# Ordinals and Knuth-Bendix Orders

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**Abstract** In this paper we consider a hierarchy of three versions of Knuth-Bendix orders. (1) We show that the standard definition can be (slightly) simplified without affecting the ordering relation. (2) For the extension of *transfinite Knuth-Bendix orders* we show that transfinite ordinals are not needed as weights, as far as termination of finite rewrite systems is concerned. (3) Nevertheless termination proving benefits from transfinite ordinals when used in the setting of *general Knuth-Bendix orders* defined over a weakly monotone algebra. We investigate the relationship to polynomial interpretations and present experimental results for both termination analysis and ordered completion. For the latter it is essential that the order is totalizable on ground terms.

**Keywords:** Knuth-Bendix order, termination, ordered completion.

## 1 Introduction

The Knuth-Bendix order (KBO) [10] is a popular criterion for automated termination analysis and theorem proving. Consequently many extensions and generalizations of this order have been proposed and investigated [2–4,6,12,13,15,18,19]. Despite the fact that this order is so well-studied we show that the definition of KBO can be simplified without affecting the ordering relation.

Concerning generalizations of this ordering, Dershowitz [2,3] suggested to extend the semantic component beyond weight functions and Middeldorp and Zantema [15] presented the *generalized Knuth-Bendix order* (GKBO) using weakly monotone algebras. Independently in the theorem proving community, McCune [14] suggested linear functions for computing weights of terms. Recently Ludwig and Waldmann [13] introduced the *transfinite Knuth-Bendix order* (TKBO), which allows linear functions over the ordinals and Kovács *et al.* [12] show that for finite signatures one can restrict to ordinals below $\omega^{\omega^\omega}$ without losing power. However, for finite rewrite systems (which is the typical case for proving termination) we show that finite weights suffice. This is in sharp contrast to GKBO where transfinite ordinals are beneficial. We also show how a restricted version of this ordering can be implemented. To this end (a fragment of) ordinal arithmetic is encoded as a constraint satisfaction problem. The usefulness of the different versions of KBO is illustrated by experimental results for both termination analysis and theorem proving.

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The remainder of this paper is organized as follows. In the next section we recall preliminaries. In Section 3 we prove that the weight of variables can be fixed a priori without affecting the power of KBO. Section 4 shows that for finite rewrite systems no transfinite ordinals are needed for TKBO. Section 5 shows the benefit of transfinite ordinals for GKBO and studies the relationship to polynomial interpretations. Implementation issues and experimental results are discussed in Section 6. Section 7 concludes.

## 2 Preliminaries

*Term Rewriting:* We assume familiarity with term rewriting and termination [21]. Let $\mathcal{F}$ be a signature and $\mathcal{V}$ a set of variables. By $\mathcal{T}(\mathcal{F},\mathcal{V})$ we denote the terms over $\mathcal{F}$ and $\mathcal{V}$. For a term $t$ let $\mathcal{P}\mathsf{os}(t)$ be the set of positions in $t$ and $\mathcal{P}\mathsf{os}_x(t)$ the set of positions of the variable $x$ in $t$. A (well-founded $\mathcal{F}$-)algebra $(\mathcal{A},>)$ consists of a non-empty carrier $A$, a well-founded relation $>$ on $A$, and an interpretation function $f_{\mathcal{A}}$ for every $f \in \mathcal{F}$. By $[\alpha]_{\mathcal{A}}(\cdot)$ we denote the usual evaluation function of $\mathcal{A}$ according to an assignment $\alpha$. An algebra $(\mathcal{A},>)$ is called (weakly) monotone if every $f_{\mathcal{A}}$ is (weakly) monotone with respect to $>$. Any algebra $(\mathcal{A},>)$ induces an order on terms, as follows: $s >_{\mathcal{A}} t$ if for any assignment $\alpha$ the condition $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ holds. The order $\geqslant_{\mathcal{A}}$ is defined similarly based on the reflexive closure of $>$. We say that a TRS $\mathcal{R}$ is compatible with relation $>$ (an algebra $\mathcal{A}$) if $\ell \to r \in \mathcal{R}$ implies $\ell > r$ ($\ell >_{\mathcal{A}} r$). It is well-known that every TRS that is compatible with a monotone algebra is terminating.

An interpretation $f_{\mathcal{A}}$ is simple if $f_{\mathcal{A}}(a_1,\dots,a_n) > a_i$ for all $a_1,\dots,a_n \in A$ and $1 \leqslant i \leqslant n$. An algebra $\mathcal{A}$ is simple if all its interpretation functions are simple. A polynomial interpretation $\mathcal{N}$ is a monotone algebra over the carrier $\mathbb{N} = \{0,1,2,\dots\}$, using $>_{\mathbb{N}}$ as ordering and where every $f_{\mathcal{N}}$ is a polynomial. A polynomial interpretation where every $f_{\mathcal{N}}$ is linear is called a linear interpretation.

*Ordinals:* We assume basic knowledge of ordinals [9]. Let $+$ and $\cdot$ denote standard addition and multiplication on ordinals (and hence also on natural numbers). Likewise let $\oplus$ and $\odot$ denote natural addition and multiplication of ordinals. Let $\mathcal{O}$ be the set of ordinals strictly less than $\epsilon_0$. Recall that every ordinal $\alpha \in \mathcal{O}$ can be uniquely represented in Cantor Normal Form (CNF):

$$\alpha = \sum_{1 \leqslant i \leqslant n} \omega^{\alpha_i} \cdot a_i$$

where $a_1,\dots,a_n \in \mathbb{N} \setminus \{0\}$ and $\alpha_1,\dots,\alpha_n \in \mathcal{O}$ are also in CNF, with $\alpha_1 > \cdots > \alpha_n$. Ordinals below $\omega$—the natural numbers—are called finite.

## 3 KBO

A precedence is a strict order on a signature. A weight function for a signature $\mathcal{F}$ is a pair $(w, w_0)$ consisting of a mapping $w\colon \mathcal{F} \to \mathbb{N}$ and a positive constant

$w_0 \in \mathbb{N}$ such that $w(c) \geqslant w_0$ for every constant $c \in \mathcal{F}$. A weight function $(w, w_0)$ is admissible for a precedence $\succ$ if for every unary $f \in \mathcal{F}$ with $w(f) = 0$ we have $f \succ g$ for all $g \in \mathcal{F} \setminus \{f\}$. The weight of a term is computed as follows: $w(x) = w_0$ for $x \in \mathcal{V}$ and $w(f(t_1, \ldots, t_n)) = w(f) + w(t_1) + \cdots + w(t_n)$. By $|t|_x$ we denote how often a variable $x$ occurs in a term $t$.

**Definition 1.** *Let $\succ$ be a precedence and $(w, w_0)$ a weight function. We define the* Knuth-Bendix order $\succ_{\mathrm{kbo}}$ *inductively as follows: $s \succ_{\mathrm{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all variables $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:*

*(1) $s = f^k(x)$ and $t = x$ for some $k > 0$, or*
*(2) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f \succ g$, or*
*(3) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $s_1 = t_1, \ldots, s_{k-1} = t_{k-1}$, and $s_k \succ_{\mathrm{kbo}} t_k$ with $1 \leqslant k \leqslant n$.*

To indicate the weight function $(w, w_0)$ used for $\succ_{\mathrm{kbo}}$ we write $\succ_{\mathrm{kbo}}^{(w,w_0)}$. A TRS $\mathcal{R}$ is called compatible with KBO if there exists a weight function $(w, w_0)$ admissible for a precedence $\succ$ such that $\mathcal{R}$ is compatible with $\succ_{\mathrm{kbo}}^{(w,w_0)}$.

**Theorem 2** ([4, 10])**.** *A TRS is terminating if it is compatible with KBO.* □

Below we examine if restricting $w_0$ to one decreases the power of KBO. We are not aware of any earlier investigations in this direction. A bit surprisingly indeed $w_0$ can be chosen one, which simplifies the definition of KBO. Note that it does not suffice to just replace $w_0$ by one. Consider the rule $\mathsf{h}(x, x) \to \mathsf{f}(x)$ and $w(\mathsf{h}) = 0$, $w(\mathsf{f}) = 2$. Using $w_0 = 3$ the constraints on the weight give $6 > 5$ but $w_0 = 1$ yields $2 \not> 3$. But a KBO proof with $w_0 > 1$ can be transformed into a KBO proof with $w_0 = 1$ by adapting (according to their arity) the weights of function symbols. Formally, we define a new weight function with

$$(w_0)^1 := 1 \qquad w^1(f) := w(f) + (n-1) \cdot (w_0 - 1) \tag{1}$$

for every $n$-ary function symbol $f$. Obviously $w^1(f) \geqslant 0$ for all $f \in \mathcal{F}$ and in particular $w^1(c) \geqslant (w_0)^1$ for constants $c \in \mathcal{F}$ since $w(c) \geqslant w_0$. Note that this transformation is not invertible since one could get negative weights for function symbols of higher arity. To cope with the reverse direction, i.e., transforming a KBO proof with arbitrary $w_0$ into a KBO proof with $w_0 = k$ for some $k \in \mathbb{N} \setminus \{0\}$ we define a weight function with

$$(w_0)_k := k \qquad w_k(f) := w^1(f) \cdot k \tag{2}$$

**Lemma 3.** *For terms $s$ and $t$ we have $w(s) > w(t)$ if and only if $w^1(s) > w^1(t)$ if and only if $w_k(s) > w_k(t)$ for any $k \in \mathbb{N} \setminus \{0\}$.*

*Proof.* The first statement follows from $w(s) = w^1(s) + w_0 - 1$, which we show by induction on the term $s$. In the base case $s \in \mathcal{V}$ and $w(s) = w_0 = w^1(s) + w_0 - 1$

since $w^1(s) = 1$. In the step case $s = f(s_1, \ldots, s_n)$ and we have

$$\begin{aligned} w(s) &= w(f) + w(s_1) + \cdots + w(s_n) \\ &= w(f) + w^1(s_1) + w_0 - 1 + \cdots + w^1(s_n) + w_0 - 1 \\ &= w^1(f) - (n-1) \cdot (w_0 - 1) + w^1(s_1) + w_0 - 1 + \cdots + w^1(s_n) + w_0 - 1 \\ &= w^1(f) + w^1(s_1) + \cdots + w^1(s_n) + w_0 - 1 = w^1(s) + w_0 - 1 \end{aligned}$$

where the induction hypothesis is used in the second step. The reasoning for the second statement is similar and based on $w_k(s) = w^1(s) \cdot k$. □

Note that Lemma 3 implies that for terms $s$ and $t$ we have $w(s) = w(t)$ if and only if $w^1(s) = w^1(t)$ if and only if $w_k(s) = w_k(t)$ for any $k \in \mathbb{N} \setminus \{0\}$.

**Corollary 4.** *We have* $\succ_{\mathrm{kbo}}^{(w,w_0)} = \succ_{\mathrm{kbo}}^{(w^1,1)} = \succ_{\mathrm{kbo}}^{(w_k,k)}$ *for all* $k \in \mathbb{N} \setminus \{0\}$. □

**Lemma 5.** *The weight function* $(w, w_0)$ *is admissible for a precedence* $\succ$ *if and only if the weight functions* $(w^1, 1)$ *and* $(w_k, k)$ *are admissible for* $\succ$.

*Proof.* The result follows from the fact that $w^1(f) = w(f)$ for unary $f \in \mathcal{F}$ and $w(f) \neq 0$ if and only if $w_k(f) = w^1(f) \cdot k \neq 0$. □

From the above results we immediately obtain that fixing the value of $w_0$ does not affect the power of KBO. Our implementation (see Section 6) benefited slightly from fixing $w_0 = 1$.

By now we have the machinery to show that KBO cannot demand lower bounds on weights (apart from unary function symbols to have weight zero). This can be supported as follows. Consider $\succ_{\mathrm{kbo}}^{(w_2,2)}$. By Corollary 4

$$\succ_{\mathrm{kbo}}^{(w_2,2)} = \succ_{\mathrm{kbo}}^{((w_2)^1,1)} = \succ_{\mathrm{kbo}}^{(((w_2)^1)_k,k)}$$

for any $k \in \mathbb{N} \setminus \{0\}$. Note that $(w_2)^1(f) \geqslant 1$ for all non-unary $f \in \mathcal{F}$ due to (1). Now choosing an appropriate $k$, all $f \in \mathcal{F}$ (with the possible exception of a unary function symbol of weight zero) satisfy $((w_2)^1)_k(f) \geqslant k$ by (2). This does not contradict [23, Theorem 1], which allows to compute an a priori upper bound on the weights.

A KBO with $w_0 = 0$ is not well-founded. The nonterminating TRS consisting of the rule $\mathsf{h}(\mathsf{h}(\mathsf{a},\mathsf{a}),\mathsf{a}) \to \mathsf{h}(\mathsf{a},\mathsf{h}(\mathsf{h}(\mathsf{a},\mathsf{a}),\mathsf{a}))$ is compatible with $\succ_{\mathrm{kbo}}^{(w,0)}$ where $w(\mathsf{a}) = w(\mathsf{h}) = 0$ and $\mathsf{h} \succ \mathsf{a}$.

## 4 Transfinite KBO

We will now consider the *transfinite Knuth-Bendix order* (TKBO) [13]. In this setting a weight function for a signature $\mathcal{F}$ is a pair $(w, w_0)$ consisting of a mapping $w\colon \mathcal{F} \to \mathcal{O}$ and a positive constant $w_0 \in \mathbb{N}$ such that $w(c) \geqslant w_0$ for every constant $c \in \mathcal{F}$. A subterm coefficient function is a mapping $s\colon \mathcal{F} \times \mathbb{N} \to \mathcal{O}$ such that for a function symbol $f$ of arity $n$ we have $s(f, i) > 0$ for all $1 \leqslant i \leqslant n$.

A TKBO where $w_0$,[1] all weights and subterm coefficients are finite will be called finite. Let $(w, w_0)$ be a weight function and $s$ a subterm coefficient function. We define the weight of a term inductively as follows: $w(t) = w_0$ for $t \in \mathcal{V}$ and $w(t) = w(t_1) \odot s(f,1) \oplus \cdots \oplus w(t_n) \odot s(f,n) \oplus w(f)$ if $t = f(t_1, \ldots, t_n)$. Given a term $t$ and a subterm coefficient function $s$, the *coefficient* of a position $p \in \mathcal{P}\mathsf{os}(t)$ is inductively defined by $\mathit{coeff}(p,t) = 1$ if $p = \epsilon$ and $s(f,i) \odot \mathit{coeff}(q,t_i)$ if $t = f(t_1, \ldots, t_n)$ and $p = iq$. The *variable coefficient* of $x \in \mathcal{V}$ is $\mathit{vcoeff}(x,t) = \bigoplus_{p \in \mathcal{P}\mathsf{os}_x(t)} \mathit{coeff}(p,t)$.

**Definition 6.** *Let $\succ$ be a precedence, $(w, w_0)$ a weight function and $s$ a subterm coefficient function. We define the* transfinite Knuth-Bendix order $\succ_{\mathrm{tkbo}}$ *inductively as follows: $s \succ_{\mathrm{tkbo}} t$ if $\mathit{vcoeff}(x,s) \geqslant \mathit{vcoeff}(x,t)$ for all variables $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:*

*(1) $s = f^k(x)$ and $t = x$ for some $k > 0$, or*
*(2) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f \succ g$, or*
*(3) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $s_1 = t_1, \ldots, s_{k-1} = t_{k-1}$, and $s_k \succ_{\mathrm{tkbo}} t_k$ with $1 \leqslant k \leqslant n$.*

A TRS $\mathcal{R}$ is called compatible with TKBO if there is a weight function $(w, w_0)$ admissible for a precedence $\succ$ and a subterm coefficient function $s$ such that $\mathcal{R}$ is compatible with $\succ_{\mathrm{tkbo}}$.

**Theorem 7** ([12,13])**.** *A TRS is terminating if it is compatible with TKBO.* □

In fact $\succ_{\mathrm{tkbo}}$ still satisfies the subterm property if admissibility is relaxed to requiring that a unary function symbol $f$ with $w(f) = 0$ and $s(f,1) = 1$ satisfies $f \succ g$ for all $g \in \mathcal{F} \setminus \{f\}$. Hence Theorem 7 remains valid. Thus in the sequel we will use this less restrictive definition of admissibility.

One motivation in [13] for allowing subterm coefficients is to cope with duplicating rules such as $\mathsf{f}(x) \to \mathsf{h}(x,x)$. We agree with this observation but show that the benefit is not limited to this case, i.e., subterm coefficients are even useful for string rewrite systems (where all function symbols are unary).

*Example 8.* Consider the SRS consisting of the following rule

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(x)))$$

For KBO this rule demands $w(\mathsf{g}) = 0$ and the admissibility condition induces $\mathsf{g} \succ \mathsf{f}$ which does not orient the rule from left to right. On the other hand the SRS is compatible with the TKBO using weights $w(\mathsf{f}) = 1$ and $w(\mathsf{g}) = 1$ and a subterm coefficient function satisfying $s(\mathsf{f},1) = 3$ and $s(\mathsf{g},1) = 1$.

In the remainder of this section we show that if a *finite* TRS is compatible with a TKBO then it also is compatible with a finite TKBO. Assume termination of a finite rewrite system $\mathcal{R}$ is shown by a TKBO with weight function $(w, w_0)$ and subterm coefficient function $s$. If some weights or subterm coefficients are

[1] Although $w_0 \in \mathbb{N}$ by definition, Theorem 13 is valid also for transfinite $w_0$.

infinite then $\mathcal{R}$ is compatible with the finite TKBO which is obtained if one "substitutes" a large enough natural number for $\omega$ in the given weight and subterm coefficient functions. The next example demonstrates the proof idea while the proof of Theorem 13 provides a suitable choice for this natural number.

*Example 9.* The TRS $\mathcal{R}$ consisting of the following three rules

$$\mathsf{f}(x) \to \mathsf{g}(x) \qquad \mathsf{h}(x) \to \mathsf{f}(\mathsf{f}(x)) \qquad \mathsf{k}(x,y) \to \mathsf{h}(\mathsf{f}(x),\mathsf{f}(y))$$

is compatible with the TKBO using the weight function

$$w_0 = 1 \qquad w(\mathsf{g}) = 0 \qquad w(\mathsf{f}) = 5 \qquad w(\mathsf{h}) = \omega \qquad w(\mathsf{k}) = \omega^2 + 1$$

and $\mathsf{g}$ greatest in the precedence. (All subterm coefficients are set to one.) Substituting 13 for $\omega$ makes $\mathcal{R}$ also compatible with the finite TKBO using

$$w_0 = 1 \qquad w(\mathsf{g}) = 0 \qquad w(\mathsf{f}) = 5 \qquad w(\mathsf{h}) = 13 \qquad w(\mathsf{k}) = 13^2 + 1 = 170$$

and $\mathsf{g}$ greatest in the precedence.

**Definition 10.** *For an ordinal $\alpha = \sum_{1 \leqslant i \leqslant n} \omega^{\alpha_i} \cdot a_i$ in CNF, let $M(\alpha)$ denote the maximal natural number occurring as coefficient, i.e., the maximum of $a_1, \ldots, a_n$ and all coefficients occurring in $\alpha_1, \ldots, \alpha_n$. (If $\alpha = 0$ then $M(\alpha) = 0$.) We denote by $\alpha(k)$ the natural number obtained when $\alpha$—considered as a function in $\omega$—is evaluated for the natural number $k$, i.e., $\alpha(k) := \sum_{1 \leqslant i \leqslant n} k^{\alpha_i(k)} \cdot a_i$.*

Let $\alpha = \omega^{\omega \cdot 3} + \omega^2 \cdot 2 + 1$. Then $M(\alpha) = 3$ and $\alpha(3) = 3^{3 \cdot 3} + 3^2 \cdot 2 + 1 = 19,702$. The next result follows from a statement in [7, p. 34].

**Lemma 11.** *Let $\alpha, \beta \in \mathcal{O}$ and $k \in \mathbb{N}$ with $k > M(\alpha), M(\beta)$. If $\alpha = \beta$ then $\alpha(k) = \beta(k)$ and if $\alpha > \beta$ then $\alpha(k) > \beta(k)$.* □

Note that the restriction on $k$ in Lemma 11 is essential: if $k = 2$, $\alpha = \omega$, and $\beta = 2$ then $\alpha > \beta$ but $\alpha(k) = \beta(k) = 2$.

**Lemma 12.** *For $\alpha, \beta \in \mathcal{O}$ we have*

*(1)* $(\alpha \oplus \beta)(k) = \alpha(k) + \beta(k)$
*(2)* $(\alpha \odot \beta)(k) = \alpha(k) \cdot \beta(k)$

*Proof.* Let $\alpha = \sum_{1 \leqslant i \leqslant n} \omega^{\alpha_i} \cdot a_i$ and $\beta = \sum_{1 \leqslant j \leqslant m} \omega^{\beta_j} \cdot b_j$ be in CNF.

(1) We may also write $\alpha = \sum_{1 \leqslant i \leqslant l} \omega^{\gamma_i} \cdot a'_i$ and $\beta = \sum_{1 \leqslant i \leqslant l} \omega^{\gamma_i} \cdot b'_i$ such that $\{\gamma_1, \ldots, \gamma_l\} = \{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_m\}$ where some $a'_i$ and $b'_i$ may be zero. We then have

$$\begin{aligned}
(\alpha \oplus \beta)(k) &= \left( \sum_{1 \leqslant i \leqslant l} \omega^{\gamma_i} \cdot (a'_i + b'_i) \right)(k) && \text{(definition of } \oplus\text{)} \\
&= \sum_{1 \leqslant i \leqslant l} k^{\gamma_i(k)} \cdot (a'_i + b'_i) \\
&= \sum_{1 \leqslant i \leqslant l} k^{\gamma_i(k)} \cdot a'_i + \sum_{1 \leqslant i \leqslant l} k^{\gamma_i(k)} \cdot b'_i \\
&= \alpha(k) + \beta(k)
\end{aligned}$$

(2) We have

$$\begin{aligned}(\alpha \odot \beta)(k) &= \left(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} \omega^{\alpha_i \oplus \beta_j} \cdot a_i \cdot b_j\right)(k) && \text{(definition of } \odot\text{)}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} k^{\alpha_i(k)+\beta_j(k)} \cdot a_i \cdot b_j && (\star)\\ &= \left(\sum_{i=1}^{n} k^{\alpha_i(k)} \cdot a_i\right) \cdot \left(\sum_{j=1}^{m} k^{\beta_j(k)} \cdot b_j\right)\\ &= \alpha(k) \cdot \beta(k)\end{aligned}$$

where in step $(\star)$ we used part (1). □

Using Lemmata 11 and 12 we can prove the following result.

**Theorem 13.** *If a finite TRS is compatible with TKBO then it is compatible with a finite TKBO.*

*Proof.* Let $\mathcal{R}$ be a finite TRS compatible with a TKBO using weight function $(w, w_0)$ and subterm coefficient function $s$. Since $\mathcal{R}$ is finite the natural number

$$k := \max\{M(w(\ell)), M(w(r)) \mid \ell \to r \in \mathcal{R}\} + 1$$

is well-defined (the maximum of the empty set is zero). We have $k > M(w(t))$ for all terms $t$ occurring in $\mathcal{R}$. Consider the weight function given by $w'(f) := \alpha(k)$ whenever $w(f) = \alpha$ and $w'_0 := \beta(k)$ if $w_0 = \beta$ together with the subterm coefficient function $s'(f, i) := \alpha(k)$ whenever $s(f, i) = \alpha$, which assigns only natural numbers as weights and subterm coefficients. For any term $t$ we then have $w'(t) = (w(t))(k)$, as is easily verified by induction: if $t \in \mathcal{V}$ then $w'(t) = w'_0 = (w_0)(k) = (w(t))(k)$ and

$$\begin{aligned}w'(f(t_1, \ldots, t_n)) &= \left(\sum_{i=1}^{n} w'(t_i) \cdot s'(f, i)\right) + w'(f)\\ &= \left(\sum_{i=1}^{n} (w(t_i))(k) \cdot (s(f, i))(k)\right) + (w(f))(k)\\ &= \left(\left(\bigoplus_{i=1}^{n} w(t_i) \odot s(f, i)\right) \oplus w(f)\right)(k)\\ &= (w(f(t_1, \ldots, t_n)))(k)\end{aligned}$$

where in the second step we used the induction hypothesis and the definition of $w'(f)$ and $s'(f, i)$, respectively, and the last but one step applies Lemma 12. Thus by Lemma 11 $w(\ell) > w(r)$ implies $w'(\ell) = (w(\ell))(k) > (w(r))(k) = w'(r)$ and $w(\ell) = w(r)$ implies $w'(\ell) = (w(\ell))(k) = (w(r))(k) = w'(r)$ for each $\ell \to r \in \mathcal{R}$. Note that admissibility is not affected. Hence $\mathcal{R}$ is compatible with the TKBO having weight function $(w'_0, w')$ and subterm coefficient function $s'$. □

We remark that Theorem 13 does not make transfinite ordinal weights in TKBO superfluous since they are beneficial for hierarchic theorem proving [13]. Furthermore, Theorem 13 only applies to finite TRSs as [12, Theorem 5.8] shows that there are TRSs over finite signatures that need transfinite ordinals (larger than $\omega^{\omega^k}$ for any $k \in \mathbb{N}$) for weights and subterm coefficients in TKBO. However, due to Theorem 13 the TRS showing the need for transfinite weights is necessarily infinite.

## 5 Generalized KBO

In this section we elaborate on the question if transfinite ordinals increase the power of KBO. As long as natural addition and multiplication of ordinals are considered, Theorem 13 shows that (for finite TRSs) coefficients beyond $\mathbb{N}$ can be ignored. Although standard addition and multiplication of ordinals are only weakly monotone, they can still be employed for KBO, as we recall in this section. To this end we consider the *generalized Knuth-Bendix order* (GKBO) [15] which computes weights of terms according to a weakly monotone simple algebra.[2] Similar extensions have been presented in [2, 3, 6, 19].

**Definition 14.** *Let $(\mathcal{A}, >)$ be a weakly monotone simple algebra and $\succ$ a precedence on $\mathcal{F}$. We define the general Knuth-Bendix order $\succ_{\text{gkbo}}$ inductively as follows: $s \succ_{\text{gkbo}} t$ if $s >_{\mathcal{A}} t$, or $s \geqslant_{\mathcal{A}} t$ and either*

*(1) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f \succ g$, or*
*(2) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $s_1 = t_1, \ldots, s_{k-1} = t_{k-1}$, and $s_k \succ_{\text{gkbo}} t_k$ with $1 \leqslant k \leqslant n$.*

**Theorem 15** ([15])**.** *A TRS $\mathcal{R}$ is terminating if it is compatible with $\succ_{\text{gkbo}}$.* □

The condition that $\mathcal{A}$ is simple ensures admissibility with $\succ$ since it e.g. rules out interpretations of the form $f_{\mathcal{A}}(x) = x$. Next we elaborate on the use of standard addition and multiplication of ordinals for GKBO.

*Example 16.* Consider the SRS $\mathcal{R}$ containing the rules

$$1\colon \mathsf{a}(x) \to \mathsf{b}(x) \qquad 2\colon \mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{c}(\mathsf{a}(x))) \qquad 3\colon \mathsf{c}(\mathsf{b}(x)) \to \mathsf{a}(x)$$

The following weakly monotone interpretation

$$\mathsf{a}_{\mathcal{O}}(x) = x + \omega + 1 \qquad \mathsf{b}_{\mathcal{O}}(x) = x + \omega \qquad \mathsf{c}_{\mathcal{O}}(x) = x + 2$$

is simple and induces a strict decrease between left- and right-hand sides:

$$x + \omega + 1 >_{\mathcal{O}} x + \omega \quad x + \omega \cdot 2 + 1 >_{\mathcal{O}} x + \omega \cdot 2 \quad x + \omega + 2 >_{\mathcal{O}} x + \omega + 1$$

[2] Note that in contrast to [15] we restrict to the case where all function symbols have lexicographic status and arguments are compared from left to right.

Hence $\mathcal{R}$ can be oriented by GKBO.

Below we show that $\mathcal{R}$ cannot be shown terminating by GKBO using a linear interpretation over $\mathbb{N}$. To this end we assume abstract interpretations $f_{\mathcal{N}}(x) = f_1 x + f_0$. Since $\mathcal{N}$ must be simple we need $f_1, f_0 \geqslant 1$. For rule (2) the constraint on variable coefficients induces $a_1 b_1 \geqslant b_1 c_1 a_1$, which requires $c_1 = 1$. Rules (1) and (3) demand $a_1 \geqslant b_1$ and $b_1 \geqslant a_1$, so $b_1 = a_1$. Rule (2) further requires $a_1 b_0 + a_0 \geqslant b_1 a_0 + b_1 c_0 + b_0$. Because $a_0 \geqslant b_0$ due to rule (1), this demands $a_0 \geqslant a_1 c_0 + b_0 = (a_1 - 1)c_0 + (c_0 + b_0)$. Rule (3) demands $c_0 + b_0 \geqslant a_0$, and hence $(a_1 - 1)c_0 = 0$. Since $\mathcal{N}$ must be simple, $c_0 > 0$ which requires $a_1 = 1$. But then rule (2) implies the constraint $b_0 + a_0 \geqslant a_0 + c_0 + b_0$, which again contradicts that $c_0$ is positive.

Hence one might conclude that standard addition and multiplication of ordinals is more useful for termination proving (where one usually deals with finite TRSs) than their natural counterparts. But standard addition of ordinals might cause problems for (at least) binary function symbols since the absolute positiveness approach [8] to compare polynomials no longer applies. To see this note that $f_1 \geqslant g_1$ and $f_2 \geqslant g_2$ does not imply $x \cdot f_1 + y \cdot f_2 \geqslant y \cdot g_2 + x \cdot g_1$ for all values of $x$ and $y$ if $f_1, f_2, g_1, g_2 \in \mathbb{N}$ and $x, y \in \mathcal{O}$. Next we show that a combination of standard and natural addition is helpful.

*Example 17.* Consider the TRS $\mathcal{R}$ consisting of the single rule

$$\mathsf{s}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{s}(y), \mathsf{s}(\mathsf{s}(x)))$$

The weakly monotone interpretation $\mathsf{f}_{\mathcal{O}}(x, y) = (x \oplus y) + \omega$ and $\mathsf{s}_{\mathcal{O}}(x) = x + 1$ is simple and induces a strict decrease between left- and right-hand side:

$$(x \oplus y) + \omega + 1 >_{\mathcal{O}} ((y + 1) \oplus (x + 2)) + \omega = (x \oplus y) + 3 + \omega = (x \oplus y) + \omega$$

Hence $\mathcal{R}$ can be oriented by GKBO. Again, linear interpretations with coefficients in $\mathbb{N}$ are not sufficient: Assuming abstract interpretations $\mathsf{f}_{\mathcal{N}}(x, y) = f_1 x + f_2 y + f_0$ and $\mathsf{s}_{\mathcal{N}}(x) = s_1 x + s_0$, we get the constraints

$$s_1 f_1 \geqslant f_2 s_1 s_1 \qquad s_1 f_2 \geqslant f_1 s_1 \qquad s_1 f_0 + s_0 \geqslant f_1 s_0 + f_2(s_0 + s_1 s_0) + f_0$$

Since $\mathsf{s}_{\mathcal{N}}$ and $\mathsf{f}_{\mathcal{N}}$ must be simple $s_1, f_1, f_2 \geqslant 1$. From the first two constraints we conclude $s_1 = 1$, such that the third simplifies to $f_0 + s_0 \geqslant f_0 + (f_1 + 2f_2)s_0$. This contradicts $f_1$, $f_2$, and $s_0$ being positive, which is needed for $\mathcal{N}$ being simple.

### GKBO vs. Polynomial Interpretations

Finally we investigate the relationship of polynomial interpretations and GKBO using a weakly monotone simple algebra $(\mathcal{N}, >_{\mathbb{N}})$ over $\mathbb{N}$ assigning polynomials $f_{\mathcal{N}}$ to every $f \in \mathcal{F}$. In the sequel we refer to this restricted version of GKBO by PKBO. We first show that there are TRSs that can be shown terminating by PKBO but not by polynomial interpretations.

*Example 18.* Consider the SRS $\mathcal{R}$ consisting of the rules

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{b}(\mathsf{a}(x))) \qquad \mathsf{a}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{c}(\mathsf{a}(x))) \qquad \mathsf{b}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{b}(x))$$

The GKBO with $\mathsf{a}_\mathcal{N}(x) = 3x{+}1$, $\mathsf{b}_\mathcal{N}(x) = \mathsf{c}_\mathcal{N}(x) = x{+}1$ and $\mathsf{b} \succ \mathsf{c}$ is compatible with $\mathcal{R}$. To orient the first two rules by a polynomial interpretation $\mathcal{N}$, $\mathsf{b}_\mathcal{N}$ and $\mathsf{c}_\mathcal{N}$ must be linear and monic. But then the last rule is not orientable.

The open question deals with the reverse direction, i.e., can any TRS that admits a compatible polynomial interpretation also be shown terminating by PKBO? Polynomial interpretations are monotone and hence also weakly monotone. Consequently polynomials can only exceed PKBO with respect to power if a *non-simple* interpretation can be enforced. Below we show that linear interpretations cannot enforce such an interpretation, in contrast to (non-linear) polynomial interpretations.

*Linear Interpretations:* In this subsection we refer to a PKBO where all interpretation functions are linear by LKBO. In the sequel we show that any linear interpretation can be transformed into a linear interpretation where all interpretation functions are simple.

Let $\mathcal{N}$ be a linear interpretation. For each $f_\mathcal{N}(x_1, \dots, x_n) = f_1x_1 + \cdots + f_nx_n + f_0$ and $m \in \mathbb{N}$ let $f_{\mathcal{N}_m}(x_1, \dots, x_n) = f_1x_1 + \cdots + f_nx_n + m \cdot f_0$. Let $\alpha_0$ be the assignment such that $\alpha_0(x) = 0$ for all variables $x$.

**Lemma 19.** *If $s >_\mathcal{N} t$ then $[\alpha]_{\mathcal{N}_m}(s) > [\alpha]_{\mathcal{N}_m}(t) + (m-1)$ holds for all $\alpha$.*

*Proof.* We first prove

$$[\alpha]_{\mathcal{N}_m}(t) = [\alpha]_\mathcal{N}(t) + (m-1)[\alpha_0]_\mathcal{N}(t) \tag{3}$$

by induction on $t$. In the base case $t \in \mathcal{V}$ and

$$[\alpha]_{\mathcal{N}_m}(t) = \alpha(t) = \alpha(t) + (m-1) \cdot 0 = [\alpha]_\mathcal{N}(t) + (m-1)[\alpha_0]_\mathcal{N}(t)$$

In the step case let $t = f(t_1, \dots, t_n)$. Then

$$\begin{aligned}
[\alpha]_{\mathcal{N}_m}(f(t_1, \dots, t_n)) &= m \cdot f_0 + \sum_{1 \leqslant i \leqslant n} f_i \cdot [\alpha]_{\mathcal{N}_m}(t_i) \\
&= m \cdot f_0 + \sum_{1 \leqslant i \leqslant n} f_i \cdot ([\alpha]_\mathcal{N}(t_i) + (m-1)[\alpha_0]_\mathcal{N}(t_i)) \\
&= [\alpha]_\mathcal{N}(t) + (m-1)[\alpha_0]_\mathcal{N}(t)
\end{aligned}$$

where the induction hypothesis is applied in the second step.

From the assumption $s >_\mathcal{N} t$ we obtain $[\alpha]_\mathcal{N}(s) > [\alpha]_\mathcal{N}(t)$ and in particular $[\alpha_0]_\mathcal{N}(s) \geqslant [\alpha_0]_\mathcal{N}(t) + 1$. Hence

$$\begin{aligned}
[\alpha]_{\mathcal{N}_m}(s) &= [\alpha]_\mathcal{N}(s) + (m-1)[\alpha_0]_\mathcal{N}(s) > [\alpha]_\mathcal{N}(t) + (m-1)[\alpha_0]_\mathcal{N}(s) \\
&\geqslant [\alpha]_\mathcal{N}(t) + (m-1)([\alpha_0]_\mathcal{N}(t) + 1) = [\alpha]_{\mathcal{N}_m}(t) + (m-1)
\end{aligned}$$

where the two equality steps follow from (3). ◻

**Definition 20.** *Let $\mathcal{A}$ be an algebra. We define the algebra $\mathcal{A}'$ to be as $\mathcal{A}$ but for each function symbol $g$ with $[\alpha_0]_{\mathcal{A}}(g(x_1,\dots,x_n)) = 0$ we define $g_{\mathcal{A}'}(x_1,\dots,x_n) = g_{\mathcal{A}}(x_1,\dots,x_n)+1$. For a finite TRS $\mathcal{R}$ let $M := \max\{[\alpha_0]_{\mathcal{A}'}(r) \mid \ell \to r \in \mathcal{R}\}+1$.*

**Lemma 21.** *If a finite TRS is compatible with $\mathcal{N}$ then it is compatible with $(\mathcal{N}_M)'$.*

*Proof.* A straightforward induction proof shows that any term $t$ satisfies

$$[\alpha]_{\mathcal{N}_m}(t) + [\alpha_0]_{\mathcal{N}'}(t) \geqslant [\alpha]_{(\mathcal{N}_m)'}(t) \tag{4}$$

Compatibility yields $\ell >_{\mathcal{N}} r$ for every $\ell \to r \in \mathcal{R}$. The claim follows from $[\alpha]_{(\mathcal{N}_M)'}(\ell) \geqslant [\alpha]_{\mathcal{N}_M}(\ell) > [\alpha]_{\mathcal{N}_M}(r)+M-1 \geqslant [\alpha]_{\mathcal{N}_M}(r)+[\alpha_0]_{\mathcal{N}'}(r) \geqslant [\alpha]_{(\mathcal{N}_M)'}(r)$ where Lemma 19 is used in the second step, $M > [\alpha_0]_{\mathcal{N}'}(r)$ in the third step, and the last step is an application of (4). □

As $(\mathcal{N}_M)'$ is simple we obtain the following result from Lemma 21.

**Theorem 22.** *If a finite TRS is compatible with a linear interpretation then it is compatible with an LKBO.*

*Proof.* Let $\mathcal{R}$ be a finite TRS that is compatible with a linear interpretation $\mathcal{N}$ and by Lemma 21 also with $(\mathcal{N}_M)'$. By construction $(\mathcal{N}_M)'$ is simple and all its interpretation functions are linear. Since $(\mathcal{N}_M)'$ is monotone it is also weakly monotone. Hence based on $(\mathcal{N}_M)'$ we have $\ell \succ_{\text{gkbo}} r$ for every $\ell \to r \in \mathcal{R}$ and empty precedence $\succ$. □

*Non-linear Interpretations:* To show that PKBO does not subsume polynomial interpretations we give a TRS that can be shown terminating by a polynomial interpretation but not by PKBO. The reason for failure is that compatibility with PKBO can enforce interpretation functions that are not simple.

**Theorem 23.** *The TRS $\mathcal{R}$ consisting of $\mathsf{s}(x) \to \mathsf{t}(\mathsf{t}(\mathsf{t}(x)))$ and the rules*

$$\begin{array}{llll}
\mathcal{R}_1 & \mathsf{f}(0) \to 0 & \mathsf{f}(\mathsf{s}(0)) \to \mathsf{s}(0) & \mathsf{f}(\mathsf{s}^2(0)) \to \mathsf{s}^6(0) \\
\mathcal{R}_1 & \mathsf{s}^2(0) \to \mathsf{f}(0) & \mathsf{s}^3(0) \to \mathsf{f}(\mathsf{s}(0)) & \mathsf{s}^8(0) \to \mathsf{f}(\mathsf{s}^2(0)) \\
\mathcal{R}_2 & \mathsf{g}(x) \to \mathsf{h}(x,x) & \mathsf{s}(x) \to \mathsf{h}(x,0) & \mathsf{s}(x) \to \mathsf{h}(0,x) \\
\mathcal{R}_3 & \mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(x))) & \mathsf{g}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{g}(x))) & \mathsf{h}(\mathsf{f}(x),\mathsf{g}(x)) \to \mathsf{f}(\mathsf{s}(x))
\end{array}$$

*can be shown terminating by a polynomial interpretation but not by PKBO.*

*Proof.* The TRS $\mathcal{R}$ is compatible with the following polynomial interpretation:

$$\begin{array}{lll}
\mathsf{f}_{\mathcal{N}}(x) = 2x^2 - x + 1 & \mathsf{s}_{\mathcal{N}}(x) = x + 1 & \mathsf{g}_{\mathcal{N}}(x) = 4x + 5 \\
\mathsf{h}_{\mathcal{N}}(x,y) = x + y & \mathsf{t}_{\mathcal{N}}(x) = x & 0_{\mathcal{N}} = 0
\end{array}$$

To see that $\mathcal{R}$ cannot be shown terminating by PKBO we adopt the idea from [16] where the shape and the coefficients of a compatible polynomial interpretation can be determined by the rewrite rules in $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$. However, in our setting

we have to re-inspect the results from [16] since PKBO allows $\ell \geqslant_{\mathcal{A}} r$ in contrast to a polynomial interpretation which requires $\ell >_{\mathcal{A}} r$ for all rules. Furthermore monotonicity has to be replaced by weak monotonicity and the algebra has to be simple.

Next we investigate which interpretation functions are enforced by the rules in $\mathcal{R}$. Inspecting [16, Lemma 16] using the first two rules from $\mathcal{R}_3$ we obtain that $\mathsf{s}_{\mathcal{N}}$ and $\mathsf{g}_{\mathcal{N}}$ must be linear. Moreover $\mathsf{s}_{\mathcal{N}}(x) = x + d$ for some $d \in \mathbb{N}$ and $\mathsf{f}_{\mathcal{N}}$ is not linear. Since $\mathcal{N}$ must be simple we have $d > 0$. From [16, Lemma 21] and $\mathcal{R}_2$ we obtain $\mathsf{h}_{\mathcal{N}}(x, y) = x + y + p$. Now [16, Lemma 20] and the last rule of $\mathcal{R}_3$ limit the degree of $\mathsf{f}_{\mathcal{N}}$ to at most two.

Next we focus on [16, Lemma 18]. Let $z = \mathsf{0}_{\mathcal{N}}$. The last rule of $\mathcal{R}_1$ yields

$$z + 8d \geqslant f_{\mathcal{N}}(z + 2d)$$

Note that $\mathsf{f}(x) >_{\mathcal{N}} x$ since $\mathsf{f}_{\mathcal{N}}$ must be simple. Since the degree of $\mathsf{f}_{\mathcal{N}}$ is two we have $\mathsf{f}_{\mathcal{N}}(x) = ax^2 + bx + c$ with $a \geqslant 1$. For well-definedness of $\mathsf{f}_{\mathcal{N}}$ we need $a \geqslant -b$ and $c \geqslant 0$. Next we show that $d \leqslant 2$. To this end we assume $d \geqslant 3$ and arrive at a contradiction. Now

$$\begin{aligned}\mathsf{f}_{\mathcal{N}}(z + 2d) &= a(z^2 + 4zd + 4d^2) + b(z + 2d) + c = f(z) + a(4zd + 4d^2) + 2bd \\ &> z + a(4zd + 4d^2) + 2bd \geqslant z + 4ad^2 + 2bd = z + d(4ad + 2b) \\ &= z + d((4d - 2)a + 2(a + b)) \geqslant z + 10d\end{aligned}$$

which is a contradiction to the constraint $z + 8d \geqslant \mathsf{f}_{\mathcal{N}}(z + 2d)$ from above. Hence $d \leqslant 2$. But then $\mathsf{s}(x) \to \mathsf{t}(\mathsf{t}(\mathsf{t}(x)))$ requires $\mathsf{t}_{\mathcal{N}}(x) = x$. Hence PKBO cannot prove termination of the TRS $\mathcal{R}$. □

## 6 Implementation and Evaluation

To establish a termination proof by KBO the task is to search for suitable weights and a precedence. For efficiently finding a compatible KBO by linear programming we refer to [23]. TKBO with finite weights can easily be encoded in non-linear integer arithmetic (similarly to [23]) for which powerful but (necessarily) incomplete tools exist [24]. Since these tools are typically overflow-safe the problems sketched in [12, 13] do not appear in our setting (termination proving).

In the remainder of this section we sketch how one can implement a version of GKBO using transfinite ordinal weights (below $\omega^\omega$) with the standard addition and multiplication of ordinals. This is sound by Theorem 15, provided the interpretation functions are weakly monotone and simple. We restrict ourselves to string rewrite systems and interpretations of the (canonical) form

$$f_{\mathcal{O}}(x) = x \cdot f' + \omega^d \cdot f_d + \cdots + \omega^1 \cdot f_1 + f_0 \tag{5}$$

where $f', f_d, \ldots, f_0 \in \mathbb{N}$. As illustration, we abstractly encode the rule

$$\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(x)))$$

with $d = 1$. For the left-hand side we get

$$x \cdot b' \cdot a' + \omega^1 \cdot b_1 \cdot a' + b_0 \cdot a' + \omega^1 \cdot a_1 + a_0$$

which can be written in the canonical form

$$x \cdot b' \cdot a' + \omega^1 \cdot (b_1 \cdot a' + a_1) + (a_1 > 0 \,?\, 0 : b_0 \cdot a') + a_0$$

where the $(\cdot\,?\,\cdot:\cdot)$ operator implements if-then-else, i.e., if $a_1$ is greater than zero then the summand $b_0 \cdot a'$ vanishes. To determine whether

$$x \cdot l' + \omega^1 \cdot l_1 + l_0 \geqslant x \cdot r' + \omega^1 \cdot r_1 + r_0$$

for all values of $x$, we use the criterion $l' \geqslant r' \wedge (l_1 > r_1 \vee (l_1 = r_1 \wedge l_0 \geqslant r_0))$. Finally, $f' \geqslant 1 \wedge (f_1 \geqslant 1 \vee f_0 \geqslant 1)$ ensures that the interpretation $f_{\mathcal{O}}$ is simple while the interpretation functions are then weakly monotone for free. Hence the search for suitable weights, subterm coefficients, and the precedence can be encoded in non-linear integer arithmetic, similar as in [24].

*Termination:* For termination analysis we considered the 1416 TRSs and 720 SRSs from TPDB 7.0.2.[3] The experiments[4] have been performed single-threaded with T$_{\text{T}}$T$_2$ [11]. The leftmost part of Table 1 shows how many TRSs can be proved terminating (column yes) and the average duration of finding a termination proof (column time) in seconds.[5] Coefficients for polynomials and weights/subterm coefficients for (T)KBO have been represented by at most six bits (to get a maximum number of termination proofs). As termination criteria we considered Theorem 2 (row KBO) and Theorem 7 using finite weights and subterm coefficients (row TKBO). We list the data for linear interpretations (POLY) and the lexicographic path order (LPO) for reference. For SRSs the entry TKBO$_\omega$ corresponds to an implementation of Theorem 15 using interpretation functions as in (5) with $d = 1$ while for KBO$_\omega$ in addition we fixed $f' = 1$ in (5). Different values for $d$ did not increase the number of systems proved terminating in this setting. However, it is possible to construct systems that need an arbitrarily large $d$ (based on the derivational complexity of the system). In both categories (TRS and SRS) TKBO subsumes KBO and POLY (which is no surprise in light of Theorem 22). Hence the additional systems stem from LPO.

*Ordered Completion:* Ordered completion [1] is one of the most frequently used calculi in equational theorem proving. Classical ordered completion tools require a reduction order as input that can be extended to a total order on ground terms. This parameter is critical for the success of a run, but a suitable choice is hardly predictable in advance. *Ordered multi-completion with termination tools* [22] addresses this challenge by employing automatic termination tools and exploring different orientations in parallel, instead of sticking to one fixed ordering. This

[3] `http://termcomp.uibk.ac.at/status/downloads/tpdb-7.0.2.tar.gz`
[4] Details are available from `http://colo6-c703.uibk.ac.at/ttt2/tkbo/`.
[5] This includes the "start-up time" of T$_{\text{T}}$T$_2$ which is around 0.3 seconds.

Table 1: Termination and Ordered Completion

| | Termination | | | | Ordered Completion | | |
|---|---|---|---|---|---|---|---|
| | 1416 TRSs | | 720 SRSs | | 42 TPTP theories | | |
| method | yes | time | yes | time | yes | time | tc |
| KBO/KBO$_\omega$ | 107/- | 0.5 | 33/34 | 0.5/0.7 | 31/- | 16 | 19% |
| TKBO/TKBO$_\omega$ | 192/- | 0.9 | 43/44 | 1.7/2.9 | 34/- | 56 | 35% |
| POLY | 149 | 0.9 | 22 | 1.6 | 15 | 4 | 70% |
| LPO | 159 | 0.5 | 5 | 0.5 | 26 | 37 | 7% |
| $\sum$ | 262 | - | 45 | - | 34 | | |

approach is implemented in the tool OMKBTT [22]. However, only termination techniques that guarantee *total termination* [5] are applicable in order to obtain a TRS that is indeed ground-complete. In practice, applicable termination techniques are thus restricted to classical reduction orders such as LPO, KBO or polynomial interpretations. We thus compared the power of OMKBTT using TKBO besides other reduction orders on a test set of 42 theories underlying TPTP [20]. Indeed TKBO is able to produce ground-complete systems for more problems than any of the other reduction orders. The right part of Table 1 shows that TKBO significantly extends the class of orientable TRSs, although more time is required (column time). The column tc indicates the percentage of the execution time spent on termination checks.

## 7 Conclusion

In this paper we considered three variants of the Knuth-Bendix order and showed that some extensions do not add power (as far as termination proving of finite TRSs is considered) while others do. We have implemented the finite version of TKBO [12, 13] as an SMT problem in non-linear integer arithmetic. Since our solver uses arbitrary precision arithmetic, overflows (as reported in [12, 13]) are not an issue. However, since already standard KBO can demand arbitrarily large weights (see [23, Example 2]) overflows are not specific to TKBO (as the discussions in [12,13] convey). We have also implemented a KBO using ordinal weights, which has been identified as one challenge in [12]. Also Vampire [17] uses ordinal numbers (see [12, Section 7]), but only for weights of predicate symbols. Since they occur at the root only no ordinal arithmetic is needed but only comparison. Hence the same effect could be achieved by allowing a (quasi-)precedence on predicate symbols.

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