Revisiting Matrix Interpretations for Polynomial Derivational Complexity of Term Rewriting*

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Abstract. Matrix interpretations can be used to bound the derivational complexity of term rewrite systems. In particular, triangular matrix interpretations over the natural numbers are known to induce polynomial upper bounds on the derivational complexity of (compatible) rewrite systems. Using techniques from linear algebra, we show how one can generalize the method to matrices that are not necessarily triangular but nevertheless polynomially bounded. Moreover, we show that our approach also applies to matrix interpretations over the real (algebraic) numbers. In particular, it allows triangular matrix interpretations to infer tighter bounds than the original approach.

Key words: derivational complexity, polynomial matrix interpretations

1 Introduction

Many powerful techniques for establishing termination of term rewrite systems have been developed in the course of time, most of which have been automated successfully, as is evident in the results of the (annual) international competition for termination and complexity tools.[1] Moreover, Hofbauer and Lautemann observe in [9] that "proving termination with one of these specific techniques in general proves more than just the absence of infinite derivations. It turns out that in many cases such a proof implies an upper bound on the maximal length of derivations", which they consider as a natural measure for the complexity of (terminating) term rewrite systems. More precisely, the resulting notion of *derivational complexity* relates the length of a longest derivation to the size of its initial term. For example, polynomial interpretations imply a double-exponential upper bound on the derivational complexity [9]. However, since term rewriting is a model of computation and algorithms of polynomial complexity are widely accepted as *feasible*, one is especially interested in polynomial derivational complexity. But currently only few techniques for establishing feasible upper complexity bounds are known. Commonly, they are stripped-down variants of existing termination techniques. For example, if a term rewrite system can be shown terminating by a matrix interpretation (over the natural numbers) [5, 10] that orients all rewrite rules strictly, then its derivational complexity is

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[1] `http://termcomp.uibk.ac.at`

at most exponential. However, by restricting the shape of the matrices to upper triangular form, one obtains a method for establishing polynomial derivational complexity [13], where the degree of the polynomial depends on the dimension of the matrices. Using match-bounds [7] or arctic matrix interpretations [12], linear derivational complexity can be inferred.

In this paper we investigate the method of (triangular) matrix interpretations that is widely used in current automated termination and complexity tools. Using techniques from linear algebra, we show how one can generalize the method of triangular matrix interpretations, as introduced in [13], to matrix interpretations that are not necessarily triangular but nevertheless induce polynomial upper bounds on the derivational complexity of compatible term rewrite systems. Moreover, we show that our approach also applies to matrix interpretations over the real (algebraic) numbers. In particular, we also show how one can infer tighter bounds from triangular matrix interpretations by examining the diagonal structure of upper triangular (complexity) matrices.

The remainder of this paper is organized as follows. Section 2 introduces basic notions of term rewriting and some mathematical prerequisites. In Section 3, we review matrix interpretations in the context of complexity analysis of term rewriting, before presenting our main result in Section 4. In Section 5, we give details on implementation-specific issues. Finally, we provide experimental results in Section 6, before concluding in Section 7.

2 Preliminaries

We assume familiarity with the basics of term rewriting [2, 17]. Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a fixed-arity signature. The set of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$. The *size* $|t|$ of a term t is defined as the number of symbols occurring in it and the *depth* of t is defined as follows: if t is a variable or a constant, then $\mathrm{depth}(t) := 0$, otherwise $\mathrm{depth}(f(t_1,\dots,t_n)) := 1 + \max\{\mathrm{depth}(t_i) \mid 1 \leqslant i \leqslant n\}$. A *rewrite rule* is a pair of terms written as $l \to r$, such that l is not a variable and all variables in r are contained in l. A *term rewrite system* $\mathcal{R}$ (TRS for short) over $\mathcal{T}(\mathcal{F},\mathcal{V})$ is a set of rewrite rules. For complexity analysis we assume TRSs to be finite. The rewrite relation induced by $\to$ is denoted by $\to_{\mathcal{R}}$. As usual, $\to_{\mathcal{R}}^*$ denotes the reflexive transitive closure of $\to_{\mathcal{R}}$ and $\to_{\mathcal{R}}^n$ its n-th iterate. A term $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ is called a *normal form* if there is no term t such that $s \to_{\mathcal{R}} t$.

The *derivation height* of a term t with respect to a TRS $\mathcal{R}$ is defined as follows: $\mathsf{dh}(t, \to_{\mathcal{R}}) := \max\{n \mid \exists u\; t \to_{\mathcal{R}}^n u\}$. The *derivational complexity* function of a terminating TRS $\mathcal{R}$ computes the maximal derivation height of all terms up to a given size, i.e., $\mathsf{dc}_{\mathcal{R}} \colon \mathbb{N} \setminus \{0\} \to \mathbb{N}, k \mapsto \max\{\mathsf{dh}(t, \to_{\mathcal{R}}) \mid |t| \leqslant k\}$. Sometimes we say that $\mathcal{R}$ has linear, quadratic, etc. derivational complexity if $\mathsf{dc}_{\mathcal{R}}(k)$ can be bounded by a linear, quadratic, etc. polynomial in k.

An important concept for establishing termination of TRSs is the notion of well-founded monotone algebras. An $\mathcal{F}$-*algebra* $\mathcal{A}$ consists of a non-empty carrier A and interpretation functions $f_{\mathcal{A}} \colon A^n \to A$ for every n-ary $f \in \mathcal{F}$.

By $[\alpha]_{\mathcal{A}}(\cdot)\colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to A$ we denote the usual evaluation function of $\mathcal{A}$ with respect to a variable assignment $\alpha\colon \mathcal{V} \to A$. A *well-founded monotone $\mathcal{F}$-algebra* is a pair $(\mathcal{A}, >_A)$, where $\mathcal{A}$ is an $\mathcal{F}$-algebra and $>_A$ is a well-founded order on A such that every $f_{\mathcal{A}}$ is strictly monotone in all arguments (with respect to $>_A$). A well-founded monotone algebra naturally induces an order $\succ_{\mathcal{A}}$ on terms: $s \succ_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) >_A [\alpha]_{\mathcal{A}}(t)$ for all assignments α of elements of A to the variables in s and t. Finally, it is well-known that a TRS $\mathcal{R}$ is terminating if and only if it is *compatible* with a well-founded monotone algebra $(\mathcal{A}, >_A)$, where compatibility means that $l \succ_{\mathcal{A}} r$ for every rewrite rule $l \to r \in \mathcal{R}$.

Linear Algebra. As usual, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of natural, integer, rational and real numbers. Given some $D \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and $m \in D$, $>_D$ denotes the natural order of the respective domain and $D_m := \{x \in D \mid x \geqslant m\}$; e.g., $\mathbb{R}_0$ refers to the set of all non-negative real numbers. For any ring R (e.g., $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$), we denote the ring of all n-dimensional square matrices over R by $R^{n\times n}$, and $R[x_1,\ldots,x_n]$ denotes the associated *polynomial ring* in n *indeterminates* $x_1,\ldots,x_n$. In the special case $n = 1$, a polynomial $P \in R[x]$ can be written as follows: $P(x) = \sum_{k=0}^{d} a_k x^k$ ($d \in \mathbb{N}$). For the largest k such that $a_k \neq 0$, we call $a_k x^k$ the *leading term* of P, a_k its *leading coefficient* and k its *degree*. P is said to be *monic* if its leading coefficient is one. Moreover, it is said to be *linear*, *quadratic*, *cubic* etc. if its degree is one, two, three etc.

We say that a matrix is *non-negative* if all its entries are non-negative. Abusing notation, we denote the set of all non-negative n-dimensional square matrices of $\mathbb{Z}^{n\times n}$ by $\mathbb{N}^{n\times n}$. An *upper triangular* matrix is a matrix, where all entries below the main diagonal are zero. An *upper triangular complexity* matrix is a non-negative upper triangular matrix whose diagonal entries are at most one and whose top-left entry is exactly one. As usual, we denote the *transpose* of a matrix (vector) A by A^T. The *characteristic polynomial* of a square matrix $A \in R^{n\times n}$ is defined as $\chi_A(\lambda) := \det(\lambda I_n - A)$, where I_n denotes the n-dimensional identity matrix and det the determinant of a matrix. It is monic and its degree is n. The equation $\chi_A(\lambda) = 0$ is called the *characteristic equation* of A. The *eigenvalues* of A are precisely the solutions of its characteristic equation, and the *spectral radius* $\rho(A)$ of A is the maximum of the absolute values of all eigenvalues. By m_λ we denote the *multiplicity* of the eigenvalue λ. A non-zero vector x is an *eigenvector* of A if $Ax = \lambda x$ for some eigenvalue λ of A. The Cayley-Hamilton theorem [15] states that every matrix satisfies its own characteristic equation, that is, $\chi_A(A) = 0$, and it holds for square matrices over commutative rings.

Recurrence Relations. Informally, a recurrence relation is an equation that recursively defines a sequence; each element of the sequence is defined as a function of the preceding elements. For example, the Fibonacci numbers are defined by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. Solving a recurrence relation means obtaining a closed-form solution; in this example, a non-recursive function of n.

A *linear homogeneous recurrence relation with constant coefficients* is an equation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$, where the $d \geqslant 1$ *coefficients* $c_1,\ldots,c_d$ are constants with $c_d \neq 0$. The same coefficients yield the *characteristic polynomial* $\chi(\lambda) := \lambda^d - c_1\lambda^{d-1} - c_2\lambda^{d-2} - \cdots - c_d$ whose d roots play a key role in the solution of a recurrence relation (cf. [3,4]). To be precise, if $\lambda_1, \lambda_2, \ldots, \lambda_r$ $(1 \leqslant r \leqslant d)$ are the distinct (possibly complex) roots of the characteristic polynomial such that λ_i is of multiplicity m_i $(i = 1, 2, \ldots, r)$, then the general solution of the recurrence relation is given by

$$a_n = \sum_{i=1}^{r}(c_{i1} + c_{i2}n + \cdots + c_{im_i}n^{m_i-1})\lambda_i^n$$

where the c_{ik}'s are (complex) constants. Any real solution is of this form as well, with the imaginary part zero. Moreover, if the coefficients of $\chi(\lambda)$ are real numbers, its non-real roots always come in conjugate pairs; i.e., if $\lambda_j := r_j(\cos(\phi_j) + i\sin(\phi_j))$ is a root of $\chi(\lambda)$, then so is its complex conjugate $\lambda_j^* := r_j(\cos(\phi_j) - i\sin(\phi_j))$. In this case, avoiding the use of complex numbers, the most general real solution can be written as

$$\begin{aligned} a_n = &\sum_i (c_{i1} + c_{i2}n + \cdots + c_{im_i}n^{m_i-1})\lambda_i^n \\ &+ \sum_j (d_{j1} + d_{j2}n + \cdots + d_{jm_j}n^{m_j-1})r_j^n\cos(n\phi_j) \\ &+ \sum_j (d'_{j1} + d'_{j2}n + \cdots + d'_{jm_j}n^{m_j-1})r_j^n\sin(n\phi_j) \end{aligned}$$

where the c_{ik}'s, d_{jk}'s and d'_{jk}'s are real constants, the λ_i's the distinct real roots of $\chi(\lambda)$ and the λ_j's, $\lambda_j := r_j(\cos(\phi_j) + i\sin(\phi_j))$, the distinct complex roots (modulo conjugates).

3 Matrix Interpretations and Derivational Complexity

Next we review the method of matrix interpretations in the context of complexity analysis of term rewriting. Matrix interpretations [5,10] were originally introduced over the natural numbers. Later on they were lifted to the reals [1,6,21] using the same technique that was already used to lift polynomial interpretations from $\mathbb{N}$ to $\mathbb{R}$ (cf. [8]). Similarly, the first results relating matrix interpretations and derivational complexity of TRSs (cf. [13], triangular matrix interpretations) are based on matrix interpretations over the natural numbers. But these results have never been lifted to the reals. In the next section we shall see, however, how this follows from a more general result that holds for matrix interpretations over both $\mathbb{N}$ and $\mathbb{R}$, the foundations of which are laid in the present chapter.

Let $\mathcal{F}$ denote a signature. A *matrix interpretation* $\mathcal{M}$ over $\mathbb{N}$ is a well-founded monotone algebra, where the carrier M is the set $\mathbb{N}^n$ for some fixed dimension $n \in \mathbb{N} \setminus \{0\}$. The well-founded order $>_M$ on M is defined as follows:

$$(x_1, x_2, \ldots, x_n)^T >_M (y_1, y_2, \ldots, y_n)^T :\Longleftrightarrow x_1 >_{\mathbb{N}} y_1 \wedge x_2 \geqslant_{\mathbb{N}} y_2 \wedge \cdots \wedge x_n \geqslant_{\mathbb{N}} y_n$$

For each k-ary function symbol $f \in \mathcal{F}$, we choose an interpretation function

$$f_{\mathcal{M}} \colon (\mathbb{N}^n)^k \to \mathbb{N}^n, (\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}) \mapsto F_1\boldsymbol{x_1} + \cdots + F_k\boldsymbol{x_k} + \boldsymbol{f}$$

where $\boldsymbol{f} \in \mathbb{N}^n$ and $F_1, \ldots, F_k \in \mathbb{N}^{n \times n}$. In addition, we require $(F_i)_{1,1} \geqslant_{\mathbb{N}} 1$ for all $i = 1, \ldots, k$ to achieve strict monotonicity of $f_{\mathcal{M}}$ in all arguments. Finally, a *triangular matrix interpretation* over $\mathbb{N}$ is a matrix interpretation over $\mathbb{N}$, where all matrices are upper triangular complexity matrices.

When extending matrix interpretations from $\mathbb{N}$ to $\mathbb{R}$, the main problem is the non-well-foundedness of $>_{\mathbb{R}}$. This problem is overcome by $>_{\mathbb{R},\delta}$, which is defined as follows: given some fixed positive real number δ, $x >_{\mathbb{R},\delta} y$ if and only if $x - y \geqslant_{\mathbb{R}} \delta$ for all $x, y \in \mathbb{R}$. Thus $>_{\mathbb{R},\delta}$ is well-founded on subsets of $\mathbb{R}$ that are bounded from below. Then a *matrix interpretation* $\mathcal{M}$ over $\mathbb{R}$ is a well-founded monotone algebra, where the carrier M is the set $\mathbb{R}_0^n$ for some fixed dimension $n \in \mathbb{N} \setminus \{0\}$. The well-founded order $>_M$ on M is defined as follows:

$$(x_1, x_2, \ldots, x_n)^T >_M (y_1, y_2, \ldots, y_n)^T :\Longleftrightarrow x_1 >_{\mathbb{R},\delta} y_1 \wedge x_2 \geqslant_{\mathbb{R}} y_2 \wedge \cdots \wedge x_n \geqslant_{\mathbb{R}} y_n$$

For each k-ary function symbol f, we choose an interpretation function

$$f_{\mathcal{M}} \colon (\mathbb{R}_0^n)^k \to \mathbb{R}_0^n, (\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}) \mapsto F_1\boldsymbol{x_1} + \ldots + F_k\boldsymbol{x_k} + \boldsymbol{f}$$

where $\boldsymbol{f} \in \mathbb{R}_0^n$ and $F_1, \ldots, F_k$ are non-negative matrices in $\mathbb{R}^{n \times n}$ with $(F_i)_{1,1} \geqslant_{\mathbb{R}} 1$ for all $i = 1, \ldots, k$ in order to achieve strict monotonicity of $f_{\mathcal{M}}$ in all arguments. Again, a *triangular matrix interpretation* over $\mathbb{R}$ is a matrix interpretation over $\mathbb{R}$, where all matrices are upper triangular complexity matrices.

Remark 1. Concerning polynomial interpretations, it was recently shown in [14] that it suffices to consider the set $\mathbb{R}_{\mathsf{alg}}$ of real *algebraic*[2] numbers instead of the entire set $\mathbb{R}$ of real numbers. To be precise, it was shown that polynomial termination over $\mathbb{R}$ is equivalent to polynomial termination over $\mathbb{R}_{\mathsf{alg}}$. Observing that the technique of [14] readily applies to matrix interpretations as well, we may draw the conclusion that matrix interpretations over $\mathbb{R}$ are equivalent to matrix interpretations over $\mathbb{R}_{\mathsf{alg}}$ with respect to proving termination of TRSs.

Matrix interpretations over $\mathbb{R}$ can be used to bound the derivational complexity of compatible TRSs.[3] Let $\mathcal{M}$ be a matrix interpretation over $\mathbb{R}$ that is compatible with some TRS $\mathcal{R}$. Then any rewrite sequence

$$t = t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} t_4 \to_{\mathcal{R}} \cdots$$

gives rise to a strictly decreasing sequence of vectors of non-negative real numbers

$$[\alpha]_{\mathcal{M}}(t) >_M [\alpha]_{\mathcal{M}}(t_1) >_M [\alpha]_{\mathcal{M}}(t_2) >_M [\alpha]_{\mathcal{M}}(t_3) >_M [\alpha]_{\mathcal{M}}(t_4) >_M \cdots$$

[2] A real number is said to be algebraic if it is a root of a non-zero polynomial in one variable with integer coefficients.

[3] The reasoning presented in the sequel readily includes matrix interpretations over $\mathbb{N}$ as a special case (by letting $\delta = 1$ and observing that $x >_{\mathbb{N}} y$ if and only if $x \geqslant_{\mathbb{N}} y+1$).

for all variable assignments α. In particular, by definition of $>_M$, the first components of these vectors form a sequence of non-negative real numbers that is strictly decreasing with respect to the order $>_{\mathbb{R},\delta}$, and every rewrite step causes a decrease of at least δ. Hence, the first component of the vector $\frac{1}{\delta} \cdot [\alpha]_{\mathcal{M}}(t)$ gives an upper bound on $\mathsf{dh}(t, \to_{\mathcal{R}})$. So if we manage to bound (the first component of) this vector for all terms t up to a given (but arbitrary) size k, then we have actually established an upper bound on the derivational complexity of $\mathcal{R}$. Moreover, as we are only interested in the asymptotic growth of $\frac{1}{\delta} \cdot [\alpha]_{\mathcal{M}}(t)$ with respect to the size of t, we may neglect the multiplicative factor $\frac{1}{\delta}$ because δ is a constant. As already observed in [13], this problem essentially reduces to bounding the entries of finite matrix products of the form $M_1 \cdot M_2 \cdot \ldots \cdot M_k$, $M_i \in \mathcal{M}$. Such products arise naturally when evaluating terms in a matrix interpretation; e.g., if $t := \mathsf{f}(\mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{c})$ then $[\alpha]_{\mathcal{M}}(t) = F_1 G_1 \boldsymbol{a} + F_1 G_2 \boldsymbol{b} + F_1 \boldsymbol{g} + F_2 \boldsymbol{c} + \boldsymbol{f}$. As in [13], we reduce this problem to the analysis of the growth of the powers of a single matrix. To this end, we note that for all $1 \leqslant i, j \leqslant n$, $(M_1 \cdot M_2 \cdot \ldots \cdot M_k)_{i,j} \leqslant (A^k)_{i,j}$, where the matrix A is the component-wise maximum of all matrices occurring in $\mathcal{M}$; i.e., $A_{i,j} := \max\{B_{i,j} \mid B \in \mathcal{M}\}$ for all $1 \leqslant i, j \leqslant n$. If $|t| \leqslant k$ then the length of each product is at most $\mathrm{depth}(t)$ $(\leqslant k)$ and the number of products equals the number of subterms of t, which is also bounded by k. Thus any lemma stating that the entries of the matrix A^k are polynomially bounded in k of degree $d - 1$ can readily be used as the basis of a corresponding theorem that establishes a polynomial upper bound of degree d on the derivational complexity of all TRSs that are compatible with the matrix interpretation $\mathcal{M}$. In [13], for example, this is achieved by restricting the shape of the matrices to upper triangular form.

Lemma 2 ([13, Lemma 5]). *Let $A \in \mathbb{N}^{n \times n}$ be an upper triangular complexity matrix and $k \in \mathbb{N}$. Then $(A^k)_{i,j} \in O(k^{n-1})$ for all $1 \leqslant i, j \leqslant n$.*

Theorem 3 ([13, Theorem 6]). *If a TRS $\mathcal{R}$ is compatible with a triangular matrix interpretation of dimension n, then $\mathsf{dc}_{\mathcal{R}}(k) \in O(k^n)$.*

However, we claim that Lemma 2 only gives a rough estimate of the growth of the entries of the matrix A^k, i.e., the degree of the polynomial bound can be lowered in many cases. To this end, we provide a more concise analysis of the growth of A^k in the next section, obtaining a replacement for Lemma 2, which allows us to tighten the bounds established by Theorem 3. In particular, our refinement holds for matrix interpretations over both $\mathbb{N}$ and $\mathbb{R}$. Moreover, we remark that the restriction of the shape of the matrices is another source for improvement. Clearly, there are also non-triangular matrices that exhibit polynomial growth, but in general non-triangular matrix interpretations do not induce polynomial (but rather exponential) upper bounds on the derivational complexity of compatible TRSs. So in order to be useful in (automated) complexity analysis of term rewriting, a characterization of polynomially bounded matrices is required such that, when searching for a compatible matrix interpretation for a given TRS, it is guaranteed beforehand that the search process only considers such matrices. This is the main goal of the following sections.

4 Main Result

In this section we elaborate on how to lift the restriction to upper triangular matrices. To this end, we leverage the Cayley-Hamilton theorem and the theory of linear homogeneous recurrence relations to completely characterize the growth of the powers of real square matrices (independently of the shape of the matrices). In particular, we show that the key point with respect to polynomial boundedness of such matrices is the nature of their eigenvalues. According to the discussion in Section 3, our results apply to matrix interpretations over $\mathbb{N}$ and $\mathbb{R}$ alike.

Lemma 4. *Let $A \in \mathbb{R}_0^{n \times n}$. Then $\rho(A) \leqslant 1$ if and only if all entries of A^k ($k \in \mathbb{N}$) are asymptotically bounded by a polynomial in k of degree d, where $d := \max_\lambda(0, m_\lambda - 1)$ and λ are the eigenvalues with absolute value exactly one.*

Proof. First, let us assume that $\rho(A) > 1$, i.e., A has an eigenvalue λ of absolute value strictly greater than one. For any eigenvector x associated to λ, we have $Ax = \lambda x$ and hence $A^k x = \lambda^k x$. Since x is non-zero by definition and $|\lambda| > 1$, there is at least one component of $\lambda^k x$ whose absolute value grows exponentially in k. But this can only be the case if at least one entry of A^k grows exponentially in k as well. Conversely, if $\rho(A) \leqslant 1$, we have to show that the entries of A^k are polynomially bounded. Since A is a real $n \times n$ matrix, its characteristic polynomial $\chi_A(\lambda)$ is a monic polynomial of degree n with real coefficients. Without loss of generality, it can be written as $\chi_A(\lambda) = \lambda^t \cdot p(\lambda)$, $0 \leqslant t \leqslant n$, where t is maximal and p is a monic polynomial of degree $n-t$. By the Cayley-Hamilton theorem, A satisfies its own characteristic equation, that is, $\chi_A(A) = 0$. Clearly, if $t = n$ then $A^k = 0$ for all $k \geqslant n$ and $d = 0$, such that the claim follows trivially. If $t < n$ we rearrange the equation $\chi_A(A) = 0$ into the form $A^n = c_1 A^{n-1} + c_2 A^{n-2} + \cdots + c_{n-t} A^t$ with coefficients $c_1, \ldots, c_{n-t}$, readily obtaining a recursive equation for the powers of A, namely, for all $k \geqslant n \in \mathbb{N}$ $A^k = c_1 A^{k-1} + c_2 A^{k-2} + \cdots + c_{n-t} A^{k-(n-t)}$. Thus we establish the following recurrence relation

$$A_k = c_1 A_{k-1} + c_2 A_{k-2} + \cdots + c_{n-t} A_{k-(n-t)} \tag{1}$$

and note that the sequence $(A_j)_{j \geqslant t}$ where $A_j := A^j$ satisfies it by construction. This is a linear homogeneous recurrence relation with constant coefficients and characteristic polynomial $\chi(\lambda) = p(\lambda)$. Since the coefficients of $\chi(\lambda)$ are real numbers, the non-real roots (eigenvalues) always come in conjugate pairs; i.e., if $\lambda_j := r_j(\cos(\phi_j) + i \sin(\phi_j))$ is a root of $\chi(\lambda)$, then so is its complex conjugate $\lambda_j^* := r_j(\cos(\phi_j) - i \sin(\phi_j))$. Thus the general solution of (1) can be written as

$$\begin{aligned} A_k = {} & \sum_i (C_{i,0} + C_{i,1} k + \cdots + C_{i,m_i - 1} k^{m_i - 1}) \lambda_i^k \\ & + \sum_j (D_{j,0} + D_{j,1} k + \cdots + D_{j,m_j - 1} k^{m_j - 1}) r_j^k \cos(k \phi_j) \\ & + \sum_j (D'_{j,0} + D'_{j,1} k + \cdots + D'_{j,m_j - 1} k^{m_j - 1}) r_j^k \sin(k \phi_j) \end{aligned} \tag{2}$$

where the λ_i's are the distinct real roots of $\chi(\lambda)$, each having multiplicity m_i, and the λ_j's, $\lambda_j := r_j(\cos(\phi_j) + i\sin(\phi_j))$, the distinct complex roots (modulo conjugates), each having multiplicity m_j. By assumption, the absolute values of all eigenvalues are at most one; hence, $|\lambda_i| \leqslant 1$ and $r_j \leqslant 1$ in (2), such that the asymptotic growth of the entries of the matrix A^k is polynomial rather than exponential. In particular, the degree d of the polynomial bound is at most $m-1$, where m is the largest of the multiplicities of the eigenvalues with absolute value exactly one. If there are no such eigenvalues, then $\rho(A) < 1$ and $\lim_{k\to\infty} A^k = 0$, such that $d = 0$. □

Example 5. Consider the 4×4 matrix $A := (A_{i,j})_{1\leqslant i,j\leqslant 4}$ with all entries zero except $A_{1,1} = A_{2,4} = A_{3,2} = A_{4,3} = 1$. It has one real eigenvalue $\lambda_1 = 1$ of multiplicity two and a pair of complex conjugate eigenvalues $\lambda_2 = \frac{1}{2}(-1 + i\sqrt{3})$ and $\lambda_2^* = \frac{1}{2}(-1 - i\sqrt{3})$ of multiplicity one, all of which have absolute value exactly one. Hence, the spectral radius $\rho(A)$ of A is also one. According to Lemma 4, the entries of the matrix A^k, $k \in \mathbb{N}$, are bounded by a linear polynomial in k. The actual bound, however, is even lower since $A^4 = A$, such that the powers of A are trivially bounded by a constant, and we can use the method outlined in the proof of Lemma 4 to show this. To this end, we note that the characteristic polynomial of A is $\chi_A(\lambda) = \lambda^4 - \lambda^3 - \lambda + 1$. Thus, by the Cayley-Hamilton theorem, we obtain the recursive equation $A^k = A^{k-1} + A^{k-3} - A^{k-4}$ for all $k \geqslant 4 \in \mathbb{N}$, the general solution of which can be written as

$$A^k = (C_0 + C_1 k)\lambda_1^k + D\, r^k \cos(k\phi) + D'\, r^k \sin(k\phi) \tag{3}$$

where $r(\cos(\phi) + i\sin(\phi)) = \lambda_2$, that is, $r = 1$ and $\phi = \frac{2\pi}{3}$. In the next step, the exact values of the four constants C_0, C_1, D and D' can be determined, for example, by letting $k = 4, 5, 6, 7$ in (3) and solving the resulting systems of linear equations. In doing so, one learns that C_1 is zero, which means that the linear summand in (3) vanishes. Further, we obtain $A^k = C_0 + D\cos(k\phi) + D'\sin(k\phi)$,

$$C_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad D := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \quad D' := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \end{pmatrix}$$

which explains why the powers of A are bounded by a constant. In particular, the periodic nature of the sequence $(A^k)_{k\in\mathbb{N}}$ becomes evident.

On the basis of Lemma 4, we now establish the following theorem concerning complexity analysis of TRSs that holds for matrix interpretations over $\mathbb{N}$ and $\mathbb{R}$.

Theorem 6. *Let $\mathcal{R}$ be a TRS and $\mathcal{M}$ a compatible matrix interpretation of dimension n. Further, let A denote the component-wise maximum of all matrices occurring in $\mathcal{M}$. If the spectral radius of A is at most one, then $\mathsf{dc}_{\mathcal{R}}(k) \in O(k^{d+1})$, where $d := \max_\lambda(0, m_\lambda - 1)$ and λ are the eigenvalues of A with absolute value exactly one.*

Remark 7. Actually the d in Theorem 6 can be strengthened to $\max_\lambda(0, m_\lambda) - 1$ because the pathological case $\rho(A) < 1$ implies $\mathsf{dc}_\mathcal{R}(k) \in O(k^0)$.

The next example shows why triangular matrices may fail. Similar (but larger) systems are contained in TPDB [18], e.g., TRS/Cime_04/dpqs.xml.

Example 8. Consider the TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{c}(\mathsf{f}(x))), \mathsf{c}(\mathsf{c}(x)) \to x\}$ which is compatible with the matrix interpretation

$$\mathsf{f}_\mathcal{M}(\boldsymbol{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathsf{c}_\mathcal{M}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \boldsymbol{x}$$

The eigenvalues of the component-wise maximum matrix are -1, 1 and 1; hence, Theorem 6 deduces a quadratic upper bound on the derivational complexity of $\mathcal{R}$. There cannot exist a *triangular* matrix interpretation compatible with $\mathcal{R}$ since the second rule demands that all diagonal entries in $\mathsf{c}_\mathcal{M}$ are non-zero, but then the first rule can no longer be oriented.

Next we specialize Theorem 6 to triangular matrix interpretations. In such interpretations all matrices are upper triangular complexity matrices whose diagonal entries are restricted to the closed interval $[0, 1]$ and whose top-left entry is always one. Hence, this is also true for the component-wise maximum matrix A. Since the diagonal entries of a triangular matrix give the multiset of its eigenvalues, the matrix A is therefore guaranteed to have spectral radius one.

Theorem 9. *Let $\mathcal{R}$ be a TRS and $\mathcal{M}$ a compatible triangular matrix interpretation over $\mathbb{N}$ or $\mathbb{R}$ of dimension n. Further, let A denote the component-wise maximum of all matrices occurring in $\mathcal{M}$, and let d denote the number of ones occurring along the diagonal of A. Then $\mathsf{dc}_\mathcal{R}(k) \in O(k^d)$.*[4]

Note that the bound established by Theorem 9 for matrix interpretations over $\mathbb{N}$ is at least as tight as the one of Theorem 3 since $d \leqslant n$.

Example 10. The TRS $\mathcal{R} = \{\mathsf{a}(\mathsf{b}(\mathsf{a}(x))) \to \mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{a}(x)))), \mathsf{b}(\mathsf{b}(\mathsf{b}(x))) \to \mathsf{b}(\mathsf{b}(x))\}$[5] is compatible with the triangular matrix interpretation

$$\mathsf{a}_\mathcal{M}(\boldsymbol{x}) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathsf{b}_\mathcal{M}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The diagonal of the component-wise maximum of the two matrices has the shape $(1, 0, 0)$. Hence, $\mathcal{R}$ has (at most) linear derivational complexity by Theorem 9, whereas the bound established by Theorem 3 is cubic. Incidentally, the bound inferred from Theorem 9 is even optimal since it is easy to see that the derivational complexity of $\mathcal{R}$ is at least linear. It is easy to show that there are no triangular matrix interpretations of dimension one and two compatible with $\mathcal{R}$.

The final example shows the benefit of matrix interpretations over $\mathbb{R}$.

[4] Independently in [19, Proposition 7.6] the same result has been established for $\mathbb{N}$.
[5] TPDB problem TRS/Zantema_04/z126.xml

Example 11. Consider the TRS $\mathcal{R}$.[6] There exists a matrix interpretation (see website in Footnote 8) compatible with $\mathcal{R}$ such that the diagonal of the component-wise maximum matrix has the shape $(1, \frac{1}{2}, 0)$. Due to Theorem 9, the derivational complexity of $\mathcal{R}$ is at most linear. Our implementation could find a triangular matrix interpretation of the same dimension over $\mathbb{N}$ compatible with $\mathcal{R}$ establishing a quadratic but not a linear bound.

5 Implementation Issues

In Theorem 6, we consider some TRS together with a compatible matrix interpretation and demand that the component-wise maximum matrix A has spectral radius at most one. So we have to make sure that the absolute values of all its eigenvalues (real and complex ones) are at most one. However, since A is a non-negative real square matrix, we only have to ensure this condition for all (non-negative) real eigenvalues of A. This follows directly from the Perron-Frobenius theorem ([16], weak form), which states that the spectral radius of a non-negative real square matrix is an eigenvalue of the matrix; i.e., there exists a non-negative real eigenvalue that dominates in absolute value all eigenvalues.

Concerning the automation of Theorem 6, the main problem that has to be dealt with is the following. Given some square matrix A with unknown entries, all of which are supposed to be non-negative real (or integer) numbers, we need a set of constraints, expressed in terms of the unknown entries, that enforce $\rho(A) \leqslant 1$; e.g., for which non-negative values of a, b, c and d has the matrix

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

spectral radius at most one? In the sequel, we present three different approaches.

(A) The first approach is based on the explicit calculation of the eigenvalues of A, i.e., the explicit calculation of the roots of the characteristic polynomial $\chi_A(\lambda)$. For the two-dimensional case, we have $\chi_A(\lambda) = \lambda^2 - (a + d)\lambda + ad - bc$, and by the quadratic formula we obtain the roots $\lambda_{1,2} = \frac{a+d}{2} \pm \frac{\sqrt{(a-d)^2+4bc}}{2}$, both of which are real because all matrix entries are non-negative. In particular, λ_2 ($\geqslant \lambda_1$) is non-negative, such that it suffices to require $\lambda_2 \leqslant 1$ according to the Perron-Frobenius theorem. Simplifying this condition as much as possible, we infer that the matrix A has spectral radius at most one if and only if $a+d \leqslant 2$ and $a + d \leqslant ad - bc + 1$. This explicit approach also applies to matrices of dimension three and four since there exist formulas for the solution of arbitrary cubic and quartic polynomial equations with symbolic coefficients (though the respective calculations are tedious). However, for equations of degree five or higher, there are no formulas that express the solutions of such equations in terms of their coefficients using only the four basic arithmetic operations and radicals (n-th roots, for some integer n).

[6] TPDB problem TRS/Secret_05_SRS/matchbox2.xml

(B) Next we present an alternative and simpler approach for three-dimensional matrices. To this end, let A be some arbitrary three-dimensional non-negative real square matrix with entries $a, b, \ldots, i$ and characteristic polynomial $\chi_A(\lambda)$

$$\lambda^3-(a+e+i)\lambda^2+(ei-fh+ai-cg+ae-bd)\lambda-(aei+bfg+cdh-ceg-bdi-afh)$$

which we abbreviate by $\lambda^3 + p\lambda^2 + q\lambda + r$. By the Perron-Frobenius theorem, it suffices to constrain the real roots of $\chi_A(\lambda)$ to the closed interval $[-1, 1]$. To this end, we make use of the well-known fact that a cubic polynomial like $\chi_A(\lambda)$ either has only one real root (and two complex conjugate roots) if its *discriminant* $D := p^2q^2 - 4q^3 - 4p^3r - 27r^2 + 18pqr$ is negative or three (not necessarily distinct) real roots if $D \geqslant 0$. Visualizing the geometric shape of $\chi_A(\lambda)$, it is not hard to see that in the latter case all three roots are in $[-1, 1]$ if and only if $\chi_A(-1) \leqslant 0$, $\chi_A(1) \geqslant 0$ and $\chi'_A(\lambda) \geqslant 0$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$ (here χ'_A denotes the first derivative of χ_A). Thus we conclude that the matrix A has spectral radius at most one if and only if

$$\begin{aligned}&(D < 0 \wedge \chi_A(-1) \leqslant 0 \wedge \chi_A(1) \geqslant 0) \vee \\ &(\chi_A(-1) \leqslant 0 \wedge \chi_A(1) \geqslant 0 \wedge \chi'_A(\lambda) \geqslant 0 \text{ for all } |\lambda| \geqslant 1)\end{aligned}$$

These are polynomial constraints in the entries of A. In particular, the constraint $\chi'_A(\lambda) = 3\lambda^2 + 2p\lambda + q \geqslant 0$ for all $|\lambda| \geqslant 1$ can be shown to be equivalent to

$$(p^2 - 3q \leqslant 0) \vee (-3 \leqslant p \leqslant 3 \wedge -(q + 3) \leqslant 2p \leqslant q + 3)$$

by means of the quadratic formula. Here the term $p^2 - 3q$ is essentially the discriminant of $\chi'_A(\lambda)$; if it is negative, then $\chi'_A(\lambda)$ has no real root, such that the constraint holds trivially, otherwise it has two real roots λ_1 and λ_2. In case $\lambda_1 = \lambda_2$, the constraint also holds because then $\chi'_A(\lambda) = 3 \cdot (\lambda - \lambda_1)^2$. Finally, if $\lambda_1 \neq \lambda_2$, then both must necessarily lie in the closed interval $[-1, 1]$ for the constraint to hold, which is ensured by the second disjunct in the above formula.

(C) Last but not least, we present a generic method that works for matrices with unknown entries of any dimension. To this end, let A be an n-dimensional square matrix whose entries are supposed to be real numbers (not necessarily non-negative). Its characteristic polynomial is a monic polynomial of degree n, which can be written as $\chi_A(\lambda) = \lambda^n + \sum_{i=0}^{n-1} c_i\lambda^i$, where the coefficients c_i, $0 \leqslant i \leqslant n-1$, are polynomial expressions in the entries of A. Since all coefficients are supposed to be real numbers, $\chi_A(\lambda)$ can always be factored as

$$\chi_A(\lambda) = (\lambda - r)^b \cdot \prod_j (\lambda^2 + p_j\lambda + q_j)^{m_j} \tag{4}$$

where $b = 0$ if n is even, $b = 1$ otherwise, $m_j \geqslant 1$ ($m_j \in \mathbb{N}$) is the multiplicity of the quadratic factor $\lambda^2 + p_j\lambda + q_j$, and $r, p_j, q_j \in \mathbb{R}$. Thus the absolute values of all roots (real and complex ones) of $\chi_A(\lambda)$ are at most one if and only if $|r| \leqslant 1$ (in case $b = 1$) and the absolute values of the roots of all quadratic factors are at most one. So when does the latter condition hold for a given quadratic factor $\lambda^2 + p_j\lambda + q_j$? By the quadratic formula, we obtain the roots $\lambda_{1,2} := -\frac{p_j}{2} \pm \frac{\sqrt{p_j^2-4q_j}}{2}$. If the discriminant $p_j^2 - 4q_j$ is negative, both roots are complex, i.e., $\lambda_{1,2} := -\frac{p_j}{2} \pm i\frac{\sqrt{4q_j-p_j^2}}{2}$ and have absolute value $|\lambda_1| = |\lambda_2| = \sqrt{q_j}$. Hence, we demand $\sqrt{q_j} \leqslant 1$, or equivalently, $q_j \leqslant 1$. In the other case, if $p_j^2 - 4q_j \geqslant 0$, both roots are real, and the constraints $|\lambda_1| \leqslant 1$ and $|\lambda_2| \leqslant 1$ simplify to

$$-2 \leqslant p_j \leqslant 2 \text{ and } -(q_j+1) \leqslant p_j \leqslant q_j+1$$

As a consequence, the matrix $A \in \mathbb{R}^{n\times n}$ with characteristic polynomial (4) has spectral radius at most one if and only if $b = 1$ implies $-1 \leqslant r \leqslant 1$ and for all quadratic factors $\lambda^2 + p_j\lambda + q_j$ in (4),

$$(p_j^2-4q_j < 0 \wedge q_j \leqslant 1) \vee (p_j^2-4q_j \geqslant 0 \wedge -2 \leqslant p_j \leqslant 2 \wedge -(q_j+1) \leqslant p_j \leqslant q_j+1)$$

Non-negative Integer Matrices. If all matrix entries are non-negative integers, one can also apply a totally different approach. It is based on graph theory and the following lemma, which is an immediate consequence of [11, Corollary 1].[7]

Lemma 12. *Let $A \in \mathbb{N}^{n\times n}$. Then $\rho(A) > 1$ if and only if $(A^k)_{i,i} > 1$ for some $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$.*

Viewing $A \in \mathbb{N}^{n\times n}$ as the adjacency matrix of a directed weighted graph G_A of n vertices numbered from 1 to n, such that for every positive entry $A_{i,j}$ there is an edge from vertex i to vertex j of weight $A_{i,j}$, the condition $(A^k)_{i,i} > 1$ for some $i \in \{1, \ldots, n\}$ mentioned in the previous lemma holds if and only if

1. there is a cycle in G_A containing at least one edge of weight $w > 1$, or
2. there are (at least) two different paths (cycles) from some vertex to itself.

This is due to the well-known fact that the entry $(A^k)_{i,j}$ equals the sum of the weights of all distinct paths in G_A of length k from vertex i to vertex j, where the weight w of a path is the product of the weights of its edges (in particular, $w \geqslant 1$). Hence, we have $\rho(A) \leqslant 1$ if and only if neither of the two conditions holds. Since every cycle of G_A is composed of simple cycles, that is, cycles with no repeated vertices (aside from the necessary repetition of the start and end vertex), we may restrict to simple cycles for both conditions.

Next we make two important observations. First, for $A \in \mathbb{N}^{n\times n}$, G_A cannot have a simple cycle containing an edge of weight greater than one if every matrix in the set $\{A, A^2, \ldots, A^n\}$ has diagonal entries less than or equal to one. Concerning the second condition, let us assume that there are two different simple cycles C_1 and C_2 of length l_1 and l_2, $1 \leqslant l_1, l_2 \leqslant n$, from some vertex i to itself. Considering all paths of length $\mathsf{lcm}(l_1, l_2)$, the least common multiple of l_1 and l_2, we clearly have $(A^{\mathsf{lcm}(l_1,l_2)})_{i,i} > 1$. In addition, we also have $(A^{l_1+l_2})_{i,i} > 1$ because there are two different cycles, each of weight at least one, from vertex i to itself of length $l_1 + l_2$, namely, the concatenation of C_1 and C_2 as well as the concatena-

[7] The joint spectral radius of a singleton set $\{A\}$ of matrices coincides with $\rho(A)$.

tion of C_2 and C_1. Hence, we can detect the existence of the cycles C_1 and C_2 by examining the diagonal entries of all matrices in the set $\{A, A^2, \ldots, A^m\}$, where $m := \min(l_1 + l_2, \mathsf{lcm}(l_1, l_2))$. More generally, we can detect any pair of cycles satisfying condition 2 by examining the diagonal entries of the matrices in the set $\{A, A^2, \ldots, A^{p(n)}\}$, where $p(n) := \max\{\min(l_1 + l_2, \mathsf{lcm}(l_1, l_2)) \mid 1 \leqslant l_1, l_2 \leqslant n\}$. The left part of the table below shows the values of $p(n)$ for various values of n.

n	1	2	3	4	5	6
$p(n)$	1	2	5	7	9	11

n	1	2	3	4	5	6
$q(n)$	1	2	3	5	7	9

In particular, we observe that $p(n) \geqslant n$ for $n \geqslant 1$, and we draw the following conclusion. If every matrix in the set $\{A, A^2, \ldots, A^{p(n)}\}$ has diagonal entries less than or equal to one, then neither condition 1 nor condition 2 can hold, which implies $\rho(A) \leqslant 1$. The converse is obvious.

Now let us apply this result to matrix interpretations. By definition, all matrices of a matrix interpretation $\mathcal{M}$ must have a top-left entry of at least one. Hence, this is also true for the maximum matrix A of $\mathcal{M}$. In other words, in G_A, vertex 1 has a loop (of length one) to itself. This corresponds to a dimension reduction by one for precluding all instances of condition 2. More precisely, we do not have to consider the cases $l_1 = n$ or $l_2 = n$ because then not only C_1 and C_2 but also C_1 (C_2) and the loop of vertex 1 satisfy condition 2 (for $n > 1$), and we can detect this by examining the diagonal entries of the matrix A^n, which has to be considered anyway for precluding all instances of condition 1. Therefore, if $A_{1,1} > 0$, we have $\rho(A) \leqslant 1$ if and only if every matrix in the set $\{A, A^2, \ldots, A^{q(n)}\}$ has diagonal entries less than or equal to one, where $q(n) := \max(n, p(n-1))$ for $n > 1$ and $q(1) := 1$. Some values for $q(n)$ are displayed in the right part of the above table.

6 Experimental Results

The criteria proposed in this paper have been implemented in the complexity tool CaT [20] and the 1172 non-duplicating TRSs in TPDB 7.0.2 have been considered. All tests have been performed on a server equipped with 64 GB of main memory and eight dual-core AMD Opteron® 885 processors running at a clock rate of 2.6 GHz with a time limit of 60 seconds per system.[8]

We searched for matrix interpretations of dimension $d \in \{1, \ldots, 5\}$ by encoding the constraints as an SMT problem (quantifier-free non-linear arithmetic), which is solved by bit-blasting. We used $\max(2, 6-d)$ $(7-d)$ bits to represent coefficients (intermediate results). The numerators of rational numbers are represented with the bit-width mentioned above while all denominators are 2. CaT found compatible matrix interpretations (not necessarily polynomially bounded) for 287 TRSs, giving an upper bound on the number of systems our results can apply to (if used stand-alone).

Table 1 indicates the number of systems where the labeled approach yields polynomial upper bounds on the derivational complexity. The first row shows

[8] For full details see `http://cl-informatik.uibk.ac.at/software/cat/polymatrix`.

Table 1. Polynomial bounds for 1172 systems

	$O(k)$	$O(k^2)$	$O(k^3)$	$O(k^n)$
Theorem $3\|9_{\mathbb{N}}\|9_{\mathbb{R}}$	46| 85| 88	158|184|185	177|202|196	203|205|199
A|B|C|Lemma 12	61|68| 80| 64	158|176|185|175	–|182|191|180	–|–|193|190
row 1|row 2|row 1+2	88| 80| 88	191|185|200	205|191|209	208|196|212
CaT(2009)|CaT(2010)	208|214	299|309	310|321	328|329

that the theorems proposed in this paper allow to infer tighter upper bounds from triangular matrices than [13]; e.g., the number of linear (quadratic) upper bounds increases by 84% (16%) if one compares Theorems 3 and $9_{\mathbb{N}}$. The results for (possibly) non-triangular matrix interpretations are reported in the second row. The generic method based on factoring the characteristic polynomial (C) is implemented by comparing the coefficients from the characteristic polynomial with the coefficients of equation (4). Note that only this non-triangular approach allows to add upper bounds on the multiplicity of eigenvalues to the matrix encoding, which explains the high score for linear bounds. Since encoding A (B) is becoming harder for larger dimensions, we implemented it for dimensions one and two (and three) only (explaining the – in the table). Row three relates the approaches based on triangular and non-triangular matrices. Here row 1 corresponds to the accumulated power of Theorems 3 and 9 and row 2 to A, B, C, and Lemma 12, respectively. The impact of the methods proposed in this paper when integrated into the 2009 competition version of CaT is shown in row four. CaT was the strongest (derivational) complexity prover in 2008, 2009, and 2010. Since most parts of this paper aim at tightening bounds, it is not surprising that the total number of polynomial bounds did not increase significantly.

7 Conclusion, Related and Future Work

We have presented a characterization of matrix interpretations that induce polynomial upper bounds on the derivational complexity of compatible TRSs. Contrary to previous approaches, our method applies to matrix interpretations over $\mathbb{N}$ and $\mathbb{R}$ alike and does not restrict the shape of the matrices. At the core of our method is the analysis of the growth of finite products of matrices. In particular, we estimate the growth of a product of the form $M_1 \cdot M_2 \cdot \ldots \cdot M_k$ by the growth of a (suitably chosen) matrix A^k, which is determined by its spectral radius. For future work, the investigation of joint spectral radius theory [11] looks promising since the joint spectral radius is a measure of the maximal growth of products of matrices taken from a set and has been the subject of intense research.

Concerning related work, very recently (and independently) Waldmann [19] provides a characterization of polynomially bounded matrix interpretations over $\mathbb{N}$, which extends triangular matrix interpretations. In [19] matrices are viewed as weighted (word) automata and the derivational complexity of TRSs is bounded by the growth of the weight function computed by such automata. We believe that the method is at least as powerful as our approach for matrix interpretations over ℕ. In contrast to our approach, it can handle the TRS in [19, Example 7.5], probably because it is not based on the maximum matrix. In practice, the method based on automata is much harder to implement (cf. [19, Section 8]). Unlike our approach, it only applies to matrix interpretations over ℕ; the extension to ℝ (ℚ) raises non-trivial issues (cf. [19, Section 10]).

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