Revisiting Matrix Interpretations for Proving Termination of Term Rewriting

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Abstract

Matrix interpretations are a powerful technique for proving termination of term rewrite systems, which is based on the well-known paradigm of interpreting terms into a domain equipped with a suitable well-founded order, such that every rewrite step causes a strict decrease. Traditionally, one uses vectors of non-negative numbers as domain, where two vectors are in the order relation if there is a strict decrease in the respective first components and a weak decrease in all other components. In this paper, we study various alternative well-founded orders on vectors of non-negative numbers based on vector norms and compare the resulting variants of matrix interpretations to each other and to the traditional approach. These comparisons are mainly theoretical in nature. We do, however, also identify one of these variants as a proper generalization of traditional matrix interpretations as a stand-alone termination method, which has the additional advantage that it gives rise to a more powerful implementation.

1998 ACM Subject Classification F.4.2 Grammars and Other Rewriting Systems, F.4.1 Mathematical Logic: Computational logic.

Keywords and phrases term rewriting, termination, matrix interpretations

Digital Object Identifier 10.4230/LIPIcs.RTA.2011.251

Category Regular Research Paper

1 Introduction

As far as research on termination of term rewrite systems is concerned, in recent years a lot of effort has been put on proving termination automatically. Indeed, many powerful techniques for establishing termination of term rewrite systems that have been developed in the course of time have been automated successfully, as is evident in the results of the (annual) international competition for termination tools.[1] In particular, the method of matrix interpretations greatly contributes to the success of these tools.

Matrix interpretations were originally introduced by Hofbauer and Waldmann in the context of string rewriting [8, 9], allowing them to solve hard termination problems like $\{\mathsf{aa} \to \mathsf{bc}, \mathsf{bb} \to \mathsf{ac}, \mathsf{cc} \to \mathsf{ab}\}$, Problem #104 in the RTA list of open problems.[2] One particular instance of the matrix method of [9] has been extended to term rewriting by Endrullis *et al.* in [4]. The method is based on the well-known paradigm of interpreting terms (strings) into a domain equipped with a suitable well-founded order, such that every

* Friedrich Neurauter is supported by a grant of the University of Innsbruck.
[1] http://termcomp.uibk.ac.at.
[2] http://rtaloop.mancoosi.univ-paris-diderot.fr.

22nd International Conference on Rewriting Techniques and Applications (RTA'11).
Editor: M. Schmidt-Schauß; pp. 251–266
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

rewrite step causes a strict decrease with respect to this order. In [4], the authors consider the set of vectors of natural numbers as domain and equip it with a well-founded order that is not total, such that two vectors are in the order relation if there is a strict decrease in the respective first components and a weak decrease in all other components. Function symbols are interpreted by suitable linear mappings represented by square matrices all of whose entries are natural numbers. In [1, 5, 15], the method of Endrullis *et al.* was lifted to the non-negative rational (real) numbers using the same technique that was already used to lift polynomial interpretations from the naturals to the rationals (reals) (cf. [7]). Recently, another generalization appeared in [3] that employs matrices of natural numbers (instead of vectors) as underlying domain and associates to each function symbol a linear matrix polynomial. In principle, this approach also allows for non-linear matrix polynomials.

In this paper, we re-examine the basics of the method of Endrullis *et al.*, especially focusing on the actual role of the well-founded order on vectors of natural numbers it is based on. Obviously, there are many other orders on vectors of natural numbers having the desired properties, so why the choice of this particular order? In [4], the justification is as follows (in addition to the very convincing fact that the resulting termination method is very powerful):

> *Of course other orders on vectors could have been chosen, too, but many of them are not suitable for our purpose. For instance, choosing a lexicographic order fails because then multiplication by a constant matrix is not monotone in general.*

But still the question remains whether there exist other orders inducing variants of matrix interpretations that are also useful for proving termination of term rewrite systems. To this end, we study various alternative well-founded orders on vectors of (natural) numbers based on vector norms. The underlying idea is that every rewrite step is supposed to decrease the "length" of the associated vectors. This leads directly to the notion of normed vector spaces, norms being a suitable measure of the length or magnitude of a vector. Basically, we consider two classes of orders, weakly decreasing orders, where two vectors are comparable only if there is a weak decrease in every single component, and orders without this property. The conclusion is that the latter kind of orders induces only weak forms of matrix interpretations that are no more powerful than linear polynomial interpretations. For weakly decreasing orders (like the order in [4]), however, the situation is different. That is to say that some of them do indeed induce matrix interpretations that are useful for proving termination. In particular, one of these variants subsumes traditional matrix interpretations and has the additional advantage that it gives rise to a more powerful implementation.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary definitions and terminology concerning matrix interpretations. In Section 3, we present the orders on vectors of natural numbers considered in this paper. Sections 4 and 5 are dedicated to matrix interpretations over weakly decreasing orders and the comparison between them, while Section 6 features matrix interpretations over non-weakly decreasing orders. In Section 7 we present a generalization of traditional matrix interpretations, before concluding in Section 8.

2 Preliminaries

As usual, we denote by $\mathbb{N}$ and $\mathbb{Z}$ the sets of natural and integer numbers. Given $N \in \{\mathbb{N}, \mathbb{Z}\}$, $>_N$ denotes the standard order of the respective domain. The *cardinality* of a (finite) set S is denoted by $|S|$. For any ring R, we denote the ring of all n-dimensional square matrices over R by $R^{n \times n}$. A matrix is *non-negative* if all its entries are non-negative. Abusing notation, we denote the set of all non-negative n-dimensional square matrices of $\mathbb{Z}^{n\times n}$ by $\mathbb{N}^{n\times n}$. As usual, we denote the *transpose* of a matrix (vector) M by M^T. For any vector $\vec{x} = (x_1, \ldots, x_n)^T$, $(\vec{x})_i$ denotes its i-th component. Likewise, M_{ij} denotes the entry in the i-th row and j-th column of a matrix M, and M_{j-} (M_{-j}) refers to the j-th row (column). A *zero column* is a column, where all entries are zero. For $M \in R^{n\times n}$ and $\mathcal{I} \subseteq \{1, \ldots, n\}$, $(M)_{\mathcal{I}}$ denotes the *submatrix* of M formed by the rows and columns whose indices are in the index set $\mathcal{I}$. Finally, a *permutation matrix* is a square matrix whose entries are all 0's and 1's, with exactly one 1 in each row and exactly one 1 in each column.

We assume familiarity with the basics of term rewriting [2, 14]. Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a signature, that is, a set of function symbols equipped with fixed arities. The set of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. By $\mathcal{V}\mathrm{ar}(t)$ we denote the set of variables occurring in a term t, and $|t|_x$ denotes the number of occurrences of the variable x. A *rewrite rule* is a pair of terms (ℓ, r), conveniently written as $\ell \to r$, such that ℓ is not a variable and all variables in r are contained in ℓ. A *term rewrite system* $\mathcal{R}$ (TRS for short) over $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a set of rewrite rules. The rewrite relation induced by $\to$ is denoted by $\to_{\mathcal{R}}$. As usual, $\to^*_{\mathcal{R}}$ denotes the reflexive transitive closure of $\to_{\mathcal{R}}$ and $\to^n_{\mathcal{R}}$ its n-th iterate. For notational convenience, we sometimes drop the subscript $\mathcal{R}$ if it is clear from the context.

An important concept for establishing termination of TRSs is the notion of well-founded monotone algebras. An $\mathcal{F}$-*algebra* $\mathcal{A}$ consists of a non-empty carrier A and interpretation functions $f_{\mathcal{A}} \colon A^n \to A$ for every n-ary function symbol $f \in \mathcal{F}$. By $[\alpha]_{\mathcal{A}}(\cdot) \colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to A$ we denote the usual evaluation function of $\mathcal{A}$ with respect to a variable assignment $\alpha \colon \mathcal{V} \to A$. An interpretation $f_{\mathcal{A}} \colon A^n \to A$ is monotone with respect to a binary relation $>_A$ on A if $f_A(a_1, \ldots, a_i, \ldots, a_n) >_A f_A(a_1, \ldots, b, \ldots, a_n)$ for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i >_A b$. A *well-founded monotone* $\mathcal{F}$-*algebra* is a pair $(\mathcal{A}, >_A)$, where $\mathcal{A}$ is an $\mathcal{F}$-algebra and $>_A$ is a well-founded order on A, such that every $f_{\mathcal{A}}$ is monotone with respect to $>_A$. It is well-known that a TRS $\mathcal{R}$ is terminating if and only if it is *compatible* with a well-founded monotone algebra $(\mathcal{A}, >_A)$, where compatibility means that for every rewrite rule $\ell \to r \in \mathcal{R}$, $[\alpha]_{\mathcal{A}}(\ell) >_A [\alpha]_{\mathcal{A}}(r)$ for all variable assignments $\alpha \colon \mathcal{V} \to A$.

3 Well-founded Orders on Vectors of Natural Numbers

In this section we introduce several well-founded orders on vectors of natural numbers serving as foundation for alternative kinds of matrix interpretations. We consider two classes of orders on $\mathbb{N}^n$, $n \geqslant 1$, weakly decreasing orders and non-weakly decreasing ones.

3.1 Weakly Decreasing Orders

We call a (partial) order $>$ on $\mathbb{N}^n$ *weakly decreasing* if $(x_1, \ldots, x_n)^T > (y_1, \ldots, y_n)^T$ implies $x_i \geqslant_{\mathbb{N}} y_i$ for all $i \in \{1, \ldots, n\}$. The component-wise (partial) order on $\mathbb{N}^n$ induced by $\geqslant_{\mathbb{N}}$ is denoted by $\geqslant^w$.

▶ **Definition 3.1.** Let $\mathcal{I} \subseteq \{1, \ldots, n\}$ be a non-empty index set, and let $\vec{x} = (x_1, \ldots, x_n)^T$ and $\vec{y} = (y_1, \ldots, y_n)^T$ be vectors in $\mathbb{N}^n$. We define relations $>^w_{\mathcal{I}}$, $>^w_{\Sigma}$, $>^w_{\ell}$ and $>^w_m$ on $\mathbb{N}^n$ as follows:

- Weak decrease + strict decrease in some component(s):

$$\vec{x} >^w_{\mathcal{I}} \vec{y} :\Longleftrightarrow \vec{x} \geqslant^w \vec{y} \wedge \exists\, j \in \mathcal{I} \colon x_j >_{\mathbb{N}} y_j$$

- Weak decrease + strict decrease in sum of components:

$$\vec{x} >^w_\Sigma \vec{y} :\iff \vec{x} \geqslant^w \vec{y} \wedge \sum_{i=1}^{n} x_i >_{\mathbb{N}} \sum_{i=1}^{n} y_i$$

- Weak decrease + strict decrease in Euclidean length:

$$\vec{x} >^w_\ell \vec{y} :\iff \vec{x} \geqslant^w \vec{y} \wedge \sum_{i=1}^{n} x_i^2 >_{\mathbb{N}} \sum_{i=1}^{n} y_i^2$$

- Weak decrease + strict decrease in maximum component:

$$\vec{x} >^w_m \vec{y} :\iff \vec{x} \geqslant^w \vec{y} \wedge \max_i x_i >_{\mathbb{N}} \max_i y_i$$

It is routine to verify that all these relations are in fact well-founded orders on vectors of natural numbers.

▶ **Lemma 3.2.** *The relations $>^w_{\mathcal{I}}$, $>^w_\Sigma$, $>^w_\ell$ and $>^w_m$ are well-founded orders on $\mathbb{N}^n$.* ◀

The relations listed above are not the only well-founded orders on $\mathbb{N}^n$. Numerous variations exist. Some of these (like parameterizing $>^w_\Sigma$ or $>^w_\ell$ by an index set $\mathcal{I}$) are implicitly covered because of the lemma below, while others (like demanding a strict decrease in all components specified by an index set $\mathcal{I}$) proved to be impractical.

Intuitively, the order $>^w_{\mathcal{I}}$ is a generalization of $>^w_1$, the order used in [4], where the strict decrease is not necessarily fixed to one specific component; in particular, $>^w_1 = >^w_{\mathcal{I}}$ for $\mathcal{I} = \{1\}$. Moreover, its extension to matrices yields the main order considered in [3]. As to the remaining three orders, two vectors being in relation means that there is a strict decrease in the lengths of the vectors with respect to the Manhattan, Euclidean or maximum norm, respectively [10]. The relationship between these orders is described in the following lemma.

▶ **Lemma 3.3.** *Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be non-empty index sets, such that $\mathcal{I} = \{1, \ldots, n\}$ and $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{I}$. Then the following statements hold:*
1. *$>^w_{\mathcal{J}} \subseteq >^w_{\mathcal{K}}$ and $>^w_1 = >^w_{\{1\}}$,*
2. *$>^w_1$ and $>^w_m$ are incomparable for $n \geqslant 2$, identical otherwise,*
3. *$>^w_m \subset >^w_{\mathcal{I}} = >^w_\Sigma = >^w_\ell$ for $n \geqslant 2$, all identical otherwise, and*
4. *$>^w_{\mathcal{I}}$ is the strict part of $\geqslant^w$.*

The last item gives rise to the following corollary stressing an important aspect of some of the orders considered above.

▶ **Corollary 3.4.** *For $\mathcal{I} = \{1, \ldots, n\}$, $>^w_{\mathcal{I}}$ is the most general of the weakly decreasing proper orders on $\mathbb{N}^n$ (in the sense that it subsumes any other such order).* ◀

3.2 Non-weakly Decreasing Orders

Taking a closer look at Definition 3.1, one observes that weak decreasingness is not the essential property for obtaining well-founded orders on vectors of natural numbers, which is all we need to build matrix interpretations upon. That is to say that the last three orders remain well-founded orders on $\mathbb{N}^n$ even after dropping this property. We denote the corresponding orders by $>_\Sigma$, $>_\ell$ and $>_m$, respectively. Concerning $>^w_{\mathcal{I}}$, one must be careful when dropping weak decreasingness because the resulting relation $>_{\mathcal{I}}$ is an order only if the index set $\mathcal{I}$ is a singleton set, in which case $>_{\mathcal{I}}$ is also well-founded. In the remainder of this paper this is implicitly assumed whenever we refer to $>_{\mathcal{I}}$. Finally, we note that all four orders coincide in the one-dimensional case ($n = 1$), all being equal to $>_{\mathbb{N}}$. For $n \geqslant 2$, however, all these orders are pairwise incomparable (for all singleton sets $\mathcal{I}$).

4 Matrix Interpretations and Weakly Decreasing Orders

In this section we take the orders introduced in Definition 3.1 and build matrix interpretations on top of them. According to Lemma 3.3 (item 3), we only have to consider the family of orders $(>^w_\mathcal{I})_\mathcal{I}$ parametrized by some non-empty index set $\mathcal{I} \subseteq \{1,\dots,n\}$ and $>^w_m$, the order induced by the maximum norm. We shall see, however, that the latter kind of matrix interpretation is subsumed by an instance of the former.

Before we can go about formally defining matrix interpretations over $>^w_\mathcal{I}$ ($>^w_m$), we have to have an understanding of when a linear function is monotone with respect to the orders $\geqslant^w$ and $>^w_\mathcal{I}$ ($>^w_m$). We consider linear functions of the form $f(\vec{x}_1,\dots,\vec{x}_k) = \sum_{i=1}^k F_i\vec{x}_i + \vec{f}$, where $\vec{f} \in \mathbb{N}^n$ and $F_i \in \mathbb{N}^{n\times n}$ for all $i \in \{1,\dots,k\}$. Obviously, all such functions are monotone with respect to $\geqslant^w$. Concerning monotonicity with respect to $>^w_\mathcal{I}$, we give necessary and sufficient conditions in the lemma below. A similar lemma, showing sufficiency of the conditions, appeared in [3].

▶ **Lemma 4.1.** *Let $\mathcal{I} \subseteq \{1,\dots,n\}$ be a non-empty index set. The function $f(\vec{x}_1,\dots,\vec{x}_k) = \sum_{i=1}^k F_i\vec{x}_i + \vec{f}$ is monotone with respect to $>^w_\mathcal{I}$ if and only if for each $(F_i)_\mathcal{I}$, $i = 1,\dots,k$, all column sums are at least one.*

Proof. Let $\vec{x}_1,\dots,\vec{x}_k$ and $\vec{y}$ be arbitrary vectors in $\mathbb{N}^n$, such that $\vec{x}_i >^w_\mathcal{I} \vec{y}$ for some argument position $i \in \{1,\dots,k\}$. Then there exist a vector $\vec{d} \in \mathbb{N}^n$ and an index $j \in \mathcal{I}$, such that $\vec{x}_i = \vec{y} + \vec{d}$ and $d_j >_\mathbb{N} 0$. Now $f(\dots,\vec{x}_i,\dots) >^w_\mathcal{I} f(\dots,\vec{y},\dots)$ holds if and only if $F_i\,\vec{x}_i >^w_\mathcal{I} F_i\,\vec{y}$, which is equivalent to $F_i\,\vec{d} >^w_\mathcal{I} 0$. If all column sums of $(F_i)_\mathcal{I}$ are at least one, then we have $(F_i)_{-j} >^w_\mathcal{I} 0$, which yields $F_i\,\vec{d} >^w_\mathcal{I} 0$ because of $F_i\,\vec{d} \geqslant^w (F_i)_{-j} \cdot d_j \geqslant^w (F_i)_{-j}$.

Conversely, if $(F_i)_\mathcal{I}$ has a zero column, then let $j' \in \mathcal{I}$ denote the index of the column of F_i it originates from, and let $\vec{x}_i$ be zero everywhere except for its j'-th component, which we set to one. Then $\vec{x}_i >^w_\mathcal{I} 0$ but $f(\dots,0,\vec{x}_i,0,\dots) = (F_i)_{-j'} + \vec{f} \not>^w_\mathcal{I} \vec{f} = f(0,\dots,0)$. ◀

We are now ready to formally define matrix interpretations over instances of $>^w_\mathcal{I}$ (cf. also the E-compatible matrix interpretations in [3]).

▶ **Definition 4.2.** Let $\mathcal{F}$ denote a signature and $\mathcal{I} \subseteq \{1,\dots,n\}$ a non-empty index set. An n-dimensional *matrix interpretation* $\mathcal{M}_{>^w_\mathcal{I}}$ *over* $>^w_\mathcal{I}$ is an $\mathcal{F}$-algebra with carrier $\mathbb{N}^n$ together with the well-founded order $>^w_\mathcal{I}$, where each k-ary function symbol $f \in \mathcal{F}$ is interpreted by a function $f_\mathcal{M}\colon (\mathbb{N}^n)^k \to \mathbb{N}^n, (\vec{x}_1,\dots,\vec{x}_k) \mapsto \sum_{i=1}^k F_i\vec{x}_i + \vec{f}$ with $\vec{f} \in \mathbb{N}^n$ and $F_i \in \mathbb{N}^{n\times n}$ for all $i \in \{1,\dots,k\}$, such that $f_\mathcal{M}$ is monotone with respect to $>^w_\mathcal{I}$.

Clearly, such matrix interpretations are well-founded monotone algebras. Moreover, the notion of matrix interpretations of Endrullis *et al.* [4] is included in Definition 4.2 by choosing the special index set $\mathcal{I} = \{1\}$. In order to use a matrix interpretation $\mathcal{M}$ over $>^w_\mathcal{I}$ to establish termination of a TRS, one should be able to check whether $[\alpha]_\mathcal{M}(\ell) >^w_\mathcal{I} [\alpha]_\mathcal{M}(r)$ for all $\alpha\colon \mathcal{V} \to \mathbb{N}^n$ and rules $\ell \to r$. The following well-known lemma is helpful for this purpose.

▶ **Lemma 4.3.** *Let $\mathcal{M}$ be an $\mathcal{F}$-algebra with carrier $\mathbb{N}^n$ as in Definition 4.2 and t a term with $\mathcal{V}\mathrm{ar}(t) = \{x_1,\dots,x_m\}$. Then there exist matrices $T_1,\dots,T_m \in \mathbb{N}^{n\times n}$ and a vector $\vec{t} \in \mathbb{N}^n$, such that for any assignment $\alpha\colon \mathcal{V} \to \mathbb{N}^n$, $[\alpha]_\mathcal{M}(t) = T_1\alpha(x_1) + \dots + T_m\alpha(x_m) + \vec{t}$.* ◀

Therefore, the compatibility checks $[\alpha]_\mathcal{M}(\ell) >^w_\mathcal{I} [\alpha]_\mathcal{M}(r)$ and $[\alpha]_\mathcal{M}(\ell) \geqslant^w [\alpha]_\mathcal{M}(r)$ boil down to the comparison of such linear functions, which is decidable according to the next lemma. Here, $\geqslant$ denotes the component-wise (partial) order on $\mathbb{N}^{n\times n}$ induced by $\geqslant_\mathbb{N}$.

▶ **Lemma 4.4.** *Let $L_1, \dots, L_m$, $R_1, \dots, R_m$ and $\vec{l}$, $\vec{r}$ correspond to a rewrite rule $\ell \to r$ as in Lemma 4.3. Then, for $\triangleright \in \{>^w_{\mathcal{I}}, \geqslant^w\}$, $[\alpha]_{\mathcal{M}}(\ell) \triangleright [\alpha]_{\mathcal{M}}(r)$ for all variable assignments $\alpha\colon \mathcal{V} \to \mathbb{N}^n$ if and only if $\vec{l} \triangleright \vec{r}$ and $L_i \geqslant R_i$ for $i = 1, \dots, m$.* ◀

We close this section with the treatment of matrix interpretations over $>^w_m$. In particular, we show that they are subsumed by the instance of matrix interpretations over $>^w_{\mathcal{I}}$ one obtains by choosing $\mathcal{I} = \{1, \dots, n\}$, which is assumed to be the case in the rest of this section. According to Lemma 3.3, we have $>^w_m \subseteq >^w_{\mathcal{I}}$ for all dimensions $n \geqslant 1$. However, this does not directly imply that the same inclusion also holds for matrix interpretations based on these two orders because of the monotonicity requirement that all interpretation functions have to satisfy. If the monotonicity conditions with respect to $>^w_{\mathcal{I}}$ are more strict than the ones for $>^w_m$, then the set of potential interpretation functions is smaller, and it is therefore very well conceivable that the inclusion on the base orders does not propagate to the notions of matrix interpretations built on top of them. However, this is not the case for the two orders considered here.

▶ **Lemma 4.5.** *Let $f(\vec{x}_1, \dots, \vec{x}_k) = \sum_{i=1}^{k} F_i \vec{x}_i + \vec{f}$, where $\vec{f} \in \mathbb{N}^n$ and $F_1, \dots, F_k \in \mathbb{N}^{n \times n}$. Then monotonicity of f with respect to $>^w_m$ implies monotonicity with respect to $>^w_{\mathcal{I}}$.*

Proof. This can be shown using contraposition. Assume that f is not monotone with respect to $>^w_{\mathcal{I}}$. According to Lemma 4.1 this means that (at least) one of its matrices has a zero column. Without loss of generality, let the j-th column of some F_i, $i \in \{1, \dots, k\}$, be a zero column and let $\vec{x}_i$ be zero everywhere except for its j-th component. Then $\vec{x}_i >^w_m 0$ but $f(\dots, 0, \vec{x}_i, 0, \dots) = \vec{f} \not>^w_m \vec{f} = f(0, \dots, 0)$, i.e., f is not monotone with respect to $>^w_m$. ◀

Hence, if $\mathcal{M}_{>^w_m}$ is a matrix interpretation over $>^w_m$, consisting of a set of interpretation functions that are monotone with respect to $>^w_m$, then the same functions together with $>^w_{\mathcal{I}}$ constitute a matrix interpretation $\mathcal{M}_{>^w_{\mathcal{I}}}$ over $>^w_{\mathcal{I}}$ that is able to orient all rules orientable by $\mathcal{M}_{>^w_m}$ because of the inclusion $>^w_m \subseteq >^w_{\mathcal{I}}$. In other words, $\mathcal{M}_{>^w_{\mathcal{I}}}$ subsumes $\mathcal{M}_{>^w_m}$.

5 Comparing Matrix Interpretations over Weakly Decreasing Orders

After the discussion in the previous section the family of orders $(>^w_{\mathcal{I}})_{\mathcal{I}}$ parametrized by some non-empty index set $\mathcal{I} \subseteq \{1, \dots, n\}$ remains as a potentially interesting foundation for matrix interpretations. It includes the traditional order $>^w_1$ as well as $>^w_\Sigma = >^w_\ell$, the most general of the weakly decreasing orders on $\mathbb{N}^n$ (cf. Lemma 3.3 and Corollary 3.4). Now the purpose of this chapter is to compare the resulting variants of matrix interpretations to each other and thus also to the traditional approach.

First, we remark that we do not have to consider all possible index sets since matrix interpretations are invariant under permutations. For example, matrix interpretations over $>^w_{\{1\}}$ are equivalent to matrix interpretations over $>^w_{\{j\}}$, $j \in \{2, \dots, n\}$, with respect to termination proving power. The relevant property is that there is a strict decrease in a single fixed vector component, it is not important which component. All that matters is the cardinality of the index set $\mathcal{I}$. Hence, for n-dimensional matrix interpretations, we are left with n different index sets, and, without loss of generality, we can restrict to the sets $\mathcal{I}_d = \{1, \dots, d\}$ for $d = 1, 2, \dots, n$. By definition, the following inclusions hold: $>^w_{\mathcal{I}_1} \subset >^w_{\mathcal{I}_2} \subset \cdots \subset >^w_{\mathcal{I}_n}$. However, as explained at the end of the previous section, from this we cannot immediately conclude that the same inclusions also hold for matrix interpretations based on these orders because for $\mathcal{I} \subset \mathcal{J}$, monotonicity of a function with respect to $>^w_{\mathcal{I}}$ does not imply monotonicity with respect to $>^w_{\mathcal{J}}$ according to Lemma 4.1. In fact, the situation turns out to be a bit more intricate. To begin with, let us consider the following example.

▶ **Example 5.1.** Consider the TRS $\mathcal{R}_1 = \{\, \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})), \mathsf{g}(\mathsf{b}) \to \mathsf{g}(\mathsf{f}(\mathsf{b}))\,\}$. Termination of this system can be shown with the following 2-dimensional matrix interpretation over $>^w_{\{1,2\}}$:

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} \qquad \mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} \qquad \mathsf{a}_{\mathcal{M}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathsf{b}_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

However, one can show that there is no compatible 2-dimensional matrix interpretation over $>^w_{\{1\}}$. In the same vein, one can establish termination of the TRS $\mathcal{R}_2$

$$\begin{array}{lll} \mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{a}(\mathsf{g}(\mathsf{g}(\mathsf{f}(x))), \mathsf{g}(\mathsf{g}(\mathsf{f}(x))))) & \mathsf{a}(x,x) \to \mathsf{h}(x) & \mathsf{f}(x) \to x \\ \mathsf{h}(\mathsf{h}(x)) \to \mathsf{c}(\mathsf{h}(x)) & \mathsf{c}(x) \to x & \mathsf{g}(x) \to x \end{array}$$

via the following 2-dimensional matrix interpretation over $>^w_{\{1\}}$

$$\mathsf{a}_{\mathcal{M}}(\vec{x},\vec{y}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \mathsf{c}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \mathsf{h}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and show that there is no compatible 2-dimensional matrix interpretation over $>^w_{\{1,2\}}$.

The bottom line of this example is that if we fix the dimension, then matrix interpretations over $>^w_{\{1\}}$ are incomparable to matrix interpretations over $>^w_{\{1,2\}}$. (We are not aware of a general construction that works for any dimension.) However, without this restriction the situation is altogether different. That is to say that for dimension 3, for example, there is a compatible matrix interpretation over $>^w_{\{1\}}$ for the TRS $\mathcal{R}_1$. Likewise, there is a compatible 3-dimensional matrix interpretation over $>^w_{\{1,2\}}$ for the TRS $\mathcal{R}_2$. Indeed, that is no coincidence as will be shown in the remainder of this section. In particular, we shall see that in some sense the various instances of matrix interpretations over $>^w_{\mathcal{I}}$ are all equivalent with respect to termination proving power, no matter what the index set $\mathcal{I}$ looks like. To get to the bottom of this phenomenon, we need a couple of transformations on matrix interpretations.

As to the first transformation, let $P \in \mathbb{N}^{n \times n}$ be a nonsingular matrix and $\mathcal{M}$ some matrix interpretation consisting of a collection of interpretation functions $\{f_{\mathcal{M}}\}_{f \in \mathcal{F}}$, such that each k-ary function symbol f in the signature is interpreted by a function $f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_k) = \sum_{i=1}^{k} F_i \vec{x}_i + \vec{f}$, where $\vec{f} \in \mathbb{N}^n$ and $F_i \in \mathbb{N}^{n \times n}$ for all $i \in \{1, \ldots, k\}$. Then we associate with $\mathcal{M}$ a matrix interpretation $\Phi_P(\mathcal{M})$, where each k-ary function symbol f is interpreted by a function $f_{\Phi_P(\mathcal{M})}(\vec{x}_1, \ldots, \vec{x}_k) = \sum_{i=1}^{k} P F_i P^{-1} \vec{x}_i + P\vec{f}$.

▶ Remark. Note that in general PF_iP^{-1} is not a non-negative matrix, even if P and F_i are non-negative. As we need this property in our context, we must be careful when applying this transformation, unless P happens to be a (generalized) permutation matrix. In the remainder of this paper, non-negativity of PF_iP^{-1} is assumed or explicitly stated.

According to Lemma 4.3, the interpretation of a term with respect to $\mathcal{M}$ and a variable assignment α can be written as $[\alpha]_{\mathcal{M}}(t) = T_1\alpha(x_1) + \cdots + T_m\alpha(x_m) + \vec{t}$. By construction of $\Phi_P(\mathcal{M})$, we obtain the following lemma.

▶ **Lemma 5.2.** *Let $T_1, \ldots, T_m$ and $\vec{t}$ correspond to a term t as described in Lemma 4.3. Then $[\alpha]_{\Phi_P(\mathcal{M})}(t) = PT_1P^{-1}\alpha(x_1) + \cdots + PT_mP^{-1}\alpha(x_m) + P\vec{t}$ for any assignment α.* ◀

▶ **Corollary 5.3.** *For every ground term t, $[\alpha]_{\Phi_P(\mathcal{M})}(t) = P \cdot [\alpha]_{\mathcal{M}}(t)$.* ◀

Our next transformation associates with an n-dimensional matrix interpretation $\mathcal{M}$ (as above) an $(n+1)$-dimensional matrix interpretation $\Psi(\mathcal{M})$, where each k-ary function symbol f is interpreted by a function $f_{\Psi(\mathcal{M})}(\vec{x}_1, \dots, \vec{x}_k) = \sum_{i=1}^{k} F_i' \vec{x}_i + \vec{f}'$, such that for all $i \in \{1, \dots, k\}$,

$$\vec{f}' = \begin{pmatrix} 0 \\ \vec{f} \end{pmatrix} \text{ and } F_i' = \begin{pmatrix} f_i & 0 \\ 0 & F_i \end{pmatrix} \text{ for some } f_i \in \mathbb{N} \setminus \{0\}.$$

Moreover, we associate with $\mathcal{M}$ (resp. $\Psi(\mathcal{M})$) a linear polynomial interpretation $\mathcal{P}(\mathcal{M})$, where each k-ary function symbol f is interpreted by a linear polynomial $f_{\mathcal{P}(\mathcal{M})}(x_1, \dots, x_k) = \sum_{i=1}^{k} f_i x_i$ (with the f_i's of $\Psi(\mathcal{M})$).

▶ **Lemma 5.4.** *Let t be an arbitrary term. Then for all variable assignments $\alpha\colon \mathcal{V} \to \mathbb{N}^n$ and $\beta\colon \mathcal{V} \to \mathbb{N}$, the following statement holds:*

$$[\gamma]_{\Psi(\mathcal{M})}(t) = \begin{pmatrix} [\beta]_{\mathcal{P}(\mathcal{M})}(t) \\ [\alpha]_{\mathcal{M}}(t) \end{pmatrix} \text{ for the variable assignment } \gamma\colon \mathcal{V} \to \mathbb{N}^{n+1},\ x \mapsto \begin{pmatrix} \beta(x) \\ \alpha(x) \end{pmatrix}. \quad ◀$$

▶ **Corollary 5.5.** *For every ground term t, $[\gamma]_{\Psi(\mathcal{M})}(t) = \begin{pmatrix} 0 \\ [\alpha]_{\mathcal{M}}(t) \end{pmatrix}$.* ◀

Again, by Lemma 4.3, $[\alpha]_{\mathcal{M}}(t)$ can be written as $[\alpha]_{\mathcal{M}}(t) = T_1\alpha(x_1) + \dots + T_m\alpha(x_m) + \vec{t}$. Likewise, the interpretation of t with respect to $\mathcal{P}(\mathcal{M})$ and some variable assignment β can be written as $[\beta]_{\mathcal{P}(\mathcal{M})}(t) = t_1\beta(x_1) + \dots + t_m\beta(x_m)$, where $t_1, \dots, t_m \in \mathbb{N}$. Plugging these expressions into Lemma 5.4, we obtain the following lemma.

▶ **Lemma 5.6.** *Let $T_1, \dots, T_m$, $t_1, \dots, t_m$ and $\vec{t}$ correspond to a term t as described above. Then, in the situation of Lemma 5.4, the following statement holds:*

$$[\gamma]_{\Psi(\mathcal{M})}(t) = \sum_{i=1}^{m} \begin{pmatrix} t_i & 0 \\ 0 & T_i \end{pmatrix} \gamma(x_i) + \begin{pmatrix} 0 \\ \vec{t} \end{pmatrix} \quad ◀$$

Moreover, if all the f_i's introduced by $\Psi(\mathcal{M})$ are one, then each t_i in $[\beta]_{\mathcal{P}(\mathcal{M})}(t)$ corresponds to the number of occurrences of the associated variable x_i.

▶ **Lemma 5.7.** *Let t be an arbitrary term with $\mathcal{V}\mathrm{ar}(t) = \{x_1, \dots, x_m\}$, and let all interpretation functions in $\mathcal{P}(\mathcal{M})$ have the shape $f_{\mathcal{P}(\mathcal{M})}(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i$ (for each k-ary function symbol f). Then for any variable assignment β, $[\beta]_{\mathcal{P}(\mathcal{M})}(t) = \sum_{i=1}^{m} |t|_{x_i} \beta(x_i)$.* ◀

We are now ready to present the main results of this section comparing matrix interpretations over various instances of $>^w_{\mathcal{I}}$ with respect to proving (direct) termination of TRSs. In what follows, for a given TRS $\mathcal{R}$, $\mathcal{R} \subseteq \succcurlyeq_{\mathcal{P}(\mathcal{M})}$ abbreviates $[\beta]_{\mathcal{P}(\mathcal{M})}(\ell) \geqslant_{\mathbb{N}} [\beta]_{\mathcal{P}(\mathcal{M})}(r)$ for all variable assignments $\beta\colon \mathcal{V} \to \mathbb{N}$ and all rewrite rules $\ell \to r \in \mathcal{R}$.

▶ **Lemma 5.8.** *Let $\mathcal{M}$ be an n-dimensional matrix interpretation over $>^w_{\mathcal{I}}$, $\mathcal{I} \subseteq \{1, \dots, n\}$, and let $\mathcal{R}$ be a TRS satisfying $\mathcal{R} \subseteq \succcurlyeq_{\mathcal{P}(\mathcal{M})}$. Then compatibility of $\mathcal{R}$ with $\mathcal{M}$ implies compatibility with an $(n+1)$-dimensional matrix interpretation over $>^w_{\mathcal{J}}$, where $|\mathcal{J}| = |\mathcal{I}| + 1$, $\mathcal{J} \subseteq \{1, \dots, n+1\}$.*

Proof. Assuming that $\mathcal{M}$ is compatible with $\mathcal{R}$, we show that $\Psi(\mathcal{M})$ is compatible as well. To this end, we let $\mathcal{J} = \{1\} \cup \{\, x+1 \mid x \in \mathcal{I} \,\}$ and reason as follows. By assumption, all interpretation functions of $\mathcal{M}$ are monotone with respect to $>^w_{\mathcal{I}}$, that is, for each matrix $M \in \mathcal{M}$, all column sums of $(M)_{\mathcal{I}}$ are at least one according to Lemma 4.1. By construction of $\Psi(\mathcal{M})$, this implies that for each matrix $M' \in \Psi(\mathcal{M})$, all column sums of $(M')_{\mathcal{J}}$ are also at least one. Hence, all interpretation functions of $\Psi(\mathcal{M})$ are monotone with respect to $>^w_{\mathcal{J}}$. As to compatibility of $\Psi(\mathcal{M})$ with $\mathcal{R}$, for any rewrite rule $\ell \to r$, $[\gamma]_{\Psi(\mathcal{M})}(\ell) >^w_{\mathcal{J}} [\gamma]_{\Psi(\mathcal{M})}(r)$ holds for all variable assignments γ if and only if

$$\begin{pmatrix}[\beta]_{\mathcal{P}(\mathcal{M})}(\ell) \\ [\alpha]_{\mathcal{M}}(\ell)\end{pmatrix} >^w_{\mathcal{J}} \begin{pmatrix}[\beta]_{\mathcal{P}(\mathcal{M})}(r) \\ [\alpha]_{\mathcal{M}}(r)\end{pmatrix} \text{ for all variable assignments } \alpha \text{ and } \beta \text{ (cf. Lemma 5.4).}$$

By definition of $>^w_{\mathcal{J}}$, it remains to show that there is a weak decrease in every single component and a strict decrease in some component with index $j \in \mathcal{J}$. By compatibility of $\mathcal{M}$ with $\mathcal{R}$, we have $[\alpha]_{\mathcal{M}}(\ell) >^w_{\mathcal{I}} [\alpha]_{\mathcal{M}}(r)$ for all assignments α, which immediately establishes the latter requirement and, together with the assumption $\mathcal{R} \subseteq \succsim_{\mathcal{P}(\mathcal{M})}$, also the former. ◀

With the help of Lemma 5.7 one can replace the semantic condition $\mathcal{R} \subseteq \succsim_{\mathcal{P}(\mathcal{M})}$ by a (more familiar) syntactic condition.

▶ **Corollary 5.9.** *Let $\mathcal{R}$ be a non-duplicating TRS. Then compatibility of $\mathcal{R}$ with an n-dimensional matrix interpretation over $>^w_{\mathcal{I}}$, $\mathcal{I} \subseteq \{1, \dots, n\}$, implies compatibility with an $(n+1)$-dimensional matrix interpretation over $>^w_{\mathcal{J}}$, where $|\mathcal{J}| = |\mathcal{I}| + 1$, $\mathcal{J} \subseteq \{1, \dots, n+1\}$.*

Proof. Setting all the f_i's introduced by $\Psi(\mathcal{M})$ to one, the condition $\mathcal{R} \subseteq \succsim_{\mathcal{P}(\mathcal{M})}$ becomes equivalent to $\mathcal{R}$ being non-duplicating according to Lemma 5.7. ◀

▶ **Example 5.10.** Consider the TRS $\mathcal{R}_3 = \{\, \mathsf{f}(x) \to \mathsf{g}(\mathsf{h}(x,x)), \mathsf{g}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \,\}$. Termination can be shown with the following 2-dimensional matrix interpretation over $>^w_{\{1\}}$:

$$\mathsf{f}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix}3 & 0 \\ 2 & 0\end{pmatrix}\vec{x} + \begin{pmatrix}2 \\ 0\end{pmatrix} \qquad \mathsf{g}_{\mathcal{M}}(\vec{x}) = \begin{pmatrix}1 & 1 \\ 0 & 0\end{pmatrix}\vec{x} + \begin{pmatrix}1 \\ 0\end{pmatrix}$$

$$\mathsf{h}_{\mathcal{M}}(\vec{x},\vec{y}) = \begin{pmatrix}2 & 0 \\ 0 & 0\end{pmatrix}\vec{x} + \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\vec{y} \qquad \mathsf{a}_{\mathcal{M}} = \begin{pmatrix}0 \\ 3\end{pmatrix}$$

Moreover, the following linear polynomial interpretation orients all rules weakly:

$$\mathsf{f}_{\mathcal{P}(\mathcal{M})}(x) = 2x \qquad \mathsf{g}_{\mathcal{P}(\mathcal{M})}(x) = x \qquad \mathsf{h}_{\mathcal{P}(\mathcal{M})}(x,y) = x + y \qquad \mathsf{a}_{\mathcal{P}(\mathcal{M})} = 0$$

Hence, by (the proof of) Lemma 5.8, there exists a compatible 3-dimensional matrix interpretation over $>^w_{\{1,2\}}$.

Next we argue that the precondition $\mathcal{R} \subseteq \succsim_{\mathcal{P}(\mathcal{M})}$ in Lemma 5.8 is only technical in nature (to satisfy the formal definition of a well-founded monotone algebra); when it comes to termination proving power, it is actually superfluous.

▶ Remark. In the situation of Lemma 5.8, if $\mathcal{M}$ is compatible with $\mathcal{R}$, then, no matter whether $\mathcal{R} \subseteq \succsim_{\mathcal{P}(\mathcal{M})}$ or not, $\Psi(\mathcal{M})$ always establishes termination of $\mathcal{R}$ by proving the absence of infinite rewrite sequences of ground terms (assuming that the associated signature contains at least one constant symbol). This follows from Corollary 5.5 and compatibility of $\mathcal{M}$ with $\mathcal{R}$. Also note that in the proof of Lemma 5.8 the only purpose of the condition $\mathcal{R} \subseteq \succsim_{\mathcal{P}(\mathcal{M})}$ is to ensure a weak decrease in the first components of the vectors associated with some rewrite rule. However, for ground terms, we always have a weak decrease by Corollary 5.5.

Finally, we remark that by doubling the dimension of $\mathcal{M}$ these technicalities can be resolved.

▶ **Lemma 5.11.** *Compatibility of a TRS $\mathcal{R}$ with an n-dimensional matrix interpretation over $>^w_{\mathcal{I}}$, $\mathcal{I} \subseteq \{1,\dots,n\}$, implies compatibility with a $2n$-dimensional matrix interpretation over $>^w_{\mathcal{J}}$, where $|\mathcal{J}| = 2 \cdot |\mathcal{I}|$, $\mathcal{J} \subseteq \{1,\dots,2n\}$.*

In order to prove Lemma 5.11, we introduce a construction that combines two matrix interpretations $\mathcal{M}$ and $\mathcal{N}$ (not necessarily of the same dimension) to a matrix interpretation $\Pi(\mathcal{M},\mathcal{N})$ as follows. Assuming $\mathcal{M}$ consists of interpretation functions $f_{\mathcal{M}}(\vec{x}_1,\dots,\vec{x}_k) = \sum_{i=1}^k F_i\vec{x}_i + \vec{f}$ and $\mathcal{N}$ of interpretation functions $f_{\mathcal{N}}(\vec{x}_1,\dots,\vec{x}_k) = \sum_{i=1}^k \widetilde{F}_i\vec{x}_i + \tilde{f}$, the $\Pi(\mathcal{M},\mathcal{N})$-interpretation of each k-ary function symbol f is

$$f_{\Pi(\mathcal{M},\mathcal{N})}(\vec{x}_1,\dots,\vec{x}_k) = \sum_{i=1}^k \begin{pmatrix} F_i & 0 \\ 0 & \widetilde{F}_i \end{pmatrix} \vec{x}_i + \begin{pmatrix} \vec{f} \\ \tilde{f} \end{pmatrix}$$

▶ **Lemma 5.12.** *Let t be an arbitrary term. Then for all variable assignments α and β,*

$$[\gamma]_{\Pi(\mathcal{M},\mathcal{N})}(t) = \begin{pmatrix} [\alpha]_{\mathcal{M}}(t) \\ [\beta]_{\mathcal{N}}(t) \end{pmatrix} \quad \textit{for the variable assignment } \gamma(x) = \begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix}.$$

Proof. By induction on the structure of t. ◀

Proof of Lemma 5.11. Assuming that $\mathcal{M}$ is an n-dimensional matrix interpretation over $>^w_{\mathcal{I}}$ compatible with $\mathcal{R}$, we show that $\Pi(\mathcal{M},\mathcal{M})$ is compatible as well. To this end, we let $\mathcal{J} = \mathcal{I} \cup \{\, x+n \mid x \in \mathcal{I} \,\}$ and reason as follows. By assumption, all interpretation functions of $\mathcal{M}$ are monotone with respect to $>^w_{\mathcal{I}}$, that is, for each matrix $M \in \mathcal{M}$, all column sums of $(M)_{\mathcal{I}}$ are at least one according to Lemma 4.1. By construction of $\Pi(\mathcal{M},\mathcal{M})$, this implies that for each matrix $M' \in \Pi(\mathcal{M},\mathcal{M})$, all column sums of $(M')_{\mathcal{J}}$ are also at least one. Hence, all interpretation functions of $\Pi(\mathcal{M},\mathcal{M})$ are monotone with respect to $>^w_{\mathcal{J}}$. As to compatibility of $\Pi(\mathcal{M},\mathcal{M})$ with $\mathcal{R}$, for any rewrite rule $\ell \to r$, $[\gamma]_{\Pi(\mathcal{M},\mathcal{M})}(\ell) >^w_{\mathcal{J}} [\gamma]_{\Pi(\mathcal{M},\mathcal{M})}(r)$ holds for all variable assignments γ if and only if

$$\begin{pmatrix} [\alpha]_{\mathcal{M}}(\ell) \\ [\beta]_{\mathcal{M}}(\ell) \end{pmatrix} >^w_{\mathcal{J}} \begin{pmatrix} [\alpha]_{\mathcal{M}}(r) \\ [\beta]_{\mathcal{M}}(r) \end{pmatrix} \quad \text{for all variable assignments } \alpha \text{ and } \beta \text{ (cf. Lemma 5.12).}$$

But this follows directly from compatibility of $\mathcal{M}$ with $\mathcal{R}$ since $[\alpha]_{\mathcal{M}}(\ell) >^w_{\mathcal{I}} [\alpha]_{\mathcal{M}}(r)$ for all assignments α. ◀

Summarizing the above results, every TRS that can be proved terminating by a matrix interpretation over $>^w_{\mathcal{I}}$, for some index set $\mathcal{I}$, can also be proved terminating by a matrix interpretation over $>^w_{\mathcal{J}}$, for a larger index set $\mathcal{J}$, at the expense of an increased dimension.

Next we elaborate on the converse of this statement. To this end, let us consider some TRS $\mathcal{R}$ and a compatible n-dimensional matrix interpretation $\mathcal{M}$ over $>^w_{\mathcal{I}}$, where $|\mathcal{I}| > 1$, consisting of interpretation functions $f_{\mathcal{M}}(\vec{x}_1,\dots,\vec{x}_k) = \sum_{i=1}^k F_i\vec{x}_i + \vec{f}$ for each k-ary function symbol f in the signature. Our aim is to show that $\mathcal{R}$ is also compatible with a matrix interpretation over $>^w_{\{1\}}$ (or more generally, $>^w_{\mathcal{I}}$ for a singleton index set $\mathcal{I}$), albeit with a higher dimension.

First, we transform $\mathcal{M}$ into $\mathcal{M}' := \Psi(\mathcal{M})$, which is in turn transformed into $\mathcal{M}'' := \Phi_P(\mathcal{M}')$ for $P = I + U$, where I is the identity matrix and U is all zero except for the entries $U_{1,i+1} = 1$, for all $i \in \mathcal{I}$. As $P^{-1} = I - U$ is not non-negative, we have to ensure well-definedness of $\mathcal{M}''$, that is, make sure that all its matrices are non-negative. Now for any matrix (vector) M, PM is equal to M except for the first row, which is the sum of the rows of M with indices in $\{1\} \cup \{\, i+1 \mid i \in \mathcal{I} \,\}$. Hence,

$$P \begin{pmatrix} f_i & 0 \\ 0 & F_i \end{pmatrix} = \begin{pmatrix} f_i & \sum_{c\in\mathcal{I}}(F_i)_{c1} & \cdots & \sum_{c\in\mathcal{I}}(F_i)_{cn} \\ 0 & (F_i)_{-1} & \cdots & (F_i)_{-n} \end{pmatrix}$$

Multiplying this matrix by P^{-1} from the right has the effect of subtracting its first column from the columns with indices in $\{\, i{+}1 \mid i \in \mathcal{I} \,\}$, thus replacing $\sum_{c\in\mathcal{I}}(F_i)_{cj}$ by $\sum_{c\in\mathcal{I}}(F_i)_{cj} - f_i$ for all indices $j \in \mathcal{I}$ in the above representation. As these are the only entries that may eventually be negative, $\sum_{c\in\mathcal{I}}(F_i)_{cj} - f_i \geqslant 0$ for all $j \in \mathcal{I}$ implies well-definedness of $\mathcal{M}''$. Note, however, that if all the f_i's introduced by the transformation Ψ are one, then the latter condition is satisfied without further ado because, by assumption, all interpretation functions of $\mathcal{M}$ are monotone with respect to $>^w_{\mathcal{I}}$; hence, for all $j \in \mathcal{I}$, $\sum_{c\in\mathcal{I}}(F_i)_{cj}$ is at least one according to Lemma 4.1. Moreover, note that the top-left entry of each matrix occurring in $\mathcal{M}''$ is positive since $f_i > 0$. Consequently, all interpretation functions of $\mathcal{M}''$ are monotone with respect to $>^w_{\{1\}}$.

As to compatibility of $\mathcal{M}''$ with $\mathcal{R}$, let $\ell \to r$ be an arbitrary rule in $\mathcal{R}$, and let $[\alpha]_{\mathcal{M}}(\ell) = L_1\alpha(x_1) + \cdots + L_m\alpha(x_m) + \vec{l}$ and $[\alpha]_{\mathcal{M}}(r) = R_1\alpha(x_1) + \cdots + R_m\alpha(x_m) + \vec{r}$, where $x_1, \ldots, x_m$ are the variables occurring in ℓ and r. Likewise, let $[\beta]_{\mathcal{P}(\mathcal{M})}(\ell) = l_1\beta(x_1) + \cdots + l_m\beta(x_m)$, where $l_1, \ldots, l_m \in \mathbb{N}$, and similarly for $[\beta]_{\mathcal{P}(\mathcal{M})}(r)$. By compatibility of $\mathcal{M}$, we have $\vec{l} >^w_{\mathcal{I}} \vec{r}$ and $L_i \geqslant R_i$ for $i = 1, \ldots, m$ (cf. Lemma 4.4). Moreover, by Lemmata 5.2 and 5.6,

$$[\gamma]_{\mathcal{M}''}(\ell) = \sum_{i=1}^{m} P \begin{pmatrix} l_i & 0 \\ 0 & L_i \end{pmatrix} P^{-1}\gamma(x_i) + P \begin{pmatrix} 0 \\ \vec{l} \end{pmatrix} \quad \text{for } \gamma\colon \mathcal{V} \to \mathbb{N}^{n+1},\ x \mapsto \begin{pmatrix} \beta(x) \\ \alpha(x) \end{pmatrix}.$$

Therefore, $[\gamma]_{\mathcal{M}''}(\ell) >^w_{\{1\}} [\gamma]_{\mathcal{M}''}(r)$ holds for all variable assignments γ if and only if

$$P \begin{pmatrix} 0 \\ \vec{l} \end{pmatrix} >^w_{\{1\}} P \begin{pmatrix} 0 \\ \vec{r} \end{pmatrix} \quad \text{and} \quad P \begin{pmatrix} l_i - r_i & 0 \\ 0 & L_i - R_i \end{pmatrix} P^{-1} \geqslant 0 \text{ for } i = 1, \ldots, m.$$

The first condition follows directly from $\vec{l} >^w_{\mathcal{I}} \vec{r}$ and the shape of P. Concerning the second condition, we first rewrite the corresponding matrix to

$$\begin{pmatrix} l_i - r_i & \sum_{c\in\mathcal{I}}(L_i - R_i)_{c1} & \cdots & \sum_{c\in\mathcal{I}}(L_i - R_i)_{cn} \\ 0 & (L_i - R_i)_{-1} & \cdots & (L_i - R_i)_{-n} \end{pmatrix} P^{-1}.$$

Using the fact that $L_i \geqslant R_i$, the entire matrix is non-negative if and only if, for $i = 1, \ldots, m$, $l_i \geqslant r_i$ and $\sum_{c\in\mathcal{I}}(L_i - R_i)_{cj} \geqslant l_i - r_i$ for all $j \in \mathcal{I}$.

Based on these observations, we establish the following lemma.

▶ **Lemma 5.13.** *Let $\mathcal{M}$ be an n-dimensional matrix interpretation over $>^w_{\mathcal{I}}$, $\mathcal{I} \subseteq \{1, \ldots, n\}$, such that $|\mathcal{I}| > 1$, and let $\mathcal{R}$ be a TRS satisfying $\mathcal{R} \subseteq \succcurlyeq_{\mathcal{P}(\mathcal{M})}$. Moreover, assume that for each k-ary function symbol f, all column sums of each $(F_i)_{\mathcal{I}}$ are greater than or equal to f_i for all $i \in \{1, \ldots, k\}$, and that for each $\ell \to r \in \mathcal{R}$, all column sums of each $(L_i - R_i)_{\mathcal{I}}$ are greater than or equal to $l_i - r_i$ for all $i \in \{1, \ldots, m\}$. Then compatibility of $\mathcal{R}$ with $\mathcal{M}$ implies compatibility with an $(n+1)$-dimensional matrix interpretation over $>^w_{\mathcal{J}}$, where $|\mathcal{J}| = 1$, $\mathcal{J} \subseteq \{1, \ldots, n+1\}$.* ◀

▶ **Corollary 5.14.** *Let $\mathcal{M}$ be an n-dimensional matrix interpretation over $>^w_{\mathcal{I}}$, $\mathcal{I} \subseteq \{1, \ldots, n\}$, such that $|\mathcal{I}| > 1$, and let $\mathcal{R}$ be a non-duplicating TRS. Moreover, assume that for each $\ell \to r \in \mathcal{R}$, all column sums of each $(L_i - R_i)_{\mathcal{I}}$ are greater than or equal to $l_i - r_i$ for all $i \in \{1, \ldots, m\}$. Then compatibility of $\mathcal{R}$ with $\mathcal{M}$ implies compatibility with an $(n+1)$-dimensional matrix interpretation over $>^w_{\mathcal{J}}$, where $|\mathcal{J}| = 1$, $\mathcal{J} \subseteq \{1, \ldots, n+1\}$.*

Proof. Setting all the f_i's introduced by $\Psi(\mathcal{M})$ to one, the condition $\mathcal{R} \subseteq \succcurlyeq_{\mathcal{P}(\mathcal{M})}$ becomes equivalent to $\mathcal{R}$ being non-duplicating according to Lemma 5.7. Moreover, all column sums of each $(F_i)_{\mathcal{I}}$ are greater than or equal to $f_i = 1$ because all interpretation functions of $\mathcal{M}$ are monotone with respect to $>^w_{\mathcal{I}}$. ◀

By restricting the class of non-duplicating TRSs further, we can get rid of the condition that all column sums of $(L_i - R_i)_{\mathcal{I}}$ are greater than or equal to $l_i - r_i$.

▶ **Corollary 5.15.** *Let $\mathcal{R}$ be a TRS, such that for all $\ell \to r \in \mathcal{R}$, $|\ell|_x = |r|_x$ for all variables x. Then compatibility of $\mathcal{R}$ with an n-dimensional matrix interpretation over $>^w_{\mathcal{I}}$, where $|\mathcal{I}| > 1$ and $\mathcal{I} \subseteq \{1, \ldots, n\}$, implies compatibility with an $(n+1)$-dimensional matrix interpretation over $>^w_{\mathcal{J}}$, where $|\mathcal{J}| = 1$, $\mathcal{J} \subseteq \{1, \ldots, n+1\}$.* ◀

▶ Remark. Let all the f_i's introduced by $\Psi(\mathcal{M})$ be one. Then, in the situation of Lemma 5.13, if $\mathcal{M}$ is compatible with $\mathcal{R}$, then, no matter whether the other preconditions mentioned in the lemma are satisfied or not, $\mathcal{M}''$ always establishes termination of $\mathcal{R}$ by proving the absence of infinite rewrite sequences of ground terms. This can be seen as follows. Assume to the contrary that $t_1 \to t_2 \to t_3 \to \cdots$ is such an infinite sequence. By Corollaries 5.3 and 5.5, we have

$$[\gamma]_{\mathcal{M}''}(t_i) = P \cdot [\gamma]_{\Psi(\mathcal{M})}(t_i) = P \cdot \begin{pmatrix} 0 \\ [\alpha]_{\mathcal{M}}(t_i) \end{pmatrix} = \begin{pmatrix} \sum_{c \in \mathcal{I}}([\alpha]_{\mathcal{M}}(t_i))_c \\ [\alpha]_{\mathcal{M}}(t_i) \end{pmatrix}$$

From this and from compatibility of $\mathcal{M}$ with $\mathcal{R}$, we conclude that $[\gamma]_{\mathcal{M}''}(t_i) >^w_{\{1\}} [\gamma]_{\mathcal{M}''}(t_{i+1})$ holds for all $i \in \mathbb{N} \setminus \{0\}$ because of $[\alpha]_{\mathcal{M}}(t_i) >^w_{\mathcal{I}} [\alpha]_{\mathcal{M}}(t_{i+1})$. However, this contradicts well-foundedness of $>^w_{\{1\}}$.

6 Matrix Interpretations and Non-weakly Decreasing Orders

In this section we investigate the usefulness of the orders $>_\Sigma$, $>_\ell$, $>_m$ and $>_{\mathcal{I}}$ (where $\mathcal{I}$ is a singleton set) introduced in Subsection 3.2 for building matrix interpretations on top of them. As these orders originated from the orders introduced in Definition 3.1 by dropping the property of weak decreasingness, each of them obviously subsumes its ancestor, e.g., $>^w_\Sigma \subset >_\Sigma$, so that one is tempted to believe that these more general base orders would induce more powerful kinds of matrix interpretations. However, as already mentioned at the end of Section 4, an inclusion like $>^w_\Sigma \subset >_\Sigma$ does not necessarily propagate to the corresponding notions of matrix interpretations because of the monotonicity requirement all interpretation functions have to satisfy. Indeed, it turns out that the monotonicity conditions with respect to $>_\Sigma$, $>_\ell$, $>_m$ and $>_{\mathcal{I}}$ are much stronger than the ones associated with their respective weakly decreasing counterparts, ultimately resulting in weaker notions of matrix interpretations. In particular, we will see that matrix interpretations over $>_{\mathcal{I}}$ and $>_\Sigma$ are equivalent to linear polynomial interpretations.

As already mentioned in Subsection 3.2, all four orders are equal to $>_{\mathbb{N}}$ when the dimension n is one. Hence, matrix interpretations based on them are at least as powerful as linear polynomial interpretations. Next we show that matrix interpretations over $>_{\mathcal{I}}$ and $>_\Sigma$ are no more powerful than linear polynomial interpretations. Since matrix interpretations are invariant under permutations, we consider the index set $\mathcal{I} = \{1\}$ without loss of any generality.

▶ **Lemma 6.1.** *Let $f(\vec{x}_1, \ldots, \vec{x}_k) = \sum_{i=1}^{k} F_i \vec{x}_i + \vec{f}$, where $\vec{f} \in \mathbb{N}^n$ and $F_1, \ldots, F_k \in \mathbb{N}^{n \times n}$. Then $f(\vec{x}_1, \ldots, \vec{x}_k)$ is monotone with respect to $>_{\{1\}}$ if and only if for each F_i, $i = 1, \ldots, k$, $(F_i)_{11} \geqslant 1$ and $(F_i)_{12} = \cdots = (F_i)_{1n} = 0$.* ◀

Intuitively, this means that the first component of a function application $f(\vec{x}_1, \ldots, \vec{x}_k)$ only depends on the respective first components of its arguments, not on the other components. Based on this observation and the fact that for comparisons with $>_{\{1\}}$ only the first components matter, we associate the following linear polynomial interpretation $\mathcal{P}$ with a given matrix interpretation $\mathcal{M}$ over $>_{\{1\}}$. For each k-ary function symbol f, if $f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_k) = \sum_{i=1}^{k} F_i \vec{x}_i + \vec{f}$ is its interpretation in $\mathcal{M}$, with all matrices satisfying the conditions of Lemma 6.1, then we define its $\mathcal{P}$-interpretation as $f_{\mathcal{P}}(x_1, \ldots, x_k) = \sum_{i=1}^{k} (F_i)_{11}\, x_i + (\vec{f})_1$, which is monotone because $(F_i)_{11} \geqslant 1$. By construction, the $\mathcal{P}$-interpretation of an arbitrary term coincides with the first component of its $\mathcal{M}$-interpretation. The straightforward induction proof is omitted.

▶ **Lemma 6.2.** *Let $\mathcal{M}$ be a matrix interpretation over $>_{\{1\}}$ of dimension n, $\mathcal{P}$ the associated linear polynomial interpretation as described above and t an arbitrary term. Then for any variable assignment $\alpha\colon \mathcal{V} \to \mathbb{N}^n$, $\pi_1([\alpha]_{\mathcal{M}}(t)) = [\pi_1 \circ \alpha]_{\mathcal{P}}(t)$, where π_1 projects a vector to its first component.* ◀

Therefore, any rewrite rule $\ell \to r$ that is orientable by $\mathcal{M}$, is also orientable by $\mathcal{P}$ (since $\pi_1 \circ \alpha$ covers all assignments $\mathcal{V} \to \mathbb{N}$), which shows that matrix interpretations over $>_{\{1\}}$ ($>_{\mathcal{I}}$) are no more powerful than linear polynomial interpretations. The following lemma states that this is also the case for matrix interpretations over $>_{\Sigma}$.

▶ **Lemma 6.3.** *Let $f(\vec{x}_1, \ldots, \vec{x}_k) = \sum_{i=1}^{k} F_i \vec{x}_i + \vec{f}$, where $\vec{f} \in \mathbb{N}^n$ and $F_1, \ldots, F_k \in \mathbb{N}^{n \times n}$. Then $f(\vec{x}_1, \ldots, \vec{x}_k)$ is monotone with respect to $>_{\Sigma}$ if and only if for each F_i, $i = 1, \ldots, k$, all column sums are equal and at least one.* ◀

In analogy to the treatment of matrix interpretations over $>_{\{1\}}$, given an n-dimensional matrix interpretation $\mathcal{M}$ over $>_{\Sigma}$, we again associate a linear polynomial interpretation $\mathcal{P}$ with $\mathcal{M}$ as follows. For each k-ary function symbol f, if $f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_k) = \sum_{i=1}^{k} F_i \vec{x}_i + \vec{f}$ is its interpretation in $\mathcal{M}$, with all matrices satisfying the conditions of Lemma 6.3, then its $\mathcal{P}$-interpretation is defined as $f_{\mathcal{P}}(x_1, \ldots, x_k) = \sum_{i=1}^{k} F_i^{\Sigma} x_i + \sum_{j=1}^{n} (\vec{f})_j$, where F_i^{Σ} denotes the column sum of F_i, which is equal for all columns of F_i and at least one; hence, $f_{\mathcal{P}}$ is monotone. By construction, the $\mathcal{P}$-interpretation of an arbitrary term coincides with the sum of the components of its $\mathcal{M}$-interpretation.

▶ **Lemma 6.4.** *Let $\mathcal{M}$ be a matrix interpretation over $>_{\Sigma}$ of dimension n, $\mathcal{P}$ the associated linear polynomial interpretation as described above and t an arbitrary term. Then for any variable assignment $\alpha\colon \mathcal{V} \to \mathbb{N}^n$, $\sum_{j=1}^{n} ([\alpha]_{\mathcal{M}}(t))_j = [\alpha']_{\mathcal{P}}(t)$, where $\alpha'(x) = \sum_{j=1}^{n} (\alpha(x))_j$ for all $x \in \mathcal{V}$.* ◀

So, if a rewrite rule $\ell \to r$ is orientable by $\mathcal{M}$, i.e., $[\alpha]_{\mathcal{M}}(\ell) >_{\Sigma} [\alpha]_{\mathcal{M}}(r)$ for all variable assignments α, then it is also orientable by $\mathcal{P}$ (since α' covers all assignments $\mathcal{V} \to \mathbb{N}$), which shows that matrix interpretations over $>_{\Sigma}$ are no more powerful than linear polynomial interpretations.

Finally, concerning matrix interpretations over $>_m$ and $>_{\ell}$, the situation is similar as for $>_{\Sigma}$ and $>_{\mathcal{I}}$. That is to say that the respective monotonicity conditions are too strong, thus reducing the set of potential interpretation functions down to a size that renders matrix interpretations over $>_m$ and $>_{\ell}$ useless. For example, one can show that for monotonicity of a function $A\vec{x} + \vec{b}$ with respect to $>_m$, it is necessary that the matrix A satisfies the conditions of Lemma 6.3, that is, all column sums of A must be equal and at least one; e.g., by considering vectors $\vec{x}$ and $\vec{y}$, such that all components of $\vec{y}$ are equal to some $y \in \mathbb{N}$ and $\vec{x}$ is zero everywhere except for its j-th component, $j \in \{1, \ldots, n\}$, which contains the value $y + 1$. Similarly, one can show that for monotonicity of $A\vec{x} + \vec{b}$ with respect to $>_{\ell}$, it is necessary that all column vectors of A are non-zero and have the same (Euclidean) length; e.g., for dimension 2 and higher, by considering vectors $\vec{x} = (y \mp 1, y \pm 1, 0, \ldots, 0)^T$ and

$\vec{y} = (y, y, 0, \ldots, 0)^T$, where $y \in \mathbb{N} \setminus \{0\}$. However, these conditions are not sufficient. Even if A is the identity matrix, $A\vec{x} + \vec{b}$ is not necessarily monotone with respect to $>_\ell$.

7 Improved Matrix Interpretations

According to the results presented in Section 5, in theory the various instances of matrix interpretations over $>^w_{\mathcal{I}}$ are all equivalent *somehow* with respect to termination proving power if there is no bound on the dimension of the matrices. In practice, however, due to computational restrictions the dimension is limited. But then the various instances of matrix interpretations over $>^w_{\mathcal{I}}$ are incomparable as witnessed by Example 5.1 and by the experiments we performed. Therefore, an implementation should try all instances (cf. also [3]). Apart from parallelization, one could try to combine the constraints associated with each instance into a single disjunctive constraint and let the constraint solver figure out which instance to pursue. This approach was chosen in [3]. However, according to our experiments, it does not yield an efficient implementation (cf. experimental results below). Therefore, we propose a different approach, which generalizes traditional matrix interpretations.

Given some signature $\mathcal{F}$, we define an $\mathcal{F}$-algebra $\mathcal{M}$ with carrier $\mathbb{N}^n$, where each k-ary function symbol $f \in \mathcal{F}$ is interpreted by a linear function as in Definition 4.2 (without the monotonicity requirement). Concerning monotonicity, we demand that

- all functions are monotone with respect to $>^w_{\mathcal{I}_1}$, or
- all functions are monotone with respect to $>^w_{\mathcal{I}_2}$, or

  $\vdots$
- all functions are monotone with respect to $>^w_{\mathcal{I}_n}$.

Compatibility with a given TRS $\mathcal{R}$ is established by demanding that for every rewrite rule $\ell \to r \in \mathcal{R}$, $[\alpha]_{\mathcal{M}}(\ell) >^w_{\mathcal{I}_1} [\alpha]_{\mathcal{M}}(r)$ for all variable assignments α; i.e., every rewrite rule gives rise to a strict decrease in the first components of the vectors associated with it.

Clearly, if all interpretation functions of $\mathcal{M}$ are monotone with respect to $>^w_{\mathcal{I}_1}$, then $\mathcal{M}$ corresponds to a traditional matrix interpretation [4]. More generally, $\mathcal{M}$ always is a matrix interpretation over $>^w_{\mathcal{I}_d}$, $d \in \{1, \ldots, n\}$, in the sense of Definition 4.2 because of the inclusions $>^w_{\mathcal{I}_1} \subset >^w_{\mathcal{I}_2} \subset \cdots \subset >^w_{\mathcal{I}_n}$.

Next we provide some experimental data. We implemented the variants of matrix interpretations considered in this paper in the termination prover TTT2 [11] and analyzed their performance on TPDB[3] version 7.0.2. All tests have been performed on a laptop equipped with 2 GB of main memory and one dual-core INTEL® Core 2 Duo T7500 processor running at a clock rate of 2.2 GHz with a time limit of 60 seconds per system.[4]

Table 1 summarizes our results for establishing direct termination (using matrix interpretations as a stand-alone method). We searched for matrix interpretations of dimensions two and three by encoding the constraints as an SMT problem (quantifier-free non-linear arithmetic), which is solved by bit-blasting. The table lists the number of bits used to represent matrix/vector coefficients, the number of bits for intermediate results is one higher than that. The entry $>^{\text{ext}}_{\{1\}}$ in the first column refers to the notion of matrix interpretations presented above, whereas the entry [3] refers to the approach proposed in [3]. For the experiments presented in the table the time limit was hardly ever consumed. Typically, a termination proof is obtained in about 2 (5) seconds for dimension 2 (3). For dimensions 4

[3] Termination Problems Data Base, `http://termcomp.uibk.ac.at`.
[4] For full details see `http://colo6-c703.uibk.ac.at/ttt2/fn/matrix`.

Table 1 Experimental results for various matrix interpretations.

method	dimension	# bits	SCORE
$>^w_{\mathcal{I}_1}$	2	3	242
$>^w_{\mathcal{I}_2}$	2	3	247
$>^{\text{ext}}_{\{1\}}$	2	3	254
[3]	2	3	250

method	dimension	# bits	SCORE
$>^w_{\mathcal{I}_1}$	3	2|3	266|285
$>^w_{\mathcal{I}_2}$	3	2|3	252|264
$>^w_{\mathcal{I}_3}$	3	2|3	249|269
$>^{\text{ext}}_{\{1\}}$	3	2|3	276|287
[3]	3	2|3	267|270

and higher, however, there are many more timeouts, resulting in inferior performance scores; e.g., for matrix interpretations over $>^w_{\mathcal{I}_1}$ of dimension 4 (with 3 bits) one loses more than 40 of the 285 systems for dimension 3.

8 Conclusion and Future Work

In this paper we studied various alternative well-founded orders on vectors of natural numbers based on vector norms. Most of them turned out to be equivalent to or subsumed by an instance of $>^w_{\mathcal{I}}$, an order which already appeared in [3]. In this respect, our main contribution are the theoretical comparisons presented in Section 5, as well as the variant of matrix interpretations introduced in Section 7. We do note, however, that the situation is altogether different when switching from the natural numbers to the rationals and reals. Then it is not the case anymore that almost all of the orders of Section 3 (suitably adapted) are equivalent. In particular, one could imagine interpretation functions, all of whose matrices have entries less than one, but which are still monotone. We leave this issue for the near future. In this context, we also mention the recent work of Lucas [12] where an attempt is made to simulate matrix interpretation over the rationals by an interpretation over the naturals.

We also plan to investigate on the ramifications of the kinds of matrix interpretations proposed in this paper with respect to recent results on the derivational complexity of TRSs [13]. For example, if a matrix has a diagonal of all zeros, then its trace, the sum of the diagonal entries, is also zero. As the trace of a matrix is the sum of its eigenvalues, which have been shown to be the determining factor for the derivational complexity of TRSs, a lower trace might be beneficial in this context.

In the near future work we will address alternative matrix interpretations in the context of the DP framework [6], where it suffices to consider weakly monotone algebras. A *well-founded weakly monotone $\mathcal{F}$-algebra* $(\mathcal{A}, >, \gtrsim)$ is an $\mathcal{F}$-algebra $\mathcal{A}$ equipped with two relations $>, \gtrsim$ on A, such that $>$ is well-founded, $> \cdot \gtrsim \subseteq >$, and for every $f \in \mathcal{F}$, $f_{\mathcal{A}}$ is monotone with respect to $\gtrsim$. If, in addition, $f_{\mathcal{A}}$ is also monotone with respect to $>$ (for every $f \in \mathcal{F}$), then we obtain an *extended monotone $\mathcal{F}$-algebra* [4], the analogon of well-founded monotone algebras in the context of relative termination.

Based on the results of the previous sections, the following instances of a weakly monotone algebra $(\mathbb{N}^n, >, \gtrsim)$, where $\gtrsim\, = \,\geqslant^w$ and (1) $> \,=\, >^w_{\mathcal{I}}$ for $\mathcal{I} = \{1, \ldots, n\}$, (2) $> \,=\, >_\Sigma$, (3) $> \,=\, >_m$, or (4) $> \,=\, >_\ell$ need to be considered. As to the first instance, we note that $>$ is the strict part of $\gtrsim$ according to Lemma 3.3. Yet this is exactly the case that is considered in [4], apart from a refinement that reduces the search space in an implementation. Moreover, by Corollary 3.4, no other weakly decreasing orders need to be considered for $>$. However, observing that weak decreasingness is not really needed to obtain a weakly monotone algebra, one might as well drop it, thus obtaining a weakly monotone algebra, where $> = >_\Sigma$ (instance (2) above), which is a proper generalization of the first one since $>^w_{\mathcal{I}} = >^w_\Sigma \subset >_\Sigma$ (cf. Lemma 3.3). Similarly, one can use the non-weakly decreasing orders $>_m$ and $>_\ell$ to obtain other instances of weakly monotone algebras. They are all incomparable since $>_\Sigma$, $>_m$ and $>_\ell$ are so.

Acknowledgements We thank Bertram Felgenhauer for his helpful comments in the early stages of this work.

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