminates. These three requirements essentially carry over to the case of using polynomial interpretations as reduction pairs in the DP framework, but in a weakened form. Most notably, the interpretation functions are merely required to be weakly monotone, and for some rules $P_l \geq P_r$ suffices.

In an automated setting, termination tools are concerned with parametric polynomials whose coefficients (i.e., the parameters) are initially unknown and have to be instantiated suitably such that the resulting concrete polynomials satisfy the above conditions. In this paper, we focus on monotonicity (strict and weak) and well-definedness of linear, quadratic and cubic parametric polynomials, two conditions that are independent of the respective TRS considered. The aim is to provide exact constraints in terms of the abstract coefficients of a parametric polynomial such that monotonicity and well-definedness of the resulting concrete polynomial are guaranteed for every instantiation of the coefficients that satisfies the constraints. For example, given the parametric polynomial $ax^2 + bx + c$, we identify constraints on the parameters a, b and c such that the associated polynomial function is both well-defined and (strictly) monotone. Our approach subsumes the absolute positiveness approach [9], which is currently used in many termination tools. In contrast to the latter, negative numbers in certain coefficients can be handled without further ado. Previous work allowing negative coefficients ensures well-definedness and (weak) monotonicity by extending polynomials with "max" [8,5]. However, all our interpretation functions are polynomials and our results do also apply to strict monotonicity. Hence in the sequel we do not consider these approaches.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary definitions and terminology concerning polynomials and polynomial interpretations. The follow-up section is the main section of this paper where we present our results concerning monotonicity of linear, quadratic and cubic parametric polynomials. In Section 4, we give a constructive proof showing that our approach is more powerful than the absolute positiveness approach. Finally, Section 5 presents some experimental results.

2 Preliminaries

For any ring R, we denote the associated polynomial ring in n indeterminates $x_1, \ldots, x_n$ by $R[x_1, \ldots, x_n]$. For example, the polynomial $2x^2 - x + 1$ is an element of $\mathbb{Z}[x]$, the ring of all univariate polynomials with coefficients in $\mathbb{Z}$. Let $P := \sum_{k=0}^{n} a_k x^k$ be an element of the polynomial ring $R[x]$. For the largest k where $a_k \neq 0$, we call $a_k x^k$ the *leading term* of P, a_k its *leading coefficient* and k its *degree*. Moreover, we call a_0 the *constant coefficient* or *constant term* of P.

A *quadratic equation* is an equation of the form $ax^2 + bx + c = 0$, where x is an indeterminate, and a, b and c represent constants, with $a \neq 0$. The solutions of a quadratic equation, called *roots*, are given by the *quadratic formula*:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this formula, the expression $d := b^2 - 4ac$ underneath the square root sign is of central importance because it determines the nature of the roots; it is also called the *discriminant* of a quadratic equation. If all coefficients are real numbers, one of the following three cases applies:

1. If d is positive, there are two distinct roots, both of which are real numbers.
2. If d is zero, there is exactly one real root, called a *double root*.
3. If d is negative, there are no real roots. Both roots are complex numbers.

The key concept for using polynomial interpretations to establish (direct) termination of term rewrite systems is the notion of well-founded monotone algebras (WFMAs) since they induce reduction orders on terms. Let $\mathcal{F}$ be a signature. A *well-founded monotone $\mathcal{F}$-algebra* $(\mathcal{A}, >)$ is a non-empty algebra $\mathcal{A} = (A, \{f_A\}_{f \in \mathcal{F}})$ together with a well-founded order $>$ on the carrier A of $\mathcal{A}$ such that every algebra operation f_A is strictly monotone in all arguments, i.e., if $f \in \mathcal{F}$ has arity $n \geq 1$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) > f_A(a_1, \ldots, b, \ldots, a_n)$ for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$.

Concerning the use of polynomial interpretations in the context of the DP framework, the notion of a *well-founded weakly monotone algebra* (WFWMA) is sufficient to obtain a reduction pair. A WFWMA is just like a WFMA, with the exception that weak rather than strict monotonicity is required; i.e., $f_A(a_1, \ldots, a_i, \ldots, a_n) \geq f_A(a_1, \ldots, b, \ldots, a_n)$ whenever $a_i \geq b$. Here $\geq$ is the reflexive closure of $>$.

Given a monotone algebra $(\mathcal{A}, >)$, we define the relations $\geq_{\mathcal{A}}$ and $>_{\mathcal{A}}$ on terms as follows: $s \geq_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) \geq [\alpha]_{\mathcal{A}}(t)$ and $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$, for all assignments α of elements of A to the variables in s and t ($[\alpha]_{\mathcal{A}}(\cdot)$ denotes the usual evaluation function associated with $\mathcal{A}$). Now if $(\mathcal{A}, >)$ is a WFMA, then $>_{\mathcal{A}}$ is a reduction order that can be used to prove termination of term rewrite systems by showing that $>_{\mathcal{A}}$ orients the rewrite rules from left to right. If, on the other hand, $(\mathcal{A}, >)$ is a WFWMA, then $(\geq_{\mathcal{A}}, >_{\mathcal{A}})$ is a reduction pair that can be used to establish termination in the context of the DP framework.

3 Parametric Polynomials

Polynomial interpretations over the natural numbers are based on the well-founded algebra $(\mathcal{N}, >)$, where $>$ is the standard order on the natural numbers $\mathbb{N}$ and $\mathcal{N} = (\mathbb{N}, \{f_{\mathbb{N}}\}_{f \in \mathcal{F}})$ such that every algebra operation $f_{\mathbb{N}}$ is a polynomial with integer coefficients. Depending on whether all algebra operations are strictly or weakly monotone, $(\mathcal{N}, >)$ is either a WFMA or a WFWMA. To be precise, every n-ary function symbol $f \in \mathcal{F}$ is associated with a polynomial with integer coefficients such that the corresponding algebra operation $f_{\mathbb{N}} \colon \mathbb{N}^n \to \mathbb{N}$ is a well-defined polynomial function which is strictly or weakly monotone in all arguments. Note, however, that this does not imply that all coefficients of the polynomials must be natural numbers.

Example 1. The univariate integer polynomial $2x^2 - x + 1 \in \mathbb{Z}[x]$ gives rise to the polynomial function $f_{\mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$, $x \mapsto 2x^2 - x + 1$, which is obviously well-defined over $\mathbb{N}$. Moreover, it is also strictly monotone with respect to $\mathbb{N}$. Note, however, that monotonicity does not hold if we view $2x^2 - x + 1$ as a function over the (non-negative) real numbers.

Summing up, an n-ary polynomial function $f_{\mathbb{N}}$ used in a polynomial interpretation is an element of the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ and must satisfy:

1. *well-definedness*: $f_{\mathbb{N}}(x_1, \dots, x_n) \geq 0$ for all $x_1, \dots, x_n \in \mathbb{N}$
2. *strict (weak) monotonicity*: $f_{\mathbb{N}}(x_1, \dots, x_i, \dots, x_n) \underset{(=)}{>} f_{\mathbb{N}}(x_1, \dots, y, \dots, x_n)$ for all $i \in \{1, \dots, n\}$ and $x_1, \dots, x_n, y \in \mathbb{N}$ with $x_i > y$.

Alas, both of these properties are instances of the undecidable problem of checking *positiveness of polynomials*[1] in the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ (undecidable by reduction from Hilbert's 10-th problem).

Termination tools face the following problem. They deal with parametric polynomials, i.e., polynomials whose coefficients are unknowns (e.g., ax^2+bx+c), and the task is to find suitable integer numbers for the unknown coefficients such that the resulting polynomials induce algebra operations that satisfy both of the above properties. The solution that is used in practice is to restrict the search space for the unknown coefficients to the non-negative integers (absolute positiveness approach [9,2]) because then well-definedness and weak monotonicity are obtained for free. To obtain strict monotonicity in the i-th argument of a polynomial function $f_{\mathbb{N}}(\dots, x_i, \dots)$, at least one of the terms $(c_k x_i^k)_{k>0}$ must have a positive coefficient $c_k > 0$.

Obviously, this approach is easy to implement and works quite well in practice. However, it is not optimal in the sense that it excludes certain polynomials, like $2x^2 - x + 1$, which might be useful to prove termination of certain TRSs. So how can we do better? To this end, let us observe that in general termination tools only use restricted forms of polynomials to interpret function symbols. There are restrictions concerning the degree of the polynomials (linear, quadratic, etc.) and sometimes also restrictions that disallow certain kinds of monomials. Now the idea is as follows. Despite the fact that well-definedness and monotonicity are undecidable in general, it might be the case that they are decidable for the restricted forms of polynomials used in practice. And indeed, that is the case, as we shall see shortly.

Remark 2. Checking compatibility of a rewrite rule $l \to r$ with a polynomial interpretation means showing that the rule gives rise to a (weak) decrease; i.e., $P_l - P_r > 0$ ($P_l - P_r \geq 0$). In $\mathbb{N}$, both cases reduce to checking non-negativity of polynomials because $x > y$ if and only if $x \geq y + 1$. Since well-definedness of a polynomial as defined above is equivalent to non-negativity of a polynomial in $\mathbb{N}$, any method that ensures non-negativity of parametric polynomials can also be used for checking compatibility. However, we remark that the method presented in this paper is not ideally suited for this purpose as it also enforces strict monotonicity, which is irrelevant for compatibility.

[1] Given $P \in \mathbb{Z}[x_1, \dots, x_n]$, decide $P(x_1, \dots, x_n) > 0$ for all $x_1, \dots, x_n \in \mathbb{N}$.

In the sequel, we analyze parametric polynomials whose only restriction is a bound on the degree. We will first treat linear parametric polynomials. While this does not yield new results or insights, it is instructive to demonstrate our approach in a simple setting. This is followed by an analysis of quadratic and finally also cubic parametric polynomials, both of which yield new results. The following lemmas will be helpful in this analysis. The first one gives a more succinct characterization of monotonicity, whereas the second one relates monotonicity and well-definedness.

Lemma 3. *A (not necessarily polynomial) function $f_{\mathbb{N}} \colon \mathbb{N}^n \to \mathbb{Z}$ is strictly (weakly) monotone in all arguments if and only if*

$$f_{\mathbb{N}}(x_1, \ldots, x_i + 1, \ldots, x_n) \underset{(}{\geq}_{)} f_{\mathbb{N}}(x_1, \ldots, x_i, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in \mathbb{N}$ and all $i \in \{1, \ldots, n\}$.

Lemma 4. *Let $f \colon \mathbb{Z}^n \to \mathbb{Z}$ be the polynomial function associated with a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$, and let $f_{\mathbb{N}} \colon \mathbb{N}^n \to \mathbb{Z}$ denote its restriction to $\mathbb{N}$. Then $f_{\mathbb{N}}$ is strictly (weakly) monotone and well-defined if and only if it is strictly (weakly) monotone and $f_{\mathbb{N}}(0, \ldots, 0) \geq 0$.*

In these lemmata, as well as in the remainder of the paper, monotonicity and well-definedness refer to the two properties mentioned at the beginning of this section. In particular, monotonicity is meant with respect to all arguments.

3.1 Linear Parametric Polynomials

In this section we consider the generic linear parametric polynomial function $f_{\mathbb{N}}(x_1, \ldots, x_n) = a_n x_n + a_{n-1} x_{n-1} + \cdots + a_1 x_1 + a_0$, and derive constraints on the coefficients a_i that guarantee monotonicity and well-definedness.

Theorem 5. *The function $f_{\mathbb{N}}(x_1, \ldots, x_n) = a_n x_n + \ldots + a_1 x_1 + a_0$ ($a_i \in \mathbb{Z}$, $0 \leq i \leq n$) is strictly (weakly) monotone and well-defined if and only if $a_0 \geq 0$ and $a_i > 0$ ($a_i \geq 0$) for all $i \in \{1, \ldots, n\}$.*

Proof. Easy consequence of Lemmata 4 and 3. □

Remark 6. Note that all coefficients must be non-negative and that the constraints on the coefficients are exactly the ones one would obtain by the absolute positiveness approach. Furthermore, these constraints are optimal in the sense that they are both necessary and sufficient for monotonicity and well-definedness.

3.2 Quadratic Parametric Polynomials

Next we apply the approach illustrated by Theorem 5 to the generic quadratic parametric polynomial function

$$f_{\mathbb{N}}(x_1, \ldots, x_n) = a_0 + \sum_{j=1}^{n} a_j x_j + \sum_{1 \leq j \leq k \leq n} a_{jk} x_j x_k \ \in \mathbb{Z}[x_1, \ldots, x_n] \qquad (1)$$

Theorem 7. *The function $f_{\mathbb{N}}$ is strictly (weakly) monotone and well-defined if and only if $a_0 \geq 0$, $a_{jk} \geq 0$ and $a_j > -a_{jj}$ ($a_j \geq -a_{jj}$) for all $1 \leq j \leq k \leq n$.*

Proof. By Lemmata 3 and 4, this theorem holds if and only if $f_{\mathbb{N}}(0,\ldots,0) \geq 0$, and $f_{\mathbb{N}}(x_1,\ldots,x_i+1,\ldots,x_n) \underset{(\geq)}{>} f_{\mathbb{N}}(x_1,\ldots,x_i,\ldots,x_n)$ for all $x_1,\ldots,x_n \in \mathbb{N}$ and all $i \in \{1,\ldots,n\}$. Clearly, $f_{\mathbb{N}}(0,\ldots,0) \geq 0$ holds if and only if $a_0 \geq 0$, and the monotonicity condition $f_{\mathbb{N}}(x_1,\ldots,x_i+1,\ldots,x_n) > f_{\mathbb{N}}(x_1,\ldots,x_i,\ldots,x_n)$ yields

$$\begin{aligned} a_i(x_i+1) + a_{ii}(x_i+1)^2 + \sum_{i<k\leq n} a_{ik}(x_i+1)x_k + \sum_{1\leq j<i} a_{ji}x_j(x_i+1) \\ > a_i x_i + a_{ii}x_i^2 + \sum_{i<k\leq n} a_{ik}x_i x_k + \sum_{1\leq j<i} a_{ji}x_j x_i \end{aligned}$$

which simplifies to

$$a_i + a_{ii} + 2a_{ii}x_i + \sum_{i<k\leq n} a_{ik}x_k + \sum_{1\leq j<i} a_{ji}x_j > 0$$

This is a linear inequality that holds for all $x_1,\ldots,x_n \in \mathbb{N}$ if and only if $a_i + a_{ii} > 0$ and all other coefficients are non-negative. Taking the quantification over i into account, this proves the claim for strict monotonicity; the result for weak monotonicity follows by replacing $>$ with $\geq$ in the above calculation. □

Corollary 8. *The function $f_{\mathbb{N}}(x) = ax^2 + bx + c$ is strictly (weakly) monotone and well-defined if and only if $a \geq 0$, $c \geq 0$ and $b > -a$ ($b \geq -a$).*

Hence, in a quadratic parametric polynomial all coefficients must be non-negative except the coefficients of the linear monomials. They can be negative; for example, the polynomial $2x^2 - x + 1$ satisfies the constraints of Corollary 8; hence, it is both well-defined and strictly monotone.

Remark 9. Not only does our approach improve upon absolute positiveness for quadratic parametric polynomials, but the constraints derived from it are even optimal, i.e., necessary and sufficient for monotonicity and well-definedness.

Example 10. The polynomial function $f_{\mathbb{N}}(x_1,x_2) = 2x_1^2 + 3x_2^2 + x_1x_2 - x_1 - 2x_2 + 1$ is both well-defined and strictly monotone according to Theorem 7. Yet we can also infer this result in a more modular and probably more intuitive way by using Corollary 8. To this end, let $f_{\mathbb{N}}(x_1,x_2) = g_1(x_1) + g_2(x_2) + x_1x_2 + 1$, where $g_1(x_1) = 2x_1^2 - x_1$ and $g_2(x_2) = 3x_2^2 - 2x_2$. Clearly, by Corollary 8, $g_1(x_1)$ and $g_2(x_2)$ are both well-defined and strictly monotone. The same also holds for their sum, $g_1(x_1) + g_2(x_2)$, because $g_1(x_1)$ and $g_2(x_2)$ do not share variables. Finally, we may conclude that $f_{\mathbb{N}}$ is then also well-defined and strictly monotone by observing that the addition of monomials with non-negative coefficients (in this case: x_1x_2 and 1) is not harmful.

Another thing that is noteworthy about the previous theorem is that it subsumes the result of Theorem 5. That is to say, if we set the coefficients a_{jk} of all quadratic monomials in (1) to zero, thereby obtaining the linear parametric polynomial function $f'_{\mathbb{N}}(x_1,\ldots,x_n) = a_0 + \sum_{j=1}^{n} a_j x_j$, then the constraints generated by Theorem 7 are in fact the ones Theorem 5 would produce when applied to $f'_{\mathbb{N}}$. In theory, this means that if we want to prove termination of some TRS, then we do not have to specify a priori whether to interpret a function symbol by a linear or a quadratic parametric polynomial function; we can always go for quadratic interpretations, and it is solely determined by the constraint solving process (i.e., the process that assigns suitable integers to the abstract coefficients such that all constraints are satisfied) whether the resulting concrete polynomial function is linear or quadratic. In practice, however, this approach has an important drawback; that is, it increases both the number of abstract coefficients and the number of constraints involving these coefficients, which is detrimental to the performance of the constraint solving process.

3.3 Cubic Parametric Polynomials

Next we apply our approach to cubic parametric polynomials. First, we consider the univariate polynomial function $f_{\mathbb{N}}(x) = ax^3 + bx^2 + cx + d \in \mathbb{Z}[x]$, for which the monotonicity condition $f_{\mathbb{N}}(x+1) \underset{(-)}{>} f_{\mathbb{N}}(x)$ for all $x \in \mathbb{N}$ simplifies to

$$\forall x \in \mathbb{N} \quad 3ax^2 + (3a+2b)x + (a+b+c) \underset{(-)}{>} 0 \tag{2}$$

In the interesting case, where $a \neq 0$, the polynomial $P := 3ax^2 + (3a+2b)x + (a+b+c)$ is a quadratic polynomial in x whose geometric representation is a parabola in two-dimensional space, which has a global minimum at $x_{min} := -(3a+2b)/(6a)$. Since a is involved in the leading coefficient of P, a must necessarily be positive in order for (2) to hold. Next we focus on strict monotonicity, that is, the solution of the inequality

$$\forall x \in \mathbb{N} \quad 3ax^2 + (3a+2b)x + (a+b+c) > 0 \tag{3}$$

Now this inequality holds if and only if either $x_{min} < 0$ and $P(0) > 0$ or $x_{min} \geq 0$ and both $P(\lfloor x_{min} \rfloor) > 0$ and $P(\lceil x_{min} \rceil) > 0$. However, these constraints use the floor and ceiling functions, but we would rather have a set of polynomial constraints in a, b and c (which can easily be encoded in SAT or SMT). It is possible, however, to eliminate the floor and ceiling functions from the above constraints, but only at the expense of introducing new variables; e.g., $\lfloor x_{min} \rfloor = n$ for some $n \in \mathbb{Z}$ if and only if $n \leq x_{min} < n+1$. Thus one obtains a set of polynomial constraints in a, b, c and the additional variables. But one can also avoid the introduction of new variables with the following approach. To this end, we examine the roots of P and distinguish two possible cases:

Case 1 P has no roots in $\mathbb{R}$ (both roots are complex numbers),

Case 2 both roots of P are real numbers.

In the first case, (3) trivially holds. Moreover, this case is completely characterized by the discriminant of P being negative, i.e., $4b^2 - 3a^2 - 12ac < 0$. In the other case, when both roots r_1 and r_2 are real numbers, the discriminant is non-negative and (3) holds if and only if the closed interval $[r_1, r_2]$ does not contain a natural number, i.e., $[r_1, r_2] \cap \mathbb{N} = \varnothing$. While this condition can be fully characterized with the help of the floor and/or ceiling functions, we can also obtain a polynomial characterization as follows. We require the larger of the two roots, that is, r_2, to be negative because then (3) is guaranteed to hold. This observation leads to the constraints

$$4b^2 - 3a^2 - 12ac \geq 0 \quad \text{and} \quad r_2 = \frac{-(3a+2b) + \sqrt{4b^2 - 3a^2 - 12ac}}{6a} < 0$$

which can be simplified to

$$4b^2 - 3a^2 - 12ac \geq 0 \tag{4}$$

$$\sqrt{4b^2 - 3a^2 - 12ac} < 3a + 2b \tag{5}$$

Due to (4), (5) holds if and only if $4b^2 - 3a^2 - 12ac < (3a+2b)^2$ and $3a+2b \geq 0$, which simplifies to $a + b + c > 0$ and $3a + 2b \geq 0$. Putting everything together, we obtain the following theorem.

Theorem 11. *The function $f_{\mathbb{N}}(x) = ax^3 + bx^2 + cx + d$ is strictly monotone and well-defined if $a \geq 0$, $d \geq 0$ and either $4b^2 - 3a^2 - 12ac < 0$ or $4b^2 - 3a^2 - 12ac \geq 0$, $a + b + c > 0$ and $3a + 2b \geq 0$.*

Note that these constraints are only sufficient for monotonicity and well-definedness, they are not necessary. However, they are very close to necessary constraints, as will be explained below.

Remark 12. Weak monotonicity of $ax^3 + bx^2 + cx + d$ is obtained by similar reasoning. The only difference is that in case 2 we differentiate between distinct real roots $r_1 \neq r_2$ and a double root $r_1 = r_2$. In the latter case, which is characterized algebraically by the discriminant of P being zero, (2) holds unconditionally, whereas in the former case, where the discriminant of P is positive, it suffices to require the larger of the two roots to be negative or zero.

Theorem 13. *The function $f_{\mathbb{N}}(x) = ax^3 + bx^2 + cx + d$ is weakly monotone and well-defined if $a \geq 0$, $d \geq 0$ and either $4b^2 - 3a^2 - 12ac \leq 0$ or $4b^2 - 3a^2 - 12ac > 0$, $a + b + c \geq 0$ and $3a + 2b \geq 0$.*

In case $a = 0$, i.e., $f_{\mathbb{N}}(x) = bx^2 + cx + d$, Theorem 11 yields exactly the same constraints as Corollary 8, that is, necessary and sufficient constraints. One possible interpretation of this fact is that the simplification we made on our way to Theorem 11 did not cast away anything essential. Indeed, that is the case. To this end, we observe that the only case where (3) holds that is not covered by the constraints of Theorem 11 is when both roots r_1 and r_2 are positive and $[r_1, r_2] \cap \mathbb{N} = \varnothing$; e.g., the polynomial $2x^3 - 6x^2 + 5x$ is both strictly monotone and well-defined, but does not satisfy the constraints of Theorem 11. However, it turns out that this case is very rare; for example, empirical investigations reveal that in the set of polynomials $\{3ax^2 + (3a+2b)x + (a+b+c) \mid 1 \leq a \leq 7, -15 \leq b, c \leq 15 \ (a, b, c \in \mathbb{Z})\}$ 3937 out of a total of 6727 polynomials satisfy (3), but only 25 of them are of this special kind. In other words, the constraints of Theorem 11 comprise 3912 out of 3937, hence almost all, polynomials; and this is way more than the 1792 (= $7\times16\times16$) polynomials that the absolute positiveness approach, where a, b and c are restricted to the non-negative integers, can handle. The following table summarizes all our experiments with varying ranges for a, b and c:

a	b, c	Theorem 11
$[1, 7]$	$[-15, 15]$	3912 of 3937
$[1, 7]$	$[-31, 31]$	14055 of 14133
$[1, 15]$	$[-31, 31]$	34718 of 34980

By design, our approach covers two out of the three possible scenarios mentioned above. But which of these scenarios can the absolute positiveness approach deal with? Just like our method, it fails on all instances of the scenario where the polynomial $P := 3ax^2 + (3a+2b)x + (a+b+c)$ has two positive roots r_1 and r_2, which gives rise to the factorization $P = k(x-r_1)(x-r_2)$, $k > 0$. This expression is equivalent to $kx^2 - k(r_1 + r_2)x + kr_1r_2$, the linear coefficient $-k(r_1 + r_2)$ of which should be equal to $3a + 2b$. Now this gives rise to a contradiction because a and b are restricted to the non-negative integers whereas $-k(r_1 + r_2)$ is a negative number. Concerning the two remaining scenarios, the absolute positiveness approach can handle only some instances of the respective scenarios while failing at others. We present one failing example for either scenario:

- If $a = 1$, $b = -1$ and $c = 1$, then $P = 3x^2 + x + 1$, which has no real roots. Clearly, P is positive for all $x \in \mathbb{N}$; in fact this is even true for all $x \in \mathbb{R}$. However, the absolute positiveness approach fails because b is negative.
- If $a = 3$, $b = -1$ and $c = -1$, then $P = 9x^2 + 7x + 1$, both roots of which are negative real numbers. Clearly, P is positive for all $x \in \mathbb{N}$, but the absolute positiveness approach fails because b and c are negative.

Generalization to Multivariate Cubic Parametric Polynomials

In this subsection, we elaborate on the question how to generalize the result of Theorem 11 to the multivariate case. In general, this is always possible by a very simple approach that we already introduced in Example 10. To this end, let $f_{\mathbb{N}}(x_1, \ldots, x_n)$ denote the n-variate generic cubic parametric polynomial function, and let us note that we can write it as

$$f_{\mathbb{N}}(x_1, \ldots, x_n) = \sum_{j=1}^{n} g_j(x_j) + r(x_1, \ldots, x_n) \tag{6}$$

where $g_j(x_j)$ denotes the univariate generic cubic parametric polynomial function in x_j without constant term and $r(x_1, \ldots, x_n)$ contains all the remaining monomials. Now, let us assume that all the $g_j(x_j)$ are both strictly monotone and well-defined. Then the same also holds for their sum, $\sum_{j=1}^{n} g_j(x_j)$, because they do not share variables. But when is this also true of $f_{\mathbb{N}}$? By Lemma 3, $f_{\mathbb{N}}$ is strictly monotone in its i-th argument if and only if $f_{\mathbb{N}}(x_1, \ldots, x_i + 1, \ldots, x_n) -$

$f_{\mathbb{N}}(x_1, \ldots, x_n) > 0$ for all natural numbers $x_1, \ldots, x_n$. With the help of (6), this simplifies to: $\forall x_1, \ldots, x_n \in \mathbb{N}$

$$g_i(x_i + 1) - g_i(x_i) + r(x_1, \ldots, x_i + 1, \ldots, x_n) - r(x_1, \ldots, x_i, \ldots, x_n) > 0 \quad (7)$$

By assumption, $g_i(x_i+1) - g_i(x_i) > 0$ for all $x_i \in \mathbb{N}$, such that (7) is guaranteed to hold if the second summand, $r(x_1, \ldots, x_i+1, \ldots, x_n) - r(x_1, \ldots, x_i, \ldots, x_n)$, is non-negative for all $x_1, \ldots, x_n \in \mathbb{N}$, that is, if $r(x_1, \ldots, x_n)$ is weakly monotone in all arguments. In other words, strict monotonicity of the functions $(g_j(x_j))_{1 \le j \le n}$ implies strict monotonicity of $f_{\mathbb{N}}$, provided that $r(x_1, \ldots, x_n)$ is weakly monotone in all arguments. Moreover, if all the functions $(g_j(x_j))_{1 \le j \le n}$ are strictly monotone and well-defined, and if $r(x_1, \ldots, x_n)$ is weakly monotone and well-defined, then $f_{\mathbb{N}}$ is strictly monotone and well-defined; and note that we can easily make $r(x_1, \ldots, x_n)$ weakly monotone and well-defined by restricting all its coefficients to be non-negative. Hence, the n-variate generic cubic parametric polynomial function $f_{\mathbb{N}}(x_1, \ldots, x_n)$ is strictly monotone and well-defined if

– all the $g_j(x_j)$ satisfy the constraints of Theorem 11, and
– all coefficients of $r(x_1, \ldots, x_n)$ are non-negative.

Example 14. Consider the bivariate generic cubic parametric polynomial function $f_{\mathbb{N}}(x_1, x_2) = ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3 + ex_1^2 + fx_1x_2 + gx_2^2 + hx_1 + ix_2 + j = g_1(x_1) + g_2(x_2) + r(x_1, x_2)$, where $g_1(x_1) = ax_1^3 + ex_1^2 + hx_1$, $g_2(x_2) = dx_2^3 + gx_2^2 + ix_2$ and $r(x_1, x_2) = bx_1^2x_2 + cx_1x_2^2 + fx_1x_2 + j$. This function is both strictly monotone and well-defined if $ax_1^3 + ex_1^2 + hx_1$ and $dx_2^3 + gx_2^2 + ix_2$ satisfy the constraints of Theorem 11, and the coefficients of $r(x_1, \ldots, x_n)$ are non-negative, i.e., $b, c, f, j \ge 0$.

4 Negative Coefficients in Polynomial Interpretations

In the previous section, we have seen that in principle we may use polynomial interpretations with (some) negative coefficients for proving termination of TRSs. Now the obvious question is the following: Does there exist a TRS that can be proved terminating by a polynomial interpretation with negative coefficients according to Theorems 7 and 11, but cannot be proved terminating by a polynomial interpretation where the coefficients of all polynomials are non-negative?

To elaborate on this question, let us consider the following scenario. Assume we have a TRS whose signature contains (amongst others) the *successor* symbol s, the constant $\mathsf{0}$ and another unary symbol f, and assume that the interpretations associated with the former two are the natural interpretations $\mathsf{s}_{\mathbb{N}}(x) = x+1$ and $\mathsf{0}_{\mathbb{N}} = 0$, whereas f is supposed to be interpreted by $\mathsf{f}_{\mathbb{N}}(x) = ax^2 + bx + c$. Now the idea is to add rules to the TRS which enforce $\mathsf{f}_{\mathbb{N}}(x) = 2x^2 - x + 1$. This can be achieved as follows.

First, note that by polynomial interpolation the coefficients a, b and c of the polynomial function $\mathsf{f}_{\mathbb{N}}(x) = ax^2 + bx + c$ are uniquely determined by the image of $\mathsf{f}_{\mathbb{N}}$ at three pairwise different locations; for example, the constraints $\mathsf{f}_{\mathbb{N}}(0) = 1$,

$\mathsf{f}_{\mathbb{N}}(1) = 2$ and $\mathsf{f}_{\mathbb{N}}(2) = 7$ enforce $\mathsf{f}_{\mathbb{N}}(x) = 2x^2 - x + 1$, as desired. Next we encode these three constraints in terms of the TRS $\mathcal{R}_1$:

$$\begin{array}{ccc} \mathsf{s}^2(\mathsf{0}) \to \mathsf{f}(\mathsf{0}) & \mathsf{s}^3(\mathsf{0}) \to \mathsf{f}(\mathsf{s}(\mathsf{0})) & \mathsf{s}^8(\mathsf{0}) \to \mathsf{f}(\mathsf{s}^2(\mathsf{0})) \\ \mathsf{f}(\mathsf{0}) \to \mathsf{0} & \mathsf{f}(\mathsf{s}(\mathsf{0})) \to \mathsf{s}(\mathsf{0}) & \mathsf{f}(\mathsf{s}^2(\mathsf{0})) \to \mathsf{s}^6(\mathsf{0}) \end{array}$$

Every constraint gives rise to two rewrite rules; e.g., the constraint $\mathsf{f}_{\mathbb{N}}(0) = 1$ is expressed by $\mathsf{f}(\mathsf{0}) \to \mathsf{0}$ and $\mathsf{s}^2(\mathsf{0}) \to \mathsf{f}(\mathsf{0})$. The former encodes $\mathsf{f}_{\mathbb{N}}(0) > 0$, whereas the latter encodes $\mathsf{f}_{\mathbb{N}}(0) < 2$. So these rewrite rules are polynomially terminating by construction, with $\mathsf{f}_{\mathbb{N}}(x) = 2x^2 - x + 1$.

Moreover, we can use $\mathcal{R}_1$ to prove a more general statement that does away with one of the above assumptions. That is to say that any feasible interpretation $\mathsf{f}_{\mathbb{N}}$ must necessarily contain at least one monomial with a negative coefficient. To this end, let us observe that no linear interpretation for f is feasible because the set of points $\{(i, \mathsf{f}_{\mathbb{N}}(i))\}_{i \in \{0,1,2\}}$ is not collinear. The case when $\mathsf{f}_{\mathbb{N}}$ is quadratic was dealt with above. So let us consider interpretations of degree at least three. Then the leading term of $\mathsf{f}_{\mathbb{N}}$ has the shape ax^k, where $a \geq 1$ and $k \geq 3$. Since $\mathsf{f}_{\mathbb{N}}(2) = 7$ must be satisfied, the claim follows immediately because for $x = 2$ the leading term alone contributes a value of at least 8.

Finally, a thorough inspection of the constraints imposed by $\mathcal{R}_1$ reveals that we can also relax the restrictions concerning the interpretations of s and $\mathsf{0}$.

Lemma 15. *In any polynomial interpretation compatible with $\mathcal{R}_1$ that satisfies $\mathsf{s}_{\mathbb{N}}(x) = x + d$ for some $d \in \mathbb{N}$, $\mathsf{f}_{\mathbb{N}}$ must contain at least one monomial with a negative coefficient. In particular, $\mathsf{f}_{\mathbb{N}}$ is not linear.*

Proof. Without loss of generality, let f be interpreted by $\mathsf{f}_{\mathbb{N}}(x) = \sum_{i=0}^{n} a_i x^i$ ($a_n \geq 1$) and $\mathsf{0}$ by some natural number z. Then the compatibility requirement with respect to $\mathcal{R}_1$ gives rise to the following constraints:

$$\begin{aligned} z <{}& \quad \mathsf{f}_{\mathbb{N}}(z) &< z + 2d \\ z + d <{}& \mathsf{f}_{\mathbb{N}}(z + d) &< z + 3d \\ z + 6d <{}& \mathsf{f}_{\mathbb{N}}(z + 2d) &< z + 8d \end{aligned}$$

Hence, d must be a positive integer, i.e., $d \geq 1$. Moreover, no linear interpretation $\mathsf{f}_{\mathbb{N}}(x) = a_1 x + a_0$ satisfies these constraints. To this end, observe that by the first four constraints $a_1 = \frac{\mathsf{f}_{\mathbb{N}}(z+d) - \mathsf{f}_{\mathbb{N}}(z)}{d} < 3$, whereas by the last four constraints $a_1 = \frac{\mathsf{f}_{\mathbb{N}}(z+2d) - \mathsf{f}_{\mathbb{N}}(z+d)}{d} > 3$, which contradicts the former. In other words, the set of points $\{(z + id, \mathsf{f}_{\mathbb{N}}(z + id))\}_{i \in \{0,1,2\}}$ is not collinear. Next we focus on $\mathsf{f}_{\mathbb{N}}(z + 2d) < z + 8d$. Clearly, if the value of the leading term $a_n x^n$ at $x = z + 2d$ is greater than or equal to $z + 8d$, then $\mathsf{f}_{\mathbb{N}}$ must contain at least one monomial with a negative coefficient in order to satisfy $\mathsf{f}_{\mathbb{N}}(z + 2d) < z + 8d$. So, when is $a_n(z + 2d)^n \geq z + 8d$? Considering the worst case, i.e. $a_n = 1$, let us investigate for which integers $n \geq 2$, $z \geq 0$ and $d \geq 1$ the inequality $(z + 2d)^n \geq z + 8d$ holds. If $n \geq 3$, then it holds for all $z \geq 0$ and $d \geq 1$ by the following reasoning $(z + 2d)^n \geq z^n + (2d)^n \geq z + 8d$. For $n = 2$, $(z + 2d)^2 \geq z + 8d$ is equivalent to $z^2 + (4d - 1)z + 4d(d - 2) \geq 0$ which holds for all $z \geq 0$ and $d \geq 1$ except $z = 0$

and $d = 1$. The latter case corresponds to using the natural interpretations for the symbols s and $\mathsf{0}$, namely, $\mathsf{s}_{\mathbb{N}}(x) = x + 1$ and $\mathsf{0}_{\mathbb{N}} = 0$. But then the six rewrite rules require the constraints $\mathsf{f}_{\mathbb{N}}(0) = 1, \mathsf{f}_{\mathbb{N}}(1) = 2$ and $\mathsf{f}_{\mathbb{N}}(2) = 7$, which uniquely determine the coefficients of $\mathsf{f}_{\mathbb{N}}(x) = a_2x^2 + a_1x + a_0$ as $a_2 = 2$, $a_1 = -1$ and $a_0 = 1$ by polynomial interpolation. Hence, $\mathsf{f}_{\mathbb{N}}$ has a negative coefficient. ◻

The result of Lemma 15 relies on the assumption that the function symbol s is interpreted by a linear polynomial $\mathsf{s}_{\mathbb{N}}(x) = x + d$. Our next goal is to do away with this assumption by adding rules that enforce such an interpretation for s.

Lemma 16. *In any polynomial interpretation that is compatible with the rewrite rules* $\mathsf{g}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{g}(x)))$ *and* $\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(x)))$*,* $\mathsf{s}_{\mathbb{N}}$ *and* $\mathsf{g}_{\mathbb{N}}$ *must be linear polynomials. Moreover,* $\mathsf{s}_{\mathbb{N}}(x) = x + d$*, for some* $d > 0$*, and* $\mathsf{f}_{\mathbb{N}}$ *is not linear.*

Proof. Without loss of generality, let us assume that the leading terms of $\mathsf{s}_{\mathbb{N}}(x)$ and $\mathsf{g}_{\mathbb{N}}(x)$ are kx^i and mx^j, respectively, with $k, i, m, j \geq 1$. Then the leading term of the polynomial $P_{\text{lhs}} := \mathsf{g}_{\mathbb{N}}(\mathsf{s}_{\mathbb{N}}(x))$ associated with the left-hand side of the first rule is $m(kx^i)^j = mk^jx^{ij}$. Likewise, the leading term of the corresponding polynomial $P_{\text{rhs}} := \mathsf{s}_{\mathbb{N}}(\mathsf{s}_{\mathbb{N}}(\mathsf{g}_{\mathbb{N}}(x)))$ is $k(k(mx^j)^i)^i = k^{i+1}m^{i^2}x^{i^2j}$. Compatibility demands that the degree of the former must be greater than or equal to the degree of the latter, i.e., $ij \geq i^2j$. This condition holds if and only if $i = 1$. Repeating this reasoning for the second rule yields $j = 1$. Substituting these values into the leading terms of P_{lhs} and P_{rhs}, we get mkx and k^2mx, respectively. Hence, P_{lhs} and P_{rhs} have the same degree, such that, in order to ensure compatibility, the leading coefficient of the former must be greater than or equal to the leading coefficient of the latter, i.e., $mk \geq k^2m$. Since $m > 0$ and $k > 0$, this condition is equivalent to $k \leq 1$ and hence $k = 1$. Therefore $\mathsf{s}_{\mathbb{N}}(x) = x + d$. Clearly, $d \neq 0$. Finally, let us assume that f is interpreted by a linear polynomial $\mathsf{f}_{\mathbb{N}}$. Repeating the above reasoning for the second rule yields $\mathsf{g}_{\mathbb{N}}(x) = x + d'$. However, such an interpretation is not compatible with the first rule. Hence, $\mathsf{f}_{\mathbb{N}}$ cannot be linear. ◻

Having all the relevant ingredients at hand, we are now ready to state the main theorem of this section, which also gives an affirmative answer to the question posed at the beginning of the section; that is, there are TRSs that can be proved terminating by a polynomial interpretation with negative coefficients, but cannot be proved terminating by a polynomial interpretation where the coefficients of all polynomials are non-negative.

Theorem 17. *Consider the TRS* $\mathcal{R}_1$ *extended with the rewrite rules* $\mathsf{g}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{g}(x)))$ *and* $\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(x)))$*. In any compatible polynomial interpretation,* $\mathsf{f}_{\mathbb{N}}$ *must contain at least one monomial with a negative coefficient.*

Proof. By Lemmata 15 and 16. ◻

Specifying Interpretations by Interpolation

Now let us revisit the motivating scenario presented at the beginning of this section, in which we leveraged polynomial interpolation to create the TRS $\mathcal{R}_1$ in such a way that it enforces the function symbol f to be interpreted by $\mathsf{f}_{\mathbb{N}}(x) =$

$2x^2 - x + 1$, a polynomial of our choice. The construction presented there was based on three assumptions:

1. the successor symbol s had to be interpreted by $\mathsf{s}_{\mathbb{N}}(x) = x + 1$,
2. the constant $\mathsf{0}$ had to be interpreted by $\mathsf{0}_{\mathbb{N}} = 0$,
3. the function symbol f had to be interpreted by a quadratic polynomial.

Next we show how one can enforce all these assumptions by adding suitable rewrite rules to $\mathcal{R}_1$. This results in a TRS that is polynomially terminating, but only if the symbols s, f and $\mathsf{0}$ are interpreted accordingly (cf. Theorem 23). However, much to our surprise, most of the current termination tools with all their advanced termination techniques fail to prove this TRS terminating (cf. Section 5) in their automatic mode.

Concerning the first two of the above assumptions, it turns out that the constraints imposed by $\mathcal{R}_1$ alone suffice to do away with them, provided that the successor symbol is interpreted by a linear polynomial of the form $x + d$ (which poses no problem according to Lemma 16). This is the result of the next lemma.

Lemma 18. *In any polynomial interpretation compatible with $\mathcal{R}_1$ such that $\mathsf{s}_{\mathbb{N}}(x) = x + d$ and the degree of $\mathsf{f}_{\mathbb{N}}$ is at most two, the constant $\mathsf{0}$ must be interpreted by 0. Moreover, $d = 1$ and $\mathsf{f}_{\mathbb{N}}$ is not linear.*

Proof. Without loss of generality, $\mathsf{f}_{\mathbb{N}}(x) = ax^2 + bx + c$ subject to the constraints $a, c \geq 0$ and $a + b > 0$ (cf. Corollary 8). By Lemma 15, $\mathsf{f}_{\mathbb{N}}$ is not linear; hence $a \geq 1$. Writing z for $\mathsf{0}_{\mathbb{N}}$, the compatibility requirement yields

$$\begin{aligned} z < {} & \mathsf{f}_{\mathbb{N}}(z) < z + 2d \\ z + d < {} & \mathsf{f}_{\mathbb{N}}(z + d) < z + 3d \\ z + 6d < {} & \mathsf{f}_{\mathbb{N}}(z + 2d) < z + 8d \end{aligned}$$

Hence, d must be a positive integer, i.e., $d \geq 1$. Next we focus on the constraint $\mathsf{f}_{\mathbb{N}}(z + d) < z + 3d$ and try to derive a contradiction assuming $z \geq 1$. We reason as follows: $\mathsf{f}_{\mathbb{N}}(z+d) - \mathsf{f}_{\mathbb{N}}(z) = d(2az + ad + b) \geq d(2az + a + b) \geq d(2az + 1) \geq 3d$. Hence, $\mathsf{f}_{\mathbb{N}}(z + d) \geq \mathsf{f}_{\mathbb{N}}(z) + 3d$, which contradicts $\mathsf{f}_{\mathbb{N}}(z + d) < z + 3d$ together with the first of the above constraints $\mathsf{f}_{\mathbb{N}}(z) > z$. As a consequence, $z = \mathsf{0}_{\mathbb{N}} = 0$. Finally, it remains to show that d must be 1. We already know that d must be at least 1. So let us assume that $d \geq 2$ and derive a contradiction with respect to the constraint $\mathsf{f}_{\mathbb{N}}(z + 2d) < z + 8d$. This can be achieved as follows: $\mathsf{f}_{\mathbb{N}}(z+2d) = \mathsf{f}_{\mathbb{N}}(2d) = 4ad^2 + 2bd + c \geq 4ad^2 + 2bd = d(4ad + 2b) = d((4d-2)a + 2(a + b)) \geq d(6a + 2(a + b)) \geq d(6a + 2) \geq 8d = z + 8d$. ☐

Next we will elaborate on how to get rid of the assumption that the function symbol f has to be interpreted by a polynomial $\mathsf{f}_{\mathbb{N}}$ of degree at most two. Again, the idea is to enforce this condition by some additional rewrite rules based on the following observation. If $\mathsf{f}_{\mathbb{N}}$ is at most quadratic, then the function $\mathsf{f}_{\mathbb{N}}(x+d) - \mathsf{f}_{\mathbb{N}}(x)$ is at most linear; i.e., there is a linear function $\mathsf{r}_{\mathbb{N}}(x)$ such that $\mathsf{r}_{\mathbb{N}}(x) > \mathsf{f}_{\mathbb{N}}(x + d) - \mathsf{f}_{\mathbb{N}}(x)$, or equivalently, $\mathsf{f}_{\mathbb{N}}(x) + \mathsf{r}_{\mathbb{N}}(x) > \mathsf{f}_{\mathbb{N}}(x + d)$, for all $x \in \mathbb{N}$. This can be encoded in terms of the rewrite rule $\mathsf{h}(\mathsf{f}(x), \mathsf{r}(x)) \to \mathsf{f}(\mathsf{s}(x))$, as soon as the interpretation of h corresponds to the addition of two natural numbers. Yet this does not pose a major problem, as will be shown shortly.

Remark 19. Note that the construction motivated above is actually more general than it seems at first sight. That is, it can be used to set arbitrary upper bounds on the degree of an interpretation (cf. proof of Lemma 20). Moreover, it can easily be adapted to establish lower bounds.

Lemma 20. *Consider the rewrite rule* $\mathsf{h}(\mathsf{f}(x), \mathsf{r}(x)) \to \mathsf{f}(\mathsf{s}(x))$. *In any compatible polynomial interpretation where* $\mathsf{s}_{\mathbb{N}}(x) = x + d$ $(d \geq 1)$, $\mathsf{r}_{\mathbb{N}}$ *is some linear polynomial, and* $\mathsf{h}_{\mathbb{N}}(x, y) = x + y + p$ $(p \in \mathbb{N})$, *the degree of* $\mathsf{f}_{\mathbb{N}}$ *is at most two.*

Proof. Without loss of generality, let $\mathsf{f}_{\mathbb{N}}(x) = \sum_{i=0}^{n} a_i x^i$ $(a_n > 0)$. By compatibility with the single rewrite rule, the inequality

$$\mathsf{f}_{\mathbb{N}}(x) + \mathsf{r}_{\mathbb{N}}(x) + p > \mathsf{f}_{\mathbb{N}}(x + d) \tag{8}$$

must be satisfied for all $x \in \mathbb{N}$. Using Taylor's theorem,

$$\mathsf{f}_{\mathbb{N}}(x + d) = \sum_{k=0}^{n} \frac{d^k}{k!} \mathsf{f}_{\mathbb{N}}^{(k)}(x) = \mathsf{f}_{\mathbb{N}}(x) + d\, \mathsf{f}_{\mathbb{N}}'(x) + \frac{d^2}{2} \mathsf{f}_{\mathbb{N}}''(x) + \ldots + \frac{d^n}{n!} \mathsf{f}_{\mathbb{N}}^{(n)}(x)$$

we can simplify (8) to

$$\mathsf{r}_{\mathbb{N}}(x) + p > d\, \mathsf{f}_{\mathbb{N}}'(x) + \sum_{k=2}^{n} \frac{d^k}{k!} \mathsf{f}_{\mathbb{N}}^{(k)}(x) \tag{9}$$

As $d \geq 1$, the right-hand side of this inequality is a polynomial of degree $n - 1$ whose leading coefficient dna_n is positive, whereas the degree of the left-hand side is one. But by compatibility, the former must be greater than or equal to $n - 1$; i.e., $n \leq 2$. $\square$

It remains to show how the interpretation of h can be fixed to addition.

Lemma 21. *Consider the TRS* $\mathcal{R}_2$ *consisting of the rules*

$$\mathsf{g}(x) \to \mathsf{h}(x, x) \qquad \mathsf{s}(x) \to \mathsf{h}(x, \mathsf{0}) \qquad \mathsf{s}(x) \to \mathsf{h}(\mathsf{0}, x)$$

Any compatible polynomial interpretation that interprets s *by* $\mathsf{s}_{\mathbb{N}}(x) = x + d$ *and* g *by a linear polynomial satisfies* $\mathsf{h}_{\mathbb{N}}(x, y) = x + y + p$, $p \in \mathbb{N}$. *Moreover, if* $d = 1$, *then* $p = 0$ *and* $\mathsf{0}_{\mathbb{N}} = 0$.

Proof. Without loss of generality, let $\mathsf{0}_{\mathbb{N}} = z$ for some $z \in \mathbb{N}$. Because $\mathsf{g}_{\mathbb{N}}$ is linear, compatibility with the first rule constrains the function $h' \colon \mathbb{N} \to \mathbb{N}, x \mapsto \mathsf{h}_{\mathbb{N}}(x, x)$ to be at most linear. This can only be the case if $\mathsf{h}_{\mathbb{N}}$ contains no monomials of degree two or higher. In other words, $\mathsf{h}_{\mathbb{N}}(x, y) = p_x \cdot x + p_y \cdot y + p$, where $p \in \mathbb{N}$ (because of well-definedness), $p_x \geq 1$ and $p_y \geq 1$ (because of strict monotonicity). Then compatibility with the second rule translates to $x + d > p_x \cdot x + p_y \cdot z + p$ for all $x \in \mathbb{N}$, which holds if and only if $p_x \leq 1$ and $d > p_y \cdot z + p$. Hence, $p_x = 1$, and by analogous reasoning with respect to the third rule, $p_y = 1$. Finally, if $d = 1$ then the condition $d > p_y \cdot z + p$ simplifies to $1 > z + p$, which holds if and only if $z = 0$ and $p = 0$ (because both z and p are non-negative). $\square$

Corollary 22. *Consider the TRS $\mathcal{R}_3$ consisting of the rules*

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(x))) \qquad \mathsf{g}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{g}(x))) \qquad \mathsf{h}(\mathsf{f}(x), \mathsf{g}(x)) \to \mathsf{f}(\mathsf{s}(x))$$

Any polynomial interpretation compatible with $\mathcal{R}_2 \cup \mathcal{R}_3$ requires degree at most two for $\mathsf{f}_{\mathbb{N}}$.

Proof. By Lemma 16, $\mathsf{s}_{\mathbb{N}}(x) = x + d$ $(d \geq 1)$ and $\mathsf{g}_{\mathbb{N}}(x)$ is linear. Hence, $\mathsf{h}_{\mathbb{N}}(x, y) = x + y + p$ $(p \in \mathbb{N})$ according to Lemma 21. Finally, Lemma 20 applied to the rule $\mathsf{h}(\mathsf{f}(x), \mathsf{g}(x)) \to \mathsf{f}(\mathsf{s}(x))$ proves the claim. ☐

Now, combining Lemma 18 and Corollary 22 yields natural semantics for the symbols $\mathsf{0}$ and s and fixes $\mathsf{f}_{\mathbb{N}}(x) = 2x^2 - x + 1$, as originally desired.

Theorem 23. *Any polynomial interpretation compatible with $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ interprets $\mathsf{0}$ by 0 and s by $\mathsf{s}_{\mathbb{N}}(x) = x + 1$. Moreover, $\mathsf{f}_{\mathbb{N}}(x) = 2x^2 - x + 1$.*

Proof. From Lemmata 18 and 16 and Corollary 22, we infer that $\mathsf{0}_{\mathbb{N}} = 0$, $\mathsf{s}_{\mathbb{N}}(x) = x + 1$ and $\mathsf{f}_{\mathbb{N}}$ is a quadratic polynomial. Without loss of generality, $\mathsf{f}_{\mathbb{N}}(x) = ax^2 + bx + c$. Next we observe that $\mathcal{R}_1$ gives rise to the constraints $\mathsf{f}_{\mathbb{N}}(0) = 1$, $\mathsf{f}_{\mathbb{N}}(1) = 2$ and $\mathsf{f}_{\mathbb{N}}(2) = 7$, which uniquely determine the coefficients $a = 2$, $b = -1$ and $c = 1$ of $\mathsf{f}_{\mathbb{N}}$ by polynomial interpolation. ☐

In order for Theorem 23 to be relevant, it remains to show that there actually exists a compatible polynomial interpretation. This is achieved, e.g., by defining $\mathsf{0}_{\mathbb{N}} = 0$, $\mathsf{s}_{\mathbb{N}}(x) = x + 1$, $\mathsf{f}_{\mathbb{N}}(x) = 2x^2 - x + 1$, $\mathsf{h}_{\mathbb{N}}(x, y) = x + y$ and $\mathsf{g}_{\mathbb{N}}(x) = 4x + 5$.

Remark 24. One can show that any polynomial interpretation compatible with the TRS $\mathcal{S} := \mathcal{R}_2 \cup \mathcal{R}_3 \cup \{\, \mathsf{s}(\mathsf{s}(\mathsf{0})) \to \mathsf{f}(\mathsf{s}(\mathsf{0})) \,\}$ must interpret $\mathsf{0}$ by 0 and s by $\mathsf{s}_{\mathbb{N}}(x) = x + 1$. Thus we can take our favourite univariate polynomial P, which must of course be both strictly monotone and well-defined, and design a TRS such that the interpretation of some unary function symbol k is fixed to it. To this end, we extend $\mathcal{S}$ by suitable rewrite rules encoding interpolation constraints for the symbol k and additional rules that set an upper bound on the degree of the interpretation of k, which corresponds to the degree of P.

5 Experimental Results

We implemented the criterion from Theorem 7 in the termination prover T_TT₂.[2] The problem of finding suitable coefficients for the polynomials is formulated as a set of diophantine constraints (as in [4]) which are solved by a transformation to SAT. Simple heuristics are applied to decide which symbols should be interpreted by non-linear polynomials (e.g., defined function symbols, symbols that appear at most once on every left and right-hand side, symbols that do not appear nested). Using coefficients in $\{-8, \ldots, 7\}$ and either of the latter two heuristics, TTT2 finds a compatible interpretation (i.e., the one mentioned at the end of

[2] See `http://termination-portal.org/wiki/Category:Tools`.

Section 4) for the TRS in Theorem 23 fully automatically within five seconds. We remark that implementing Theorems 7 and 11 is about as expensive as the absolute positiveness approach, since the size of the search space is mainly determined by the degree of the polynomials.

Despite the tremendous progress in automatic termination proving during the last decade, it is remarkable that the other powerful termination tools AProVE^2 and JAMBOX^2 cannot prove this system terminating within ten minutes. The same holds for $\mathsf{T_TT_2}$ without the criterion from Theorem 7. Surprisingly, the 2006 version of TPA^2 finds a lengthy termination proof based on semantic labeling. However, it is straightforward to generate a variant of the TRS from Theorem 23 that is orientable if $\mathsf{f}_{\mathbb{N}}(0) = 0$, $\mathsf{f}_{\mathbb{N}}(1) = 1$, and $\mathsf{f}_{\mathbb{N}}(2) = 8$, suggesting $\mathsf{f}_{\mathbb{N}}(x) = 3x^2 - 2x$. While $\mathsf{T_TT_2}$ can still prove this system terminating, TPA now also fails. Moreover, due to Remark 24 one can generate myriads of TRSs that can easily be shown to be polynomially terminating but where an automated termination proof is out of reach for current termination analyzers.

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References

1. Arts, T., Giesl, J.: Termination of term rewriting using dependency pairs. TCS 236(1-2), 133–178 (2000)
2. Contejean, E., Marché, C., Tomás, A.P., Urbain, X.: Mechanically proving termination using polynomial interpretations. JAR 34(4), 325–363 (2005)
3. Dershowitz, N.: A note on simplification orderings. IPL 9(5), 212–215 (1979)
4. Fuhs, C., Giesl, J., Middeldorp, A., Schneider-Kamp, P., Thiemann, R., Zankl, H.: SAT solving for termination analysis with polynomial interpretations. In: Proc. 10th SAT. LNCS, vol. 4501, pp. 340–354 (2007)
5. Fuhs, C., Giesl, J., Middeldorp, A., Schneider-Kamp, P., Thiemann, R., Zankl, H.: Maximal termination. In: Proc. 19th RTA. LNCS, vol. 5117, pp. 110–125 (2008)
6. Giesl, J., Thiemann, R., Schneider-Kamp, P.: The dependency pair framework: Combining techniques for automated termination proofs. In: Proc. 11th LPAR. LNAI, vol. 3452, pp. 301–331 (2005)
7. Hirokawa, N., Middeldorp, A.: Automating the dependency pair method. I&C 199(1-2), 172–199 (2005)
8. Hirokawa, N., Middeldorp, A.: Tyrolean Termination Tool: Techniques and features. I&C 205(4), 474–511 (2007)
9. Hong, H., Jakuš, D.: Testing positiveness of polynomials. JAR 21(1), 23–38 (1998)
10. Lankford, D.: On proving term rewrite systems are noetherian. Tech. Rep. MTP-3, Louisiana Technical University, Ruston (1979)
11. Lucas, S.: Polynomials over the reals in proofs of termination: From theory to practice. TIA 39(3), 547–586 (2005)