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Decreasing Diagrams and Relative Termination

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Abstract In this article we use the decreasing diagrams technique to show that a left-linear and locally confluent term rewrite system $\mathcal{R}$ is confluent if the critical pair steps are relatively terminating with respect to $\mathcal{R}$. We further show how to encode the rule-labeling heuristic for decreasing diagrams as a satisfiability problem. Experimental data for both methods are presented.

Keywords Confluence · Decreasing diagrams · Relative termination · Term rewriting

1 Introduction

This article is concerned with automatically proving confluence of term rewrite systems. Unlike termination, for which the interest in automation gave and continues to give rise to new methods and tools, automated confluence analysis has received little attention. We present a new confluence criterion which is easy to implement on top of existing termination tools that support relative termination. The criterion states that a left-linear and locally confluent rewrite system is confluent if the rewrite steps that give rise to critical pairs are *relatively terminating* with respect to the given rewrite rules. This result can be viewed as a generalization of the two standard approaches for proving confluence of term rewrite systems: orthogonality and joinability of critical pairs for terminating systems. In the proof we use the conversion version [27] of *decreasing diagrams* with the predecessor labeling in which rewrite steps are labeled by a term that can be rewritten to the starting term of the step. For countable abstract rewrite systems, the decreasing diagrams technique of van Oostrom [25,27] subsumes all sufficient conditions for confluence. To use this technique for

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term rewrite systems, a well-founded order on the rewrite steps has to be supplied such that rewrite peaks can be completed into so-called decreasing diagrams.

The second result of this article is the encoding of the rule-labeling heuristic of van Oostrom [27] for linear rewrite systems as a satisfiability problem. In this heuristic rewrite steps are labeled by the applied rewrite rule. By limiting the number of steps that may be used to complete local diagrams, we obtain a finite search problem which is readily transformed into a satisfiability problem. Any satisfying assignment returned by a modern SAT or SMT solver is then translated back into a concrete rule-labeling.

The remainder of this article is organized as follows. In the next section we present a few basic definitions pertaining to term rewriting and confluence. We introduce proof terms to represent multi-steps in left-linear rewrite systems and recall the conversion version of the decreasing diagrams technique. Section 3 is devoted to our main result. We explain how the result is implemented and we present a small extension. In Section 4 we first show that it is undecidable whether confluence of a locally confluent rewrite system can be established by the rule-labeling heuristic for decreasing diagrams. By approximating conversions by valleys in an extended rewrite system and putting a bound on the number of steps to check joinability, we obtain a decidable sufficient condition. Experimental data is presented in Section 5. We also comment upon the limitations of our results. In Section 6 we mention related work before concluding in Section 7 with suggestions for future research.

A preliminary version of this article appeared in [14]. There are four major changes. First, the proof of the main theorem [14, Theorem 16] is simplified by using proof terms to represent multi-steps and adopting decreasing diagrams with respect to conversions in connection with the predecessor labeling. Second, the extension of the main theorem mentioned in [14, Section 4] was based on an incorrect claim. We use van Oostrom's orthogonalization technique to recover the result. Furthermore, the encoding for the rule labeling heuristic is extended to the conversion version of decreasing diagrams. Finally, we include an analysis of the limitations of our results to prove confluence.

2 Preliminaries

Term rewriting

We assume familiarity with the basics of term rewriting (e.g. [30]). Below we recall some important definitions needed in the remainder of the article. We only deal with first-order terms, which are built from variables and function applications. Let t be a term. The root symbol of t is denoted by $\mathrm{root}(t)$. We write $\mathcal{V}\mathsf{ar}(t)$ for the set of variables occurring in t. The sets of all variable (function) positions in t is denoted by $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$ ($\mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$). A rewrite rule $\ell \to r$ is a pair (ℓ, r) of terms with non-variable term ℓ and $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell)$. A term rewrite system (TRS for short) is a collection of rewrite rules between terms over a fixed-arity signature. A rewrite rule is left-linear (right-linear) if no variable occurs more than once in ℓ (r). A left-linear and right-linear rewrite rule is called linear. A TRS is said to be (left-/right-)linear if all rewrite rules have this property. A TRS $\mathcal{R}$ is confluent if

$$\underset{\mathcal{R}}{\overset{*}{\longleftarrow}} \cdot \underset{\mathcal{R}}{\overset{*}{\longrightarrow}} \subseteq \underset{\mathcal{R}}{\overset{*}{\longrightarrow}} \cdot \underset{\mathcal{R}}{\overset{*}{\longleftarrow}}$$

Many sufficient conditions for confluence of TRSs are based on critical pairs. Critical pairs are generated from overlaps. An overlap $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ of a TRS $\mathcal{R}$ consists of variants $\ell_1 \to r_1$ and $\ell_2 \to r_2$ of rules of $\mathcal{R}$ without common variables, a position $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$, and a most general unifier μ of ℓ_1 and $\ell_2|_p$. If $p = \varepsilon$ then we require that $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants of each other. The induced critical pair is $(\ell_2\mu[r_1\mu]_p, r_2\mu)$. Following Dershowitz [7], we write $s \leftarrow\rtimes\rightarrow t$ to indicate that (s,t) is a critical pair. We drop the subscript μ from $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ when it is not relevant for the discussion. The well-known critical pair lemma states that local confluence of $\mathcal{R}$ is equivalent to $\leftarrow\rtimes\rightarrow \subseteq \to_{\mathcal{R}}^* \cdot {}_{\mathcal{R}}^*\!\leftarrow$. Left-linear TRSs without critical pairs are called orthogonal. A critical pair $s \leftarrow\rtimes\rightarrow t$ is trivial if $s = t$. Left-linear TRSs without non-trivial critical pairs are called weakly orthogonal. Both orthogonal and weakly orthogonal TRSs are known to be confluent. Moreover, Knuth and Bendix' criterion [17] states that $\leftarrow\rtimes\rightarrow \subseteq \to_{\mathcal{R}}^* \cdot {}_{\mathcal{R}}^*\!\leftarrow$ implies confluence of terminating $\mathcal{R}$.

Proof terms

We define proof terms that witness multi-steps [30, Chapter 8]. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$. For each rule $\ell \to r \in \mathcal{R}$ we introduce a *rule symbol* $\underline{\ell \to r}$ which is a fresh (with respect to $\mathcal{F}$) function symbol whose arity is given by the number of variables in ℓ. *Proof terms* are terms over functions in $\mathcal{F}$ and rule symbols. A proof term containing exactly one rule symbol is called a *redex*. We write $\sqsubseteq$ for the smallest rewrite order on proof terms such that $\ell \sqsubseteq \underline{\ell \to r}(x_1,\dots,x_n)$ for all rules $\ell \to r \in \mathcal{R}$ with $\mathsf{var}(\ell) = (x_1,\dots,x_n)$. Here $\mathsf{var}(\ell)$ denotes a sequence consisting of all variables in $\mathcal{V}\mathsf{ar}(\ell)$ in some fixed order.

Definition 1 Let $\mathcal{R}$ be a TRS and A be a proof term. The *multi-step* relation $\multimap_A$ is defined by induction on A as follows:

- $x \multimap_x x$ for all variables x,
- $f(s_1,\dots,s_n) \multimap_{f(A_1,\dots,A_n)} f(t_1,\dots,t_n)$ if $s_i \multimap_{A_i} t_i$ for all i,
- $\ell\{x_i \mapsto s_i \mid 1 \leqslant i \leqslant n\} \multimap_{\underline{\ell \to r}(A_1,\dots,A_n)} r\{x_i \mapsto t_i \mid 1 \leqslant i \leqslant n\}$ if $\mathsf{var}(\ell) = (x_1,\dots,x_n)$ and $s_i \multimap_{A_i} t_i$ for all i.

We write $s \multimap_{\mathcal{R}} t$ (or simply $s \multimap t$) if $s \multimap_A t$ for some proof term A.

Note that every proof term A uniquely determines s and t such that $s \multimap_A t$. Proof terms A and B are *co-initial* if $s_1 \multimap_A t_1$ and $s_2 \multimap_B t_2$ with $s_1 = s_2$. It is known that the inclusion $\to_{\mathcal{R}} \subseteq \multimap_{\mathcal{R}} \subseteq \to_{\mathcal{R}}^*$ holds in general.

Example 1 Consider the left-linear TRS consisting of the rules

$$1\colon \mathsf{f}(\mathsf{a},x,y) \to \mathsf{g}(x,x,y) \qquad 2\colon \mathsf{a} \to \mathsf{b} \qquad 3\colon \mathsf{a} \to \mathsf{h}(\mathsf{a})$$

and the proof term $A = \underline{1}(\underline{2}, \mathsf{h}(\underline{3}))$, assuming $\mathsf{var}(\mathsf{f}(\mathsf{a},x,y)) = (x,y)$. We have the multi-step $\mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{h}(\mathsf{a})) \multimap_A \mathsf{g}(\mathsf{b},\mathsf{b},\mathsf{h}(\mathsf{h}(\mathsf{a})))$ and the inequalities

$$\mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{h}(\mathsf{a})) \sqsubseteq \mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{h}(\underline{3})) \sqsubseteq \underline{1}(\mathsf{a},\mathsf{h}(\underline{3})) \sqsubseteq A$$

Next we define orthogonality of proof terms (cf. [30, Definition 8.2.33]). Let $\mathcal{R}$ be a left-linear TRS. We say that an overlap $(\ell_1 \to r_1, p, \ell_2 \to r_2)$ is between co-initial redexes Δ_1 and Δ_2 if $\mathrm{root}(\Delta_i|_{qp}) = \underline{\ell_1 \to r_1}$ and $\mathrm{root}(\Delta_j|_q) = \underline{\ell_2 \to r_2}$ for some $\{i,j\} = \{1,2\}$ and position q in Δ_j. Co-initial proof terms A and B are *orthogonal* if there is no overlap between any pair of (distinct) redexes $\Delta_1 \sqsubseteq A$ and $\Delta_2 \sqsubseteq B$. The next lemma states two known properties [30, Lemma 8.8.4(v)], which can be shown by easy structural induction on proof terms.[1]

[1] In [30] the properties are stated for orthogonal TRSs.

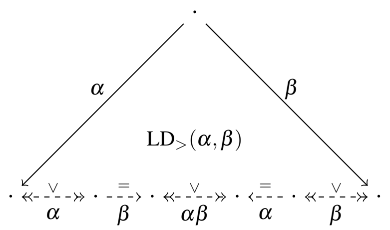


Fig. 1 Local decreasingness

Lemma 1 *Let $\mathcal{R}$ be a left-linear TRS and A,B co-initial proof terms.*

(i) If $A \sqsubseteq B$ then $\mathrel{\multimap\!\!\!\to}_B \subseteq \mathrel{\multimap\!\!\!\to}_A \cdot \mathrel{\multimap\!\!\!\to}$.
(ii) If A and B are orthogonal then ${}_A\!\mathrel{\leftarrow\!\!\!\circ\!\!-} \cdot \mathrel{\multimap\!\!\!\to}_B \subseteq \mathrel{\multimap\!\!\!\to} \cdot \mathrel{\leftarrow\!\!\!\circ\!\!-}$. □

Decreasing diagrams

We conclude this preliminary section by recalling the decreasing diagrams technique for abstract rewrite systems (ARSs) from [25, 27]. We write $\langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ to denote the ARS $\langle A, \to\rangle$ where $\to$ is the union of $\to_\alpha$ for all $\alpha \in I$.

Let $\mathcal{A} = \langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ be an ARS and let $>$ be a well-founded order on I. For every $\alpha \in I$ we write $\overset{\vee}{\to}_\alpha$ for the union of $\to_\beta$ for all $\beta < \alpha$. Moreover, we write $\overset{\vee}{\to}{}^*_\alpha$ for $(\overset{\vee}{\to}_\alpha)^*$. The union of $\overset{\vee}{\to}_\alpha$ and ${}_\alpha\overset{\vee}{\leftarrow}$ is denoted by $\overset{\vee}{\leftrightarrow}_\alpha$. If $\alpha, \beta \in I$ then $\overset{\vee}{\to}_{\alpha\beta}$ denotes the union of $\overset{\vee}{\to}_\alpha$ and $\overset{\vee}{\to}_\beta$. We say that α and β are *locally decreasing* with respect to $>$ and we write $\mathrm{LD}_>(\alpha,\beta)$ if

$${}_\alpha\!\leftarrow \cdot \to_\beta \;\subseteq\; \overset{\vee}{\leftrightarrow}{}^*_\alpha \cdot \to^=_\beta \cdot \overset{\vee}{\leftrightarrow}{}^*_{\alpha\beta} \cdot {}^=_\alpha\!\leftarrow \cdot {}^*_\beta\overset{\vee}{\leftrightarrow}$$

See Figure 1 for a graphical depiction (dashed arrows are implicitly existentially quantified and double-headed arrows denote reflexive and transitive closure).

The ARS $\mathcal{A} = \langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ is locally decreasing if there exists a well-founded order $>$ on I such that $\mathrm{LD}_>(\alpha,\beta)$ for all $\alpha, \beta \in I$ for all $\alpha, \beta \in I$. Van Oostrom [27] obtained the following result.

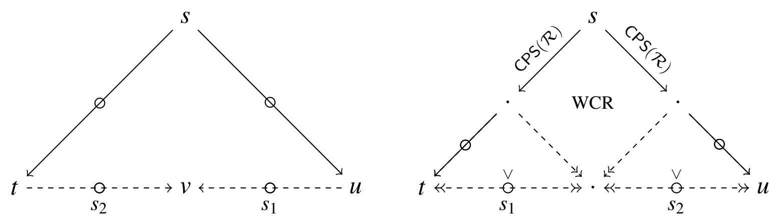
Theorem 1 *Every locally decreasing ARS with respect to conversions is confluent.* □

3 Confluence via Relative Termination

According to Newman's Lemma, an arbitrary non-confluent but locally confluent TRS admits an infinite rewrite sequence. The main result of this section (Theorem 2 below) states that if the system is in addition left-linear, there is an infinite rewrite sequence that involves infinitely many steps that were used in the generation of critical pairs. Let $\mathcal{R}$ be a TRS. We denote the set

$$\{\ell_2\mu \to \ell_2\mu[r_1\mu]_p, \ell_2\mu \to r_2\mu \mid (\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu \text{ is an overlap of } \mathcal{R}\}$$

of rewrite steps that give rise to critical pairs of $\mathcal{R}$ by $\mathsf{CPS}(\mathcal{R})$. We view $\mathsf{CPS}(\mathcal{R})$ as a TRS. Its rules are called *critical pair steps*. We say that $\mathcal{R}$ is relatively terminating with respect



(a) (b)

Fig. 2 The proof of Theorem 2.

to $\mathcal{S}$ or that $\mathcal{R}/\mathcal{S}$ is terminating if the relation $\rightarrow_{\mathcal{R}/\mathcal{S}} = \rightarrow^*_{\mathcal{S}} \cdot \rightarrow_{\mathcal{R}} \cdot \rightarrow^*_{\mathcal{S}}$ is well-founded. Our main result can now be expressed as follows: A left-linear locally confluent TRS $\mathcal{R}$ is confluent if $\mathsf{CPS}(\mathcal{R})$ is relatively terminating with respect to $\mathcal{R}$. Since $\mathsf{CPS}(\mathcal{R})$ is empty for every orthogonal TRS $\mathcal{R}$, this yields a generalization of orthogonality. A key problem when trying to prove confluence in the absence of termination is the handling of duplicating rules. Parallel rewrite steps are typically used for this purpose [15, 29]. To anticipate future developments (cf. Section 7) we use multi-steps instead.

The following lemma relates $\mathrel{\circ\!\!\!\!\!\longrightarrow}_{\mathcal{R}}$ to $\rightarrow_{\mathsf{CPS}(\mathcal{R})/\mathcal{R}}$. It is the key in our proof of the main result.

Lemma 2 *Let $\mathcal{R}$ be a left-linear TRS. If $t \mathrel{\longleftarrow\!\!\!\!\!\circ\;} s \mathrel{\circ\!\!\!\!\!\longrightarrow} u$ then*

(a) $t \mathrel{\circ\!\!\!\!\!\longrightarrow} \cdot \mathrel{\longleftarrow\!\!\!\!\!\circ\;} u$, or
(b) $t \mathrel{\longleftarrow\!\!\!\!\!\circ\;} \cdot {}_{\mathsf{CPS}(\mathcal{R})}\!\leftarrow s \rightarrow_{\mathsf{CPS}(\mathcal{R})} \cdot \mathrel{\circ\!\!\!\!\!\longrightarrow} u$.

Proof Let A and B be proof terms such that $t \;{}_A\!\mathrel{\longleftarrow\!\!\!\!\!\circ\;} s \mathrel{\circ\!\!\!\!\!\longrightarrow}_B u$. We distinguish two cases.

- If A and B are orthogonal then $t \mathrel{\circ\!\!\!\!\!\longrightarrow} v \mathrel{\longleftarrow\!\!\!\!\!\circ\;} u$ follows from Lemma 1(ii).
- Otherwise, there are overlapping redexes $\Delta_1 \sqsubseteq A$ and $\Delta_2 \sqsubseteq B$. Let v be the term that $s \mathrel{\circ\!\!\!\!\!\longrightarrow}_{\Delta_1} v$. It is easy to see that since Δ_1 and Δ_2 are overlapping, their induced steps are critical pair steps and hence $s \rightarrow_{\mathsf{CPS}(\mathcal{R})} v$. As $\Delta_1 \sqsubseteq A$, Lemma 1(i) yields $v \mathrel{\circ\!\!\!\!\!\longrightarrow} t$. Using the same reasoning for Δ_2, we obtain $s \rightarrow_{\mathsf{CPS}(\mathcal{R})} \cdot \mathrel{\circ\!\!\!\!\!\longrightarrow} u$. $\square$

We are ready to prove the main theorem. Figure 2 illustrates the two cases in the proof.

Theorem 2 *A left-linear locally confluent TRS $\mathcal{R}$ is confluent if $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ is terminating.*

Proof Since $\rightarrow^*_{\mathcal{R}}$ and $\mathrel{\circ\!\!\!\!\!\longrightarrow}^*_{\mathcal{R}}$ coincide, it is sufficient to prove confluence of $\mathrel{\circ\!\!\!\!\!\longrightarrow}_{\mathcal{R}}$. To this end, we use Theorem 1 with the *predecessor labeling* [27, Example 18] in which steps $t \mathrel{\circ\!\!\!\!\!\longrightarrow} u$ are labeled by any term s such that $s \rightarrow^* t$. Labels are compared with respect to the well-founded order $> = \rightarrow^+_{\mathsf{CPS}(\mathcal{R})/\mathcal{R}}$. Let $t \;{}_{s_1}\!\mathrel{\longleftarrow\!\!\!\!\!\circ\;} s \mathrel{\circ\!\!\!\!\!\longrightarrow}_{s_2} u$. Following Lemma 2, we distinguish two cases.

- If $t \mathrel{\circ\!\!\!\!\!\longrightarrow} v \mathrel{\longleftarrow\!\!\!\!\!\circ\;} u$ then $t \mathrel{\circ\!\!\!\!\!\longrightarrow}_{s_2} v \;{}_{s_1}\!\mathrel{\longleftarrow\!\!\!\!\!\circ\;} u$ follows from $s_2 \rightarrow^* s \rightarrow^* t$ and $s_1 \rightarrow^* s \rightarrow^* u$.
- Suppose $t \mathrel{\longleftarrow\!\!\!\!\!\circ\;} v \;{}_{\mathsf{CPS}(\mathcal{R})}\!\leftarrow s \rightarrow_{\mathsf{CPS}(\mathcal{R})} w \mathrel{\circ\!\!\!\!\!\longrightarrow} u$. From the inclusion $\rightarrow_{\mathsf{CPS}(\mathcal{R})} \subseteq \rightarrow_{\mathcal{R}}$ and local confluence of $\mathcal{R}$ we obtain $t \mathrel{\longleftarrow\!\!\!\!\!\circ\;} v \rightarrow^* \cdot {}^*\!\leftarrow w \mathrel{\circ\!\!\!\!\!\longrightarrow} u$. Because $s_1 > v$, $s_2 > w$, and $\rightarrow \subseteq \mathrel{\circ\!\!\!\!\!\longrightarrow}$, we obtain $t \mathrel{\overset{\vee}{\longleftrightarrow\!\!\!\!\!\!\!\circ\;\;}}{}^*_{s_1} \cdot \mathrel{\overset{\vee}{\longleftrightarrow\!\!\!\!\!\!\!\circ\;\;}}{}^*_{s_2} u$.

In both cases local decreasingness is established. Hence, the relation $\mathrel{\circ\!\!\!\!\!\longrightarrow}$ is confluent. $\square$

We present two examples showing the use of Theorem 2 to obtain confluence.

Example 2 Consider the TRS $\mathcal{R}$ from [12, p.28] consisting of the rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{h}(x,x)) \qquad \mathsf{g}(\mathsf{a}) \to \mathsf{g}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{h}(\mathsf{a},\mathsf{a}) \to \mathsf{g}(\mathsf{g}(\mathsf{a}))$$

The only critical pair $\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{a}))) \leftarrow\rtimes\rightarrow \mathsf{f}(\mathsf{h}(\mathsf{a},\mathsf{a}))$ is clearly joinable. The TRS $\mathsf{CPS}(\mathcal{R})$ consists of the rewrite rules

$$\mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{h}(\mathsf{a},\mathsf{a})) \qquad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{a})))$$

By taking the matrix interpretation (cf. [8])

$$\mathsf{f}_{\mathcal{M}}(\mathbf{x}) = \begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}\mathbf{x} \qquad \mathsf{g}_{\mathcal{M}}(\mathbf{x}) = \begin{pmatrix}1&0&0\\1&0&0\\1&1&0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \mathsf{a}_{\mathcal{M}} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\mathsf{h}_{\mathcal{M}}(\mathbf{x},\mathbf{y}) = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\mathbf{x} + \begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix}\mathbf{y} + \begin{pmatrix}0\\1\\0\end{pmatrix}$$

we obtain $\mathcal{R} \subseteq \geqslant_{\mathcal{M}}$ and $\mathsf{CPS}(\mathcal{R}) \subseteq >_{\mathcal{M}}$:

$$[\mathsf{f}(\mathsf{g}(\mathsf{a}))]_{\mathcal{M}} = \begin{pmatrix}2\\2\\0\end{pmatrix} > \begin{pmatrix}1\\2\\0\end{pmatrix} = [\mathsf{f}(\mathsf{h}(\mathsf{a},\mathsf{a}))]_{\mathcal{M}}$$

$$[\mathsf{f}(\mathsf{g}(\mathsf{a}))]_{\mathcal{M}} = \begin{pmatrix}2\\2\\0\end{pmatrix} > \begin{pmatrix}1\\1\\0\end{pmatrix} = [\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{a})))]_{\mathcal{M}}$$

Therefore $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ is terminating and confluence of $\mathcal{R}$ is concluded by Theorem 2.

Example 3 Consider the left-linear TRS $\mathcal{R}$

$$\begin{array}{lll} \mathsf{nats} \to \mathsf{0} : \mathsf{inc}(\mathsf{nats}) & \mathsf{inc}(x:y) \to \mathsf{s}(x) : \mathsf{inc}(y) & \mathsf{hd}(x:y) \to x \\ \mathsf{d}(x:y) \to x : (x : \mathsf{d}(y)) & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) & \mathsf{tl}(x:y) \to y \end{array}$$

which is Example 2 from [13] extended with the rule $\mathsf{d}(x:y) \to x : (x : \mathsf{d}(y))$. Since the only critical pair $\mathsf{inc}(\mathsf{tl}(\mathsf{0} : \mathsf{inc}(\mathsf{nats}))) \leftarrow\rtimes\rightarrow \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$ is joinable (cf. Example 7 below), $\mathcal{R}$ is locally confluent. The TRS $\mathsf{CPS}(\mathcal{R})$ consists of

$$\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \qquad \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{inc}(\mathsf{tl}(\mathsf{0} : \mathsf{inc}(\mathsf{nats})))$$

By taking the matrix interpretation

$$\begin{array}{lll} \mathsf{inc}_{\mathcal{M}}(\mathbf{x}) = \begin{pmatrix}1&0\\1&0\end{pmatrix}\mathbf{x} & \mathsf{hd}_{\mathcal{M}}(\mathbf{x}) = \mathbf{x} & \mathsf{0}_{\mathcal{M}} = \begin{pmatrix}0\\0\end{pmatrix} \\ \mathsf{nats}_{\mathcal{M}} = \begin{pmatrix}0\\1\end{pmatrix} & \mathsf{tl}_{\mathcal{M}}(\mathbf{x}) = \begin{pmatrix}1&1\\1&0\end{pmatrix}\mathbf{x} & \mathsf{s}_{\mathcal{M}}(\mathbf{x}) = \begin{pmatrix}1&1\\0&0\end{pmatrix}\mathbf{x} \\ \mathsf{d}_{\mathcal{M}}(\mathbf{x}) = \begin{pmatrix}1&1\\1&1\end{pmatrix}\mathbf{x} & :_{\mathcal{M}}(\mathbf{x},\mathbf{y}) = \begin{pmatrix}1&1\\1&1\end{pmatrix}\mathbf{x} + \mathbf{y} & \end{array}$$

we obtain $\mathcal{R} \subseteq \geqslant_{\mathcal{M}}$ and $\mathsf{CPS}(\mathcal{R}) \subseteq >_{\mathcal{M}}$:

$$[\mathsf{inc}(\mathsf{tl}(\mathsf{nats}))]_{\mathcal{M}} = \begin{pmatrix}1\\1\end{pmatrix} > \begin{pmatrix}0\\0\end{pmatrix} = [\mathsf{tl}(\mathsf{inc}(\mathsf{nats}))]_{\mathcal{M}} = [\mathsf{inc}(\mathsf{tl}(\mathsf{0} : \mathsf{inc}(\mathsf{nats})))]_{\mathcal{M}}$$

Hence $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ is terminating and Theorem 2 yields confluence of $\mathcal{R}$.

The following example[2] shows that left-linearity is essential in Theorem 2.

Example 4 Consider the non-left-linear TRS $\mathcal{R}$

$$\mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{f}(x,\mathsf{g}(x)) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

from [15]. Since $\mathsf{CPS}(\mathcal{R})$ is empty, termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ is trivial. However, $\mathcal{R}$ is not confluent because the term $\mathsf{f}(\mathsf{c},\mathsf{c})$ has two distinct normal forms. Moreover, considering $\mathsf{a} \mathrel{\leftarrow\!\!\circ\!\!-} \mathsf{f}(\mathsf{c},\mathsf{c}) \mathrel{-\!\!\circ\!\!\to} \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c}))$, one can see that left-linearity is essential for Lemma 2. Note that adding the non-left-linear rules to $\mathsf{CPS}(\mathcal{R})$ would not help to recover the result of Theorem 2 because $\{\mathsf{f}(x,x) \to \mathsf{a}, \mathsf{f}(x,\mathsf{g}(x)) \to \mathsf{b}\}$ is relatively terminating with respect to $\{\mathsf{c} \to \mathsf{g}(\mathsf{c})\}$.

We remark that left-linearity can be dispensed with in Theorem 2 when $\mathsf{CPS}(\mathcal{R})$ is replaced by $\mathcal{R}$. However, the resulting condition is identical to Knuth and Bendix' criterion [17] since termination of $\mathcal{R}/\mathcal{R}$ is equivalent to termination of $\mathcal{R}$.

Replacing $\mathsf{CPS}(\mathcal{R})$ in Theorem 2 by

$$\mathsf{CPS}^\flat(\mathcal{R}) = \{\ell_1\mu \to r_1\mu, \ell_2\mu \to r_2\mu \mid (\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu \text{ is an overlap of } \mathcal{R}\}$$

yields a correct but strictly weaker confluence criterion as termination of $\mathsf{CPS}^\flat(\mathcal{R})/\mathcal{R}$ implies termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ but not vice versa; $\mathsf{CPS}^\flat(\mathcal{R})/\mathcal{R}$ in Examples 2 and 3 is not terminating.

The next example explains why one cannot replace $\mathsf{CPS}(\mathcal{R})$ by one of its subsets

$$\mathsf{CPS}_1(\mathcal{R}) = \{\ell_2\mu \to \ell_2\mu[r_1\mu]_p \mid (\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu \text{ is an overlap of } \mathcal{R}\}$$

and

$$\mathsf{CPS}_2(\mathcal{R}) = \{\ell_2\mu \to r_2\mu \mid (\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu \text{ is an overlap of } \mathcal{R}\}.$$

Example 5 Consider the left-linear TRSs $\mathcal{R}_1 = \{\mathsf{a} \to \mathsf{c}, \mathsf{b} \to \mathsf{d}, \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b}), \mathsf{f}(\mathsf{b}) \to \mathsf{f}(\mathsf{a})\}$ and $\mathcal{R}_2 = \{\mathsf{f}(\mathsf{a}) \to \mathsf{c}, \mathsf{f}(\mathsf{b}) \to \mathsf{d}, \mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{a}\}$. Both TRSs are locally confluent but not confluent. We have $\mathsf{CPS}_1(\mathcal{R}_1) = \{\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{c}), \mathsf{f}(\mathsf{b}) \to \mathsf{f}(\mathsf{d})\}$ and $\mathsf{CPS}_2(\mathcal{R}_2) = \{\mathsf{f}(\mathsf{a}) \to \mathsf{c}, \mathsf{f}(\mathsf{b}) \to \mathsf{d}\}$. It is easy to see that $\mathsf{CPS}_1(\mathcal{R}_1)/\mathcal{R}_1$ and $\mathsf{CPS}_2(\mathcal{R}_2)/\mathcal{R}_2$ are terminating.

An extension of our main result is obtained by excluding critical pair steps from $\mathsf{CPS}(\mathcal{R})$ that originate from trivial overlaps. Let us denote the set $\{\ell_2\mu \to \ell_2\mu[r_1\mu]_p, \ell_2\mu \to r_2\mu \mid (\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ is an overlap of $\mathcal{R}$ such that $\ell_2\mu[r_1\mu]_p \neq r_2\mu\}$ by $\mathsf{CPS}'(\mathcal{R})$. The proof is based on the observation that Lemma 2 still holds when $\mathsf{CPS}(\mathcal{R})$ is replaced by $\mathsf{CPS}'(\mathcal{R})$. For this we use the following result [30, Proposition 8.8.23][3] which is known as the orthogonalization of weakly orthogonal proof terms. Here co-initial proof terms A and B are *weakly* orthogonal if there is no non-trivial overlap between any pair of redexes $\Delta_1 \sqsubseteq A$ and $\Delta_2 \sqsubseteq B$.

Lemma 3 *Let $\mathcal{R}$ be a left-linear TRS. If A and B are weakly orthogonal proof terms and $t \mathrel{{}_A{\leftarrow\!\!\circ\!\!-}} s \mathrel{-\!\!\circ\!\!\to_B} u$ then there are orthogonal proof terms A' and B' with $t \mathrel{{}_{A'}{\leftarrow\!\!\circ\!\!-}} s \mathrel{-\!\!\circ\!\!\to_{B'}} u$.* $\square$

Lemma 4 *Let $\mathcal{R}$ be a left-linear TRS. If $t \mathrel{\leftarrow\!\!\circ\!\!-} s \mathrel{-\!\!\circ\!\!\to} u$ then*

[2] This example contradicts [16, Theorem 4].

[3] In [30] the result is stated for weakly orthogonal TRSs but its simple proof goes through for arbitrary left-linear TRSs.

(a) $t \mathrel{\multimap\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\multimap} u$*, or*
(b) $t \mathrel{\leftarrow\!\!\!\multimap} \cdot {}_{\mathsf{CPS}'(\mathcal{R})}\!\leftarrow s \rightarrow_{\mathsf{CPS}'(\mathcal{R})} \cdot \mathrel{\multimap\!\!\!\rightarrow} u$*.*

Proof The proof is similar to the proof of Lemma 2. Let A and B be the proof terms such that $t \;{}_A\!\mathrel{\leftarrow\!\!\!\multimap}\; s \mathrel{\multimap\!\!\!\rightarrow}_B u$. If A and B are weakly orthogonal then there exist orthogonal proof terms A' and B' such that $t \;{}_{A'}\!\mathrel{\leftarrow\!\!\!\multimap}\; s \mathrel{\multimap\!\!\!\rightarrow}_{B'} u$ by Lemma 3. Hence we can complete this case as in the proof of Lemma 2. Otherwise, there is a non-trivial overlap between some redexes $\Delta_1 \sqsubseteq A$ and $\Delta_2 \sqsubseteq B$, and thus the second case in the proof of Lemma 2 goes through. ☐

Theorem 3 *A left-linear locally confluent TRS $\mathcal{R}$ is confluent if $\mathsf{CPS}'(\mathcal{R})/\mathcal{R}$ is terminating.*

Proof Simply replace $\mathsf{CPS}(\mathcal{R})$ by $\mathsf{CPS}'(\mathcal{R})$ and Lemma 2 by Lemma 4 in the proof of Theorem 2. ☐

Theorem 3 is a proper extension of Theorem 2 and the result stating that weakly orthogonal TRSs are confluent. (A left-linear TRS $\mathcal{R}$ is weakly orthogonal precisely when $\mathsf{CPS}'(\mathcal{R}) = \varnothing$.)

Remark 1 In [14] we claimed that Theorem 3 follows from the following property: Let $\mathcal{R}$ be a left-linear TRS and $\ell \rightarrow r \in \mathcal{R}$. If $\ell\sigma \mathrel{\multimap\!\!\!\rightarrow} t$ then

1. $t \in \{\ell\tau, r\tau\}$ for some τ with $\sigma \mathrel{\multimap\!\!\!\rightarrow} \tau$, or
2. $\ell\sigma \rightarrow_{\mathsf{CPS}'(\mathcal{R})} \cdot \mathrel{\multimap\!\!\!\rightarrow} t$ and $\ell\sigma \rightarrow_{\mathsf{CPS}'(\mathcal{R})} r\sigma$.

Here $\sigma \mathrel{\multimap\!\!\!\rightarrow} \tau$ is defined as $x\sigma \mathrel{\multimap\!\!\!\rightarrow} x\tau$ for all variables x. This, however, is incorrect. Consider the TRS $\mathcal{R}$ consisting of the rules $\mathsf{f}(x) \rightarrow x$ and $\mathsf{f}(\mathsf{f}(x)) \rightarrow \mathsf{f}(x)$. Note that $\mathsf{CPS}'(\mathcal{R})$ is empty. Let $\ell \rightarrow r$ be the second rule, $t = x$, and let σ be the empty substitution. We have $\ell\sigma \mathrel{\multimap\!\!\!\rightarrow} t$ but none of the above conditions holds.

Concerning the automation of Theorems 2 and 3, for checking relative termination we use the following criteria of Geser [9]:

Lemma 5 *For TRSs $\mathcal{R}$ and $\mathcal{S}$, $\mathcal{R}/\mathcal{S}$ is terminating if*

1. $\mathcal{R} = \varnothing$*, or*
2. $\mathcal{R} \cup \mathcal{S}$ *is terminating, or*
3. there exist a well-founded order $>$ and a quasi-order $\geqslant$ such that $>$ and $\geqslant$ are closed under contexts and substitutions, $\geqslant \cdot > \cdot \geqslant \;\subseteq\; >$, $\mathcal{R} \cup \mathcal{S} \subseteq \;\geqslant$, and $(\mathcal{R} \setminus >)/(\mathcal{S} \setminus >)$ is terminating. ☐

Based on this result, termination of $\mathsf{CPS}'(\mathcal{R})/\mathcal{R}$ is shown by repeatedly using the last condition to simplify $\mathsf{CPS}'(\mathcal{R})$ and $\mathcal{R}$. As soon as the first condition applies, termination is concluded. If the first condition does not apply and the third condition does not make progress, we try to establish termination of $\mathcal{R}$ (which implies termination of $\mathsf{CPS}'(\mathcal{R}) \cup \mathcal{R}$). For checking the third condition matrix interpretations and match-bound techniques [34] are used.

The final example in this section illustrates Theorem 3.

Example 6 Consider the left-linear TRS $\mathcal{R}$

$$\begin{array}{llll}
\mathsf{f}(\mathsf{a},\mathsf{b}) \rightarrow \mathsf{d} & \mathsf{a} \rightarrow \mathsf{c} & \mathsf{d} \rightarrow \mathsf{f}(\mathsf{a},\mathsf{c}) & \mathsf{f}(x,\mathsf{c}) \rightarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \\
 & \mathsf{b} \rightarrow \mathsf{c} & \mathsf{d} \rightarrow \mathsf{f}(\mathsf{c},\mathsf{b}) & \mathsf{f}(\mathsf{c},x) \rightarrow \mathsf{f}(\mathsf{c},\mathsf{c})
\end{array}$$

from [24]. One easily checks that all critical pairs are joinable. Hence $\mathcal{R}$ is locally confluent. Note that $\mathsf{CPS}'(\mathcal{R})$ consists of the rules

$$\begin{array}{lll} \mathsf{f}(\mathsf{a},\mathsf{b}) \to \mathsf{d} & \mathsf{f}(\mathsf{a},\mathsf{b}) \to \mathsf{f}(\mathsf{a},\mathsf{c}) & \mathsf{d} \to \mathsf{f}(\mathsf{a},\mathsf{c}) \\ & \mathsf{f}(\mathsf{a},\mathsf{b}) \to \mathsf{f}(\mathsf{c},\mathsf{b}) & \mathsf{d} \to \mathsf{f}(\mathsf{c},\mathsf{b}) \end{array}$$

Termination of $\mathsf{CPS}'(\mathcal{R})/\mathcal{R}$ can be shown by a simple linear polynomial interpretation:

$$\mathsf{f}_{\mathbb{N}}(x,y) = x+y \qquad \mathsf{a}_{\mathbb{N}} = \mathsf{b}_{\mathbb{N}} = 2 \qquad \mathsf{c}_{\mathbb{N}} = 0 \qquad \mathsf{d}_{\mathbb{N}} = 3$$

Hence, confluence of $\mathcal{R}$ is concluded from Theorem 3. Note that Theorem 2 is not applicable because $\mathsf{CPS}(\mathcal{R})$ contains the non-terminating rule $\mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{c})$.

4 Rule-Labeling

In this section we are concerned with the automation of Theorem 1 for proving confluence of TRSs. In [27] van Oostrom proposed the *rule-labeling heuristic* in which rewrite steps are partitioned according to the employed rewrite rules. If one can find an order on the rules of a *linear* TRS such that every critical pair is locally decreasing, confluence is guaranteed. A formalization of this heuristic is given below where ${}_{\alpha}\!\leftarrow \rtimes \rightarrow_{\beta}$ denotes the set of critical pairs obtained from overlaps (α, p, β). Let $\gtrsim$ be a quasi-order. The relation $\xrightarrow{\vee\wr}_{\alpha}$ denotes the union of $\rightarrow_{\beta}$ for all $\beta \lesssim \alpha$.

Lemma 6 *A linear TRS $\mathcal{R}$ is confluent if there exists a well-founded quasi-order $\gtrsim$ on the rules of $\mathcal{R}$ such that*

$${}_{\alpha}\!\leftarrow \rtimes \rightarrow_{\beta} \subseteq \overset{\vee}{\longleftrightarrow}{}^{*}_{\alpha} \cdot \xrightarrow{\vee\wr}{}^{=}_{\beta} \cdot \overset{\vee}{\longleftrightarrow}{}^{*}_{\alpha\beta} \cdot {}^{=}_{\alpha}\!\xleftarrow{\vee\wr} \cdot {}^{*}_{\beta}\!\overset{\vee}{\longleftrightarrow}$$

for all rewrite rules $\alpha, \beta \in \mathcal{R}$. Here $>$ denotes the strict part of $\gtrsim$. □

The heuristic readily applies to the following example from [13].

Example 7 Consider the linear TRS $\mathcal{R}$ consisting of the following five of the rewrite rules in Example 3:

$$\begin{array}{llll} 1: \mathsf{nats} \to \mathsf{0} : \mathsf{inc}(\mathsf{nats}) & 2: \quad \mathsf{inc}(x:y) \to \mathsf{s}(x) : \mathsf{inc}(y) & 4: \mathsf{hd}(x:y) \to x \\ & 3: \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) & 5: \quad \mathsf{tl}(x:y) \to y \end{array}$$

There is one critical pair: $s = \mathsf{inc}(\mathsf{tl}(\mathsf{0} : \mathsf{inc}(\mathsf{nats}))) \underset{1}{\leftarrow} \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \underset{3}{\rightarrow} \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) = t$. We have

$$s \underset{5}{\rightarrow} \mathsf{inc}(\mathsf{inc}(\mathsf{nats})) \underset{5}{\leftarrow} \mathsf{tl}(\mathsf{s}(\mathsf{0}) : \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \underset{2}{\leftarrow} \mathsf{tl}(\mathsf{inc}(\mathsf{0} : \mathsf{inc}(\mathsf{nats}))) \underset{1}{\leftarrow} t$$

Hence the critical pair is locally decreasing with respect to the rule-labeling heuristic together with the order $3 > 2,5$.

The following example (Vincent van Oostrom, personal communication) shows that linearity in Lemma 6 cannot be weakened to left-linearity.

Example 8 Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$1\colon \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{c} \qquad 2\colon \mathsf{f}(\mathsf{b},x) \to \mathsf{f}(x,x) \qquad 3\colon \mathsf{f}(x,\mathsf{b}) \to \mathsf{f}(x,x) \qquad 4\colon \mathsf{a} \to \mathsf{b}$$

There are three critical pairs:

$$\mathsf{f}(\mathsf{a},\mathsf{b}) \underset{4}{\leftarrow} \mathsf{f}(\mathsf{a},\mathsf{a}) \underset{1}{\rightarrow} \mathsf{c} \qquad \mathsf{f}(\mathsf{b},\mathsf{a}) \underset{4}{\leftarrow} \mathsf{f}(\mathsf{a},\mathsf{a}) \underset{1}{\rightarrow} \mathsf{c} \qquad \mathsf{f}(\mathsf{b},\mathsf{b}) \underset{2}{\leftarrow} \mathsf{f}(\mathsf{b},\mathsf{b}) \underset{3}{\rightarrow} \mathsf{f}(\mathsf{b},\mathsf{b})$$

Since $\mathsf{f}(\mathsf{a},\mathsf{b}) \underset{3}{\rightarrow} \mathsf{f}(\mathsf{a},\mathsf{a}) \underset{1}{\rightarrow} \mathsf{c}$ and $\mathsf{f}(\mathsf{b},\mathsf{a}) \underset{2}{\rightarrow} \mathsf{f}(\mathsf{a},\mathsf{a}) \underset{1}{\rightarrow} \mathsf{c}$, it follows that the critical pairs are locally decreasing by taking the order $4 > 2,3$. Nevertheless, the conversion $\mathsf{f}(\mathsf{b},\mathsf{b}) \leftarrow \mathsf{f}(\mathsf{b},\mathsf{a}) \leftarrow \mathsf{f}(\mathsf{a},\mathsf{a}) \rightarrow \mathsf{c}$ reveals that $\mathcal{R}$ is not confluent.

We show how to implement Lemma 6. From now on we assume that TRSs are finite. We start by observing that the condition of Lemma 6 is undecidable even for locally confluent TRSs.

Lemma 7 *The following decision problem is undecidable:*

instance: a locally confluent linear TRS $\mathcal{R}$,
question: are all critical pairs locally decreasing with respect to the rule-labeling heuristic?

Proof We provide a reduction from the problem whether two (arbitrary) combinators in combinatory logic are convertible. The undecidability of the latter is well-known [6]. So let s and t be arbitrary ground terms in combinatory logic. We extend the TRS $\mathcal{CL}$ with fresh constants a,b and the rewrite rules $\{\mathsf{a} \to s, \mathsf{b} \to t, \mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{a}\}$ to obtain the TRS $\mathcal{R}$. If s and t are convertible (in $\mathcal{CL}$) then all critical pairs of $\mathcal{R}$ are locally decreasing by ordering the rules of $\mathcal{CL}$ below the above four rules. If s and t are not convertible, then no order on the rules will make the critical pairs locally decreasing with respect to the rule-labeling heuristic. So confluence of $\mathcal{R}$ can be established by the rule-labeling heuristic if and only if the terms s and t are convertible in $\mathcal{CL}$. $\square$

We explain how to obtain a decidable approximation of Lemma 6. The basic idea is to put a bound on the length of the conversions between the terms of each critical pair that are computed. Subsequently we test whether there exists an ordering of the rules such that at least one of the computed conversions satisfies the constraints. As we see later, a semi-decision procedure is obtained by simply repeating the test with a larger bound.

A sequence $(\gamma_1,\dots,\gamma_n)$ is called a *k-conversion instance* of (s,t) with respect to $\mathcal{R}$ if $n \leqslant k$, $\gamma_1,\dots,\gamma_n \in \mathcal{R}$, and

$$s \leftrightarrow_{\gamma_1} \cdots \leftrightarrow_{\gamma_n} t$$

Still, we face the following obstacle: *variable erasing* rules like $\mathsf{hd}(x : y) \to x$ may yield infinitely many such instances, even when s, t, and k are fixed. To obtain a computable approximation, we use only variable preserving rules in both directions. To cut down the search space further, we do not allow collapsing rules to be used from right to left. So we approximate $\leftrightarrow_{\mathcal{R}}$ by $\to_{\mathcal{R}^{\leftrightarrow}}$, where $\mathcal{R}^{\leftrightarrow}$ is the union of $\mathcal{R}$ and $\{r \to \ell \mid \ell \to r \in \mathcal{R}$, r is not a variable, and $\mathcal{V}\mathsf{ar}(\ell) = \mathcal{V}\mathsf{ar}(r)\}$. Now a $2k$-conversion instance in $\mathcal{R}$ is estimated by a k-join instance in $\mathcal{R}^{\leftrightarrow}$:

$$s \to_{\gamma_1} \cdots \to_{\gamma_m} \cdot {}_{\delta_n}\!\leftarrow \cdots {}_{\delta_1}\!\leftarrow t$$

with $m,n \leqslant k$. Below we reduce the ensuing constraints on the rule labeling to *precedence constraints* of the form

$$\phi ::= \top \mid \bot \mid \phi \vee \phi \mid \phi \wedge \phi \mid \alpha > \alpha \mid \alpha \sim \alpha$$

where α stands for variables corresponding to the rules in $\mathcal{R}$, and $\sim$ corresponds to the equivalence part of the quasi-order (of which $>$ is the strict part). From the encodings of termination methods for term rewriting, we know that the satisfiability of such precedence constraints is easily determined by SAT or SMT solvers (cf. [5, 33]).

Definition 2 For terms s, t and $k \geqslant 0$, a pair $((\gamma_1,\dots,\gamma_m),(\delta_1,\dots,\delta_n))$ is called a *k-join instance* of (s,t) with respect to a TRS $\mathcal{S}$ if $m,n \leqslant k$, $\gamma_1,\dots,\gamma_m,\delta_1,\dots,\delta_n \in \mathcal{S}$, and

$$s \to_{\gamma_1} \cdots \to_{\gamma_m} \cdot {}_{\delta_n}\!\leftarrow \cdots {}_{\delta_1}\!\leftarrow t$$

The *subsequence order* $\sqsupseteq$ is defined as $(a_1,\dots,a_n) \sqsupseteq (a_{i_1},\dots,a_{i_m})$ whenever $1 \leqslant i_1 < \cdots < i_m \leqslant n$. The set of all minimal (with respect to $\sqsupseteq \times \sqsupseteq$) k-join instances of (s,t) is denoted by $J^k_{\mathcal{S}}(s,t)$. Let $\gamma_1,\dots,\gamma_n \in \mathcal{S}$, $0 \leqslant i \leqslant n$, and $1 \leqslant j \leqslant n$. We define

$$\Psi_{i,j}(\gamma_1,\dots,\gamma_n) = \begin{cases} \alpha > \gamma_j & \text{if } j < i \\ \beta \sim \gamma_j & \text{if } j = i \\ \alpha > \gamma_j \vee \beta > \gamma_j & \text{if } j > i \end{cases}$$

Moreover, the disjunction of

$$\bigwedge_{j=1}^{n} \Psi_{i,j}(\gamma_1,\dots,\gamma_n)$$

for all $0 \leqslant i \leqslant n$ is denoted by $\Phi^{\alpha}_{\beta}((\gamma_1,\dots,\gamma_n))$.

Note that Φ^{α}_{β} encodes the constraints imposed on the left part of the conclusion of a locally decreasing diagram for peaks of the form ${}_{\alpha}\!\leftarrow \cdot \to_{\beta}$. The next lemma explains why non-minimal pairs can be excluded from $J^k_{\mathcal{R}^{\leftrightarrow}}(s,t)$ and Example 9 shows the benefit of doing so.

Lemma 8 *If $\Phi^{\alpha}_{\beta}(\delta)$ is satisfiable and $\delta \sqsupseteq \gamma$ then $\Phi^{\alpha}_{\beta}(\gamma)$ is satisfiable.*

Proof Straightforward. □

Definition 3 Let $\mathcal{R}$ be a TRS. We define $\mathsf{RL}_k(\mathcal{R})$ as the conjunction of

$$\bigvee \{ \Phi^{\ell_1\to r_1}_{\ell_2\to r_2}(\gamma) \wedge \Phi^{\ell_2\to r_2}_{\ell_1\to r_1}(\delta) \mid (\gamma,\delta) \in J^k_{\mathcal{R}^{\leftrightarrow}}(\ell_2[r_1]_p\mu, r_2\mu) \}$$

for all overlaps $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ of $\mathcal{R}$ and

$$(\ell \to r) \sim (r \to \ell)$$

for all $\ell \to r \in \mathcal{R}$ with $\mathcal{V}\mathsf{ar}(\ell) = \mathcal{V}\mathsf{ar}(r)$.

We illustrate the encoding on a concrete example.

Example 9 Consider again the TRS $\mathcal{R}$ of Example 7. We already computed the critical pair $s = \mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats}))) \leftarrow\!\rtimes\!\to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) = t$ arising from the single overlap $(1, 1{\cdot}1, 3)$. We show how $\mathsf{RL}_4(\mathcal{R})$ is computed. The TRS $\mathcal{R}^{\leftrightarrow}$ is the union of $\mathcal{R}$ and the three rules

$$6{:}\ 0 : \mathsf{inc}(\mathsf{nats}) \to \mathsf{nats} \quad 7{:}\ \mathsf{s}(x) : \mathsf{inc}(y) \to \mathsf{inc}(x : y) \quad 8{:}\ \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \to \mathsf{inc}(\mathsf{tl}(\mathsf{nats}))$$

There are 426 4-join instances of (s,t) with respect to $\mathcal{R}^{\leftrightarrow}$:

$$\begin{array}{llll}
\underline{((),(8,1))} & \underline{((6),(8))} & \underline{((5),(1,2,5))} & \underline{((6,3),())} \\
((1),(8,1,1)) & ((1,5),(1,1,2,5)) & ((5,1),(1,1,2,5)) & ((6,3,1),(1)) \\
((1,1),(8,1,1,1)) & ((1,5),(1,2,1,5)) & ((5,1),(1,2,1,5)) & ((6,3,1,1),(1,1)) \\
\quad\cdots & \quad\cdots & \quad\cdots & \quad\cdots
\end{array}$$

Only the four underlined instances belong to $J^4_{\mathcal{R}^{\leftrightarrow}}(s,t)$. For example, because $(1,5) \sqsupseteq (5)$ and $(1,2,1,5) \sqsupseteq (1,2,5)$, the instance $((1,5),(1,2,1,5))$ does not belong to $J^4_{\mathcal{R}^{\leftrightarrow}}(s,t)$. It follows that $\mathsf{RL}_4(\mathcal{R})$ is the conjunction of $1 \sim 6$, $2 \sim 7$, $3 \sim 8$, and

$$\bigvee \{ \Phi^1_3(\alpha) \wedge \Phi^3_1(\beta) \mid (\alpha,\beta) \in \{((),(8,1)),((6),(8)),((5),(1,2,5)),((6,3),())\} \}$$

Here, for example, $\Phi^1_3((5)) = (3 > 5 \vee 1 > 5) \vee 3 \sim 5$ and $\Phi^3_1((1,2,5))$ is the disjunction of the following four formulas:

$$\begin{array}{lllll}
(1 > 1 \vee 3 > 1) & \wedge\ (1 > 2 \vee 3 > 2) & \wedge\ (1 > 5 \vee 3 > 5) \\
1 \sim 1 & \wedge\ (1 > 2 \vee 3 > 2) & \wedge\ (1 > 5 \vee 3 > 5) \\
\qquad 3 > 1\ \wedge\ 1 \sim 2 & & \wedge\ (1 > 5 \vee 3 > 5) \\
\qquad 3 > 1\ \wedge & 3 > 2\ \wedge\ 1 \sim 5 &
\end{array}$$

This formula is satisfied by taking (e.g.) $3 > 1$, $3 > 2$, $1 \sim 5$, $1 \sim 6$, $2 \sim 7$, and $3 \sim 8$. Hence, confluence of $\mathcal{R}$ is concluded by local decreasingness with respect to the rule labeling heuristic using at most 3 steps to close critical pairs.

Typically, there are a large number ($|\mathcal{S}|^{2k}$ in the worst case) and hence it is expensive to compute of k-join instances $J^k_{\mathcal{S}}(s,t)$ after computing all k-join instances. Instead, we compute $J^k_{\mathcal{S}}(s,t)$ iteratively, minimizing the intermediate ingredients. This is achieved by the recursive definition of $X^{(k)}$ in the following characterisation of $J^k_{\mathcal{S}}(s,t)$.

Lemma 9 *For all $k \geqslant 0$ the following identity holds:*

$$J^k_{\mathcal{S}}(s,t) = \min\ \{(\gamma,\delta) \mid (\gamma,s') \in \{((),s)\}^{(k)},\ (\delta,t') \in \{((),t)\}^{(k)},\ \textit{and } s' = t'\}$$

where

$$X^{(k)} = \begin{cases} X & \textit{if } k = 0 \\ \min\ \left(X \cup \{(\gamma\alpha,t) \mid (\gamma,s) \in X,\ \alpha \in \mathcal{S},\ \textit{and } s \to_\alpha t\}^{(k-1)}\right) & \textit{otherwise} \end{cases}$$

Proof Straightforward. □

In the definition of $X^{(k)}$, $\gamma\alpha$ denotes the result of appending α to the sequence of rewrite rules γ, and $\min X$ computes the set of all minimal elements in X with respect to $\sqsupseteq \times =$.

Theorem 4 *A linear TRS $\mathcal{R}$ is confluent if $\mathsf{RL}_k(\mathcal{R})$ is satisfiable for some $k \geqslant 0$.* □

The following example shows that the equivalence constraints $(\ell \to r) \sim (r \to \ell)$, which express that the orientation of rewrite rules has no influence on the label, are essential for the soundness of $\mathsf{RL}_k(\mathcal{R})$.

Example 10 Consider the non-confluent linear TRS $\mathcal{R}$ consisting of the rules $\mathsf{a} \to \mathsf{b}$ and $\mathsf{a} \to \mathsf{c}$. If one would drop the equivalence constraints $(\mathsf{a} \to \mathsf{b}) \sim (\mathsf{b} \to \mathsf{a})$ and $(\mathsf{a} \to \mathsf{c}) \sim (\mathsf{c} \to \mathsf{a})$ from $\mathsf{RL}_1(\mathcal{R})$, the resulting formula would be satisfied by the order $(\mathsf{a} \to \mathsf{b}) > (\mathsf{b} \to \mathsf{a}) > (\mathsf{c} \to \mathsf{a})$.

A natural idea to reduce the size of the encoding further is to restrict the search space to valleys in $\mathcal{R}$ rather than its extension $\mathcal{R}^{\leftrightarrow}$ (which model conversions). In this way we get an approximation of the original version of decreasing diagrams [25]: A linear TRS $\mathcal{R}$ is confluent if there exists a well-founded quasi-order $\gtrsim$ on the rules of $\mathcal{R}$ such that

$$ {}_{\alpha}{\leftarrow} \rtimes {\to}_{\beta} \subseteq \xrightarrow{\vee}{}^{*}_{\alpha} \cdot \xrightarrow{\vee\wr}{}^{=}_{\beta} \cdot \xrightarrow{\vee}{}^{*}_{\alpha\beta} \cdot {}_{\alpha\beta}^{\;*}\xleftarrow{\vee} \cdot {}_{\alpha}^{=}\xleftarrow{\vee\wr} \cdot {}_{\beta}^{*}\xleftarrow{\vee} $$

for all rewrite rules $\alpha, \beta \in \mathcal{R}$. According to [27, Proof of Theorem 3], this version and Lemma 6 are equally powerful for obtaining confluence, complexity considerations left aside.

Definition 4 Let $\mathcal{R}$ be a TRS. We define $\mathsf{RLV}_k(\mathcal{R})$ as the conjunction of

$$ \bigvee \{ \Phi^{\ell_1 \to r_1}_{\ell_2 \to r_2}(\gamma) \wedge \Phi^{\ell_2 \to r_2}_{\ell_1 \to r_1}(\delta) \mid (\gamma, \delta) \in J^k_{\mathcal{R}}(\ell_2[r_1]_p\mu, r_2\mu) \} $$

for all overlaps $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ of $\mathcal{R}$.

Theorem 5 *A linear TRS $\mathcal{R}$ is confluent if $\mathsf{RLV}_k(\mathcal{R})$ is satisfiable for some $k \geqslant 0$.* $\square$

We conclude the section by commenting upon the relative completeness of our encodings. Since the conversion version and the valley version are equally powerful, if Lemma 6 applies then Theorems 4 and 5 apply as well, despite the approximations of $\leftrightarrow_{\mathcal{R}}$ by $\to_{\mathcal{R}^{\leftrightarrow}}$ and $\to_{\mathcal{R}}$. So if Theorem 4 is applicable then also Theorem 5 can be used to establish confluence. Of course, the minimal k to satisfy $\mathsf{RLV}_k(\mathcal{R})$ may be larger than the one to satisfy $\mathsf{RL}_k(\mathcal{R})$. This is nicely illustrated in Example 13 in Section 6.

5 Assessment

All described techniques have been implemented in Saigawa, an open source confluence tool.[4] We used the tool to test our methods on a collection of 212 TRSs, consisting of the 106 TRSs in the ACP[5] (see Section 6) distribution, the TRSs of Examples 3, 5, 8, and 11, the TRSs $\mathcal{R}_5$ and $\mathcal{R}_{10}$ of Example 13, as well as those TRSs in version 8.0 of the Termination Problems Data Base[6] that are either non-terminating or not known to be terminating. (Rewrite systems with extra variables in right-hand sides of rewrite rules are excluded.) For reference, ACP proves that 68 of the 212 TRSs are not confluent. Of the remaining 144 TRSs, local confluence can be shown for 134 TRSs by means of

$$ \leftarrow \rtimes \to \;\subseteq\; \bigcup_{i,j \leqslant 5} \to^{i} \cdot \, {}^{j}\!\leftarrow \qquad (*) $$

and in addition it is known that the TRS $\mathcal{R}_{10}$ is locally confluent. Moreover, of these 135 locally confluent TRSs, 101 are left-linear and 56 are linear.

Table 1 summarizes the results.[7] The following techniques are used to produce the columns:

[4] http://www.jaist.ac.jp/project/saigawa/

[5] http://www.nue.riec.tohoku.ac.jp/tools/acp/

[6] http://termcomp.uibk.ac.at/status/downloads/tpdb-8.0.tar.gz

[7] The detailed results are available from http://www.jaist.ac.jp/project/saigawa/.

Table 1 Summary of experimental results (212 TRSs).

	(a)	(b)	(c)	(d)	(e)	(f)	(d,e,f)	(a,d,e,f)
YES	25	41	62	66	50	49	86	91
timeout (120 s)	0	1	1	1	4	0	–	–

Table 2 Summary of experimental results for the rule-labeling heuristic (75 TRSs).

	Theorem 4					Theorem 5				
k	1	2	3	4	5	1	2	3	4	5
YES	38	48	49	50	46	34	46	47	48	49
timeout (120 s)	0	0	2	4	8	0	0	0	0	0

Table 3 Data on examples from this article.

	(a)	(b)	(c)	(d)	(e)	(f)
Example 2	*0.73*	*0.08*	**10.48**	**15.54**	*0.08*	*0.08*
Example 3	*0.74*	*0.07*	**16.14**	**22.29**	*0.08*	*0.08*
Example 6	*0.73*	*0.08*	*0.08*	**4.18**	**1.64**	**0.14**
Example 7	*0.80*	*0.08*	**5.45**	**8.29**	**0.15**	**0.20**
Example 11	*0.74*	*0.08*	*0.09*	*0.07*	*0.08*	*0.08*
Example 13 ($\mathcal{R}_5$)	**0.80**	*0.08*	**9.99**	**13.60**	**0.98**	**0.14**
Example 13 ($\mathcal{R}_{10}$)	**1.46**	*0.08*	*0.08*	*0.07*	**3.87**	*0.13*

(a) Knuth and Bendix' criterion [17],
(b) orthogonality,
(c) Theorem 2,
(d) Theorem 3,
(e) Theorem 4 with $k = 4$,
(f) Theorem 5 with $k = 5$.

To obtain the data in columns (a), (c), and (d) we used the open source termination tool $\mathsf{T_TT_2}$ [18] to check the (relative) termination requirements. Since local confluence is undecidable for non-terminating TRSs, it is approximated by ($*$). For the data in columns (e) and (f) the SMT solver MiniSmt [35] is used. Columns (d,e,f) and (a,d,e,f) indicate the total number of different TRSs that can be handled by the respective techniques. We remark that the TRSs handled by (d) include those handled by (c). On our collection, the TRSs handled by (f) are also covered by (e).

Table 2 provides experimental data for Theorems 4 and 5 with $1 \leqslant k \leqslant 5$. We used the subset of 75 linear TRSs of the collection used for Table 1. Due to the larger number of conversion instances, Theorem 4 produces several timeouts for $k \geqslant 3$.

Individual data on the confluent TRSs from this article are presented in Table 3. The numbers in the table indicate runtime. Times in **boldface** denote that confluence was shown, while *italics* denote failure.

From Table 1 we observe that numerous confluent TRSs cannot be handled by Theorems 2 and 3. Among the 101 left-linear locally confluent TRSs, Theorem 3 failed to show confluence of 35 TRSs. The reason is the relative termination requirement. We indicate two different patterns.

Example 11 Consider the left-linear TRS $\mathcal{R}'$ obtained from the TRS $\mathcal{R}$ of Example 3 by replacing the rule $\mathsf{nats} \to 0 : \mathsf{inc}(\mathsf{nats})$ with the two rules

$$\mathsf{nats} \to \mathsf{from}(0) \qquad\qquad \mathsf{from}(x) \to x : \mathsf{inc}(\mathsf{from}(x))$$

The TRS $\mathcal{R}'$ is locally confluent but $\mathsf{CPS}(\mathcal{R}')/\mathcal{R}' = \mathsf{CPS}'(\mathcal{R}')/\mathcal{R}'$ is non-terminating:

$$\begin{aligned}
&\mathsf{from}(\mathsf{inc}(\mathsf{tl}(\mathsf{nats})))\\
&\quad \to_{\mathcal{R}'} \quad \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) : \mathsf{inc}(\mathsf{from}(\mathsf{inc}(\mathsf{tl}(\mathsf{nats}))))\\
&\quad \to_{\mathsf{CPS}(\mathcal{R}')} \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) : \mathsf{inc}(\mathsf{from}(\mathsf{inc}(\mathsf{tl}(\mathsf{nats}))))\\
&\quad \to_{\mathcal{R}'} \quad \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) : \mathsf{inc}(\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) : \mathsf{inc}(\mathsf{from}(\mathsf{inc}(\mathsf{tl}(\mathsf{nats})))))\\
&\quad \to_{\mathsf{CPS}(\mathcal{R}')} \cdots
\end{aligned}$$

Nevertheless, $\mathcal{R}'$ is easily seen to be confluent by observing that the two sides of the rule $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$ are convertible with respect to the other rules, which do not admit critical pairs.

The culprit in the above example is the recursive rule $\mathsf{from}(x) \to x : \mathsf{inc}(\mathsf{from}(x))$. The substitution of the left-hand side ℓ of an arbitrary rule $\ell \to r \in \mathsf{CPS}'(\mathcal{R}')$ for the variable x enables an infinite rewrite sequence in $\mathsf{CPS}'(\mathcal{R}')/\mathcal{R}'$ in which after each application of the rule instance $\mathsf{from}(\ell) \to \ell : \mathsf{inc}(\mathsf{from}(\ell))$, the first occurrence of ℓ is rewritten to r.

Out of the 56 locally confluent linear TRS, Theorem 4 only misses 7 TRSs. Four of these contain so-called AC rules that specify the associativity and commutativity of a function f:

$$f(x,y) \to f(y,x) \qquad\qquad f(x,f(y,z)) \to f(f(x,y),z)$$

TRSs that contain AC rules are beyond the results presented in this paper.[8] As a matter of fact, none of the Theorems 3, 4, and 5 is applicable when the TRS $\mathcal{R}$ under consideration contains AC rewrite rules. The reason is that AC rules make $\mathsf{CPS}'(\mathcal{R})/\mathcal{R}$ non-terminating and the critical pair $f(f(y,z),x) \leftarrow f(x,f(y,z)) \to f(x,f(z,y))$ is not locally decreasing (using the AC rules for f).

It is not difficult to generalize the above observations as relative non-termination criteria.

Lemma 10 *For TRSs $\mathcal{R}$ and $\mathcal{S} \neq \varnothing$, $\mathcal{S}/\mathcal{R}$ is non-terminating if*

1. *$\mathcal{S}$ is non-terminating, or*
2. *$t \to^{+}_{\mathcal{R}\cup\mathcal{S}} C[t\sigma]$ and $x \in \mathcal{V}\mathsf{ar}(x\sigma) \cap \mathcal{V}\mathsf{ar}(C)$ for some term t, context C, substitution σ, and variable x.*

Proof The claim is trivial when the first condition holds. Assume the second condition. Since rewriting does not introduce variables, we have $x \in \mathcal{V}\mathsf{ar}(t)$. Because $\mathcal{S} \neq \varnothing$, it contains a rule $\ell \to r$. Consider the substitutions $\mu = \{x \mapsto \ell\}$ and $\nu = \{x \mapsto r\}$. For every substitution τ with $x \in \mathcal{V}\mathsf{ar}(x\tau)$ we have $t\tau\mu \to^{+}_{\mathcal{R}\cup\mathcal{S}} C\tau\mu[t\sigma\tau\mu] \to^{+}_{\mathcal{S}} C\tau\nu[t\sigma\tau\mu]$ because $x \in \mathcal{V}\mathsf{ar}(C\tau)$. Thus $t\tau\mu \to^{+}_{\mathcal{S}/\mathcal{R}} C\tau\nu[t\sigma\tau\mu]$, and moreover, $x \in \mathcal{V}\mathsf{ar}(x\sigma\tau)$. Hence by repeating the above reasoning it follows that $\mathcal{S}/\mathcal{R}$ is non-terminating. ◻

As a consequence of Lemma 10, $\mathsf{CPS}'(\mathcal{R})/\mathcal{R}$ is non-terminating if $\mathsf{CPS}'(\mathcal{R})$ contains a rule of the form $\ell \to C[\ell\sigma]$ or if $\mathcal{R}$ contains such a rule with $x \in \mathcal{V}\mathsf{ar}(x\sigma) \cap \mathcal{V}\mathsf{ar}(C)$ (and $\mathsf{CPS}'(\mathcal{R})$ is non-empty). This condition holds for 21 out of the 35 TRSs that Theorem 3 misses, and hence it provides an effective criterion to avoid spending resources searching for a relative termination proof of $\mathsf{CPS}'(\mathcal{R})/\mathcal{R}$, which is bound to fail.

[8] In [3] a new technique for confluence of TRSs with AC rules is presented.

Lemma 10(2) makes the limitation of Theorem 3 manifest. Let $\mathcal{R}$ be a left-linear and locally confluent TRS. If $\mathsf{CPS}'(\mathcal{R})/\mathcal{R}$ is terminating and $t \to_{\mathcal{R}}^{+} C[t\sigma]$ with $x \in \mathcal{V}\mathsf{ar}(x\sigma) \cap \mathcal{V}\mathsf{ar}(C)$ for some term t, context C, substitution σ, and variable x, then $\mathcal{R}$ is weakly orthogonal.

6 Related Work

In his PhD thesis [9, Chapter 4], Geser presents a number of results connecting relative termination to confluence. Besides a couple of abstract results, he presents two conditions for the confluence of the union of two TRSs $\mathcal{R}$ and $\mathcal{S}$ such that $\mathcal{R}/\mathcal{S}$ is terminating.

The first one states that the union of a left-linear TRS $\mathcal{R}$ and a confluent TRS $\mathcal{S}$ is confluent provided $\mathcal{R}/\mathcal{S}$ is terminating and both

$$\xleftarrow[\mathcal{S}]{} \rtimes \xrightarrow[\mathcal{R}]{} \;\subseteq\; \xrightarrow[\mathcal{S}]{*} \cdot \xleftarrow[\mathcal{R}\cup\mathcal{S}]{*} \;\cup\; \xrightarrow[\mathcal{R}]{} \cdot \xrightarrow[\mathcal{R}\cup\mathcal{S}]{*} \cdot \xleftarrow[\mathcal{R}\cup\mathcal{S}]{*} \tag{1}$$

and

$$\xleftarrow[\mathcal{R}]{} \rtimes \xrightarrow[\mathcal{R}]{} \;\subseteq\; \xrightarrow[\mathcal{R}\cup\mathcal{S}]{*} \cdot \xleftarrow[\mathcal{R}\cup\mathcal{S}]{*} \tag{2}$$

Like Theorem 2 this result generalizes joinability of critical pairs for terminating left-linear TRSs (take $\mathcal{S} = \varnothing$). To use this condition to establish confluence of a given TRS, the challenge is to partition the rules into $\mathcal{R}$ and $\mathcal{S}$; placing all rules in $\mathcal{S}$ deflates the point of using the condition. Assuming local confluence, the partitioning must satisfy three properties: $\mathcal{R}/\mathcal{S}$ is terminating, $\mathcal{S}$ can be shown to be confluent by other means, and condition (1) holds for critical pairs between $\mathcal{R}$ and $\mathcal{S}$. Geser [9, p. 68] presents a non-left-linear combination which can be handled by his result.

Geser's second result states that the union of a left-linear TRS $\mathcal{R}$ and a linear TRS $\mathcal{S}$ is confluent provided $\mathcal{R}/\mathcal{S}$ is terminating and both

$$\xleftarrow[\mathcal{S}]{} \rtimes \xrightarrow[\mathcal{R}\cup\mathcal{S}]{} \;\subseteq\; \xrightarrow[\mathcal{S}]{=} \cdot \xleftarrow[\mathcal{R}\cup\mathcal{S}]{*} \;\cup\; \xrightarrow[\mathcal{R}]{} \cdot \xrightarrow[\mathcal{R}\cup\mathcal{S}]{*} \cdot \xleftarrow[\mathcal{R}\cup\mathcal{S}]{*}$$

and

$$\xleftarrow[\mathcal{R}]{} \rtimes \xrightarrow[\mathcal{R}]{} \;\subseteq\; \xrightarrow[\mathcal{R}\cup\mathcal{S}]{*} \cdot \xleftarrow[\mathcal{R}\cup\mathcal{S}]{*}$$

Due to the finite number of ways to partition a TRS, this result can be used as a stand-alone criterion for finite TRSs. Unlike Theorem 2 the result does not generalize orthogonality (if $\mathcal{S} = \varnothing$ then $\mathcal{R}$ must be terminating), but it is not subsumed by Theorem 2, as shown in the following example.

Example 12 Consider the TRS $\mathcal{R}$ consisting of the single rewrite rule $\mathsf{a} \to \mathsf{f}(\mathsf{b})$ and the TRS $\mathcal{S}$ consisting of the rules $\mathsf{a} \to \mathsf{f}(\mathsf{a})$ and $\mathsf{f}(x) \to x$. One easily checks that the conditions of Geser's second result are fulfilled. Note that his first result is also applicable if one shows confluence of $\mathcal{S}$ by e.g. orthogonality. Because $\mathsf{CPS}(\mathcal{R} \cup \mathcal{S})$ contains the non-terminating rule $\mathsf{a} \to \mathsf{f}(\mathsf{a})$, Theorem 2 is not applicable.

In 2009 the first confluence tool made its appearance: ACP [4] implements Knuth and Bendix' criterion as well as a variation for overlay systems based on innermost termination due to Ohlebusch [23, p.126], several critical pair criteria for left-linear TRSs (e.g. [15, 24, 26, 31]), and divide and conquer techniques based on persistence [2], layer-preservation [22], and commutativity [29]. The latest version of ACP [1] also supports the rule-labeling heuristic for the original version of decreasing diagrams (cf. Theorem 5). We use an example from [27] to illustrate that there are situations where the conversion version (cf. Theorem 4) is to be preferred.

Example 13 Consider the confluent TRSs $\mathcal{R}_n$ consisting of the rewrite rules

$$\mathsf{a}_i \to \mathsf{b}_i \qquad\qquad \mathsf{b}_i \to \mathsf{b}_{i+1}$$
$$\mathsf{a}_i \to \mathsf{c}_i \qquad\qquad \mathsf{c}_i \to \mathsf{c}_{i+1}$$

for all $1 \leqslant i \leqslant n$ and with $\mathsf{b}_{n+1} = \mathsf{c}_{n+1}$. Since $\mathsf{b}_i \to \mathsf{b}_{i+1} \leftarrow \mathsf{a}_{i+1} \to \mathsf{c}_{i+1} \leftarrow \mathsf{c}_i$ for all $i < n$ and $\mathsf{b}_i \to \mathsf{b}_{i+1} = \mathsf{c}_{i+1} \leftarrow \mathsf{c}_i$ for $i = n$, $\mathsf{RL}_2(\mathcal{R}_n)$ is satisfiable for all n. On the other hand, $\mathsf{RLV}_k(\mathcal{R}_n)$ is satisfiable only when $k \geqslant n$.

A major strength of ACP lies in an extended version of the rule-labeling heuristic for possibly non-right-linear TRSs. In this version the label information is extended by counting certain function symbols along the path from the root of the starting term to the root of the contracted redex, following a suggestion in [27].[9] ACP solves these problems by using an SMT solver. While our approach checks satisfiability after computing (and minimizing) all k-join instances, ACP adopts a generate and test approach to avoid the computation of all join instances.

ACP can show confluence of 109 TRSs of the collection in Section 5. Of these 109 TRSs, 93 are left-linear and 52 are linear. This includes all TRSs that are covered by (a,d,f), but it is easy to find examples that can be handled by our techniques but not by ACP. For instance, if we add the rules $\mathsf{b}_7 \to \mathsf{b}_0$ and $\mathsf{b}_7 \to \mathsf{c}_0$ to the TRS $\mathcal{R}_6$ of Example 13 to make it non-terminating, ACP fails (after more than 20 minutes of CPU time and producing more than 56 million lines of output)[10] whereas Theorem 4 succeeds for $k = 2$ in a fraction of a second. ACP's failure is due to the fact that it does not test RLV_k for large k, while its divide and conquer techniques cause a combinatorial explosion of non-confluent subproblems.

7 Conclusion

In this article we presented a new confluence result for TRSs based on the decreasing diagrams technique: A left-linear locally confluent TRS is confluent if its critical pair steps are relatively terminating with respect to its rewrite rules. Moreover, for linear TRSs we showed how the rule-labeling heuristic can be implemented by means of an encoding as a satisfiability problem.

As future work we plan to investigate whether the former result can be strengthened by decreasing the set $\mathsf{CPS}(\mathcal{R})$ of critical pair steps that need to be relatively terminating with respect to $\mathcal{R}$. We anticipate that some of the many critical pair criteria for confluence that have been proposed in the literature (e.g. [15, 24, 26]) can be used for this purpose. The idea here is to exclude the critical pair steps that give rise to critical pairs whose joinability can be shown by the conditions of the considered criterion. It would be of particular interest to extend Okui's criterion based on simultaneous critical pairs [24], because it can handle AC rules.

Another direction for future work is the extension of Theorems 2 and 3 to higher-order pattern rewrite systems (PRSs) as defined by Mayr and Nipkow [19]. For higher-order

[9] Very recently, a stronger semantic approach was announced in [32].

[10] ACP version 0.20.

rewrite systems several confluence criteria are known (e.g. [19, 20, 26]), including orthogonality and joinability of critical pairs for terminating systems. We expect that Theorem 2 can be extended from first-order TRSs to PRSs without much effort. When it comes to automation, however, much research remains to be done.

The results presented in this paper are restricted to left-linear TRSs. We are aware of two approaches to tackle non-left-linear TRSs. By relaxing overlaps to so-called E-overlaps one can formulate direct sufficient conditions for confluence (e.g. [10, 11, 28]). Another approach is based on decomposition techniques (e.g. [2, 22]). In particular it is worthwhile to investigate whether our methods can be used to establish commutativity instead of confluence. Commutative versions of orthogonality [29] and decreasing diagrams [27] are known. Moreover, the results of Geser mentioned in Section 6 illustrate how relative termination can be used in this setting.

Last but not least, in order to certify the output of confluence tools, we plan to formalize the confluence results presented in this paper in the Isabelle proof assistant [21].

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