Deciding Confluence of Ground Term Rewrite Systems in Cubic Time*

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Abstract

It is well known that the confluence property of ground term rewrite systems (ground TRSs) is decidable in polynomial time. For an efficient implementation, the degree of this polynomial is of great interest. The best complexity bound in the literature is given by Comon, Godoy and Nieuwenhuis (2001), who describe an $O(n^5)$ algorithm, where n is the size of the ground TRS. In this paper we improve this bound to $O(n^3)$. The algorithm has been implemented in the confluence tool CSI.

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1 Introduction

It is well known that confluence of ground TRSs can be decided in polynomial time. In this paper, we are interested in the degree of the associated polynomial.

To derive a polynomial time decision procedure for confluence of ground TRSs, Comon et al. [3] use an approach based on a transformation by Plaisted [9] that flattens the TRS. Then they test *deep joinability* of sides of rules. The authors sketch an implementation with complexity $O(n^5)$, where n is the size of the given TRS. Tiwari [10] and Godoy et al. [6] base their approach on a rewrite closure that constructs tree transducers—the given TRS $\mathcal{R}$ is converted into two TRSs $\mathcal{F}$ and $\mathcal{B}$ such that $\mathcal{F}$ and $\mathcal{B}^{-1}$ are left-flat, right-constant, $\mathcal{F}$ is terminating, and $\to_{\mathcal{R}}^* = \to_{\mathcal{F}}^* \cdot \to_{\mathcal{B}}^*$. They then consider *top-stabilizable* terms to derive conditions for confluence. Tiwari obtains a bound of $O(n^9)$ (but a more careful implementation would end up with $O(n^6)$), while Godoy et al. obtain a bound of $O(n^6)$. The algorithm of [3] is limited to ground TRSs, but [10] extends the algorithm to certain shallow, linear systems, and [5] treats shallow, linear systems in full generality.[1] In these extensions, however, the exponent depends on the maximum arity of the function symbols of the given TRS. In our work we combine ideas from [3, 10, 6] in order to improve the complexity bound to $O(n^3)$. The key ingredients are a Plaisted-style rewrite closure, which results in TRSs $\mathcal{F}$ and $\mathcal{B}$ of only quadratic size, and top-stabilizability, which is cheaper to test than deep joinability.

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[1] The same claim can be found in [6]. However, rule splitting, a key step in the proof of their Lemma 3.1, only works if left-hand side and right-hand side variables are disjoint for every rule.

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The remainder of this paper is structured as follows: After some preliminaries in Section 2 we describe the confluence check in Section 3. Some experimental results are presented in Section 4. Finally we conclude in Section 5.

2 Preliminaries

A signature is a set of function symbols $\mathcal{F} = \mathcal{F}^{(0)} \cup \mathcal{F}^{(1)} \cup \dots$, where $\mathcal{F}^{(i)}$ is the set of function symbols of arity i, and the sets $(\mathcal{F}^{(i)})_{i \in \mathbb{N}}$ are pairwise disjoint. The ground terms $\mathcal{T}(\mathcal{F})$ over $\mathcal{F}$ are constructed inductively in the usual way: If $t_1, \dots, t_n \in \mathcal{T}(\mathcal{F})$ and $f \in \mathcal{F}^{(n)}$, then $f(t_1, \dots, t_n) \in \mathcal{T}(\mathcal{F})$. A position p of a term is a sequence of natural numbers addressing a subterm $t|_p$. Replacement of subterms $t[u]_p$, and the size of terms $|t|$ have their standard definitions [2]. A term t together with a position p defines a context $C[\cdot] = t[\cdot]_p$. Contexts can be instantiated, $C[s] = t[s]_p$. The elements of $\mathcal{F}^{(0)} \subseteq \mathcal{F}$ are called constants. A term is flat if it is either a constant or a function symbol applied to constants.

A set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F})^2$ of rules is a ground term rewrite system (TRS). If $(\ell, r) \in \mathcal{R}$, we also write $\ell \to r \in \mathcal{R}$ and sometimes $\ell \approx r \in \mathcal{R}$. By $\mathcal{R}^-$, $\mathcal{R}^\pm$, $|\mathcal{R}|$, $\|\mathcal{R}\|$ we denote $\mathcal{R}^{-1}$ (where we view $\mathcal{R}$ as a relation on ground terms), $\mathcal{R} \cup \mathcal{R}^{-1}$, the number of rules in $\mathcal{R}$, and the total size of the rules, $\sum_{\ell \to r \in \mathcal{R}}(|\ell| + |r|)$, respectively. Any ground TRS $\mathcal{R}$ induces a rewrite relation $\to_\mathcal{R}$ on ground terms: $s \to_\mathcal{R} t$ if there is a context $C[\cdot]$ such that $C[\ell] = s$ and $C[r] = t$ for some $\ell \to r \in \mathcal{R}$. Properties like flatness extend to rules and TRSs. For example, a rule is left-flat if its left-hand side is flat; a TRS is left-flat if all its rules are.

Given a rewrite relation $\to$, we write $\to^*$ and $\leftrightarrow$ for its reflexive, transitive closure and its symmetric closure, respectively. A rewrite relation $\to$ is confluent if ${}^*\!\leftarrow \cdot \to^* \subseteq \to^* \cdot {}^*\!\leftarrow$. It is terminating if there are no infinite rewrite sequences $t_0 \to t_1 \to t_2 \to \dots$. Two terms s and t are convertible if $s \leftrightarrow^* t$. They are joinable, denoted by $s \downarrow t$, if $s \to^* \cdot {}^*\!\leftarrow$.

In the complexity analysis we make use of the fact that systems of Horn clauses can be solved linear time, see Dowling and Gallier [4]. We work with propositional variables. For each variable A, there is a positive atom A and a negative atom $\neg A$. A Horn clause is a disjunction of atoms, with at most one positive atom. Horn clauses can be written as implications—the clause $\neg A \vee \neg B \vee C$ is equivalent to $A \wedge B \to C$. This implication is equivalent to the following inference rule:

$$\frac{A \quad B}{C}$$

3 Testing Confluence

We are given a finite ground TRS $\mathcal{R}_0$ over a finite signature $\mathcal{F}$. We may assume without loss of generality that $\mathcal{R}_0$ is curried, with a binary function symbol $\circ$ (representing function application) and no other non-constant function symbols. To curry an arbitrary ground TRS $\mathcal{R}$, one introduces a fresh binary function symbol $\circ$ and replaces all function applications $f(t_1, \dots, t_n)$ by $(\dots((f \circ t_1) \circ t_2) \dots) \circ t_n$. The original function symbols become constants in the curried TRS. It is well-known that currying preserves (non-)confluence (e.g., [7]) and can be performed in linear time, increasing the size $\|\mathcal{R}\|$ of the TRS by a constant factor.

Furthermore we assume that $\mathcal{F}$ is minimal, i.e., only function symbols occurring in $\mathcal{R}_0$ are elements of $\mathcal{F}$. We can make this assumption because (non-)confluence is preserved under signature extension (this follows from the modularity of confluence, [11]). Let K be a countably infinite set of fresh constants (disjoint from $\mathcal{F}^{(0)}$), and let u, v, w denote elements of $\mathcal{F}^{(0)} \cup K$. We call $u \to v$ a C-rule (constant rule), and $u \circ v \to w$ a D-rule (decreasing rule).

The construction proceeds in four phases: First the TRS is flattened preserving confluence and non-confluence, then we determine its rewrite closure and congruence closure, and finally these closures are used for testing confluence of the flattened TRS.

▶ **Example 3.1.** We will decide confluence of $\mathcal{R} = \{\mathsf{a} \to \mathsf{b}, \mathsf{a} \to \mathsf{f}(\mathsf{a}), \mathsf{b} \to \mathsf{f}(\mathsf{f}(\mathsf{b}))\}$ and $\mathcal{R}' = \mathcal{R} \cup \{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \to \mathsf{b}\}$. We start with the curried ground TRSs $\mathcal{R}_0 = \{\mathsf{a} \to \mathsf{b}, \mathsf{a} \to \mathsf{f} \circ \mathsf{a}, \mathsf{b} \to \mathsf{f} \circ (\mathsf{f} \circ \mathsf{b})\}$ and $\mathcal{R}_0' = \mathcal{R}_0 \cup \{\mathsf{f} \circ (\mathsf{f} \circ (\mathsf{f} \circ \mathsf{b})) \to \mathsf{b}\}$.

3.1 Flattening

First, we flatten the TRS $\mathcal{R}_0$, as follows. We start with $(\mathcal{R}_0, \varnothing)$ and exhaustively apply the rules

$$
\begin{aligned}
(\mathcal{R}[u \circ v], \mathcal{E}) &\vdash_{\mathrm{ext}} (\mathcal{R}[w], \mathcal{E} \cup \{u \circ v \approx w\}) \\
(\mathcal{R}[t], \mathcal{E} \cup \{t \approx u\}) &\vdash_{\mathrm{simp}} (\mathcal{R}[u], \mathcal{E} \cup \{t \approx u\}),
\end{aligned}
$$

where $w \in K$ is fresh in $\vdash_{\mathrm{ext}}$, and $\mathcal{R}[\cdot]$ is a context of $\mathcal{R}$, i.e., a context $\ell[\cdot]$ (or $r[\cdot]$) of a side of a rule $\ell \to r$ in $\mathcal{R}$, so that $\mathcal{R}[u]$ is obtained by replacing $\ell \to r$ by $\ell[u] \to r$ (or $\ell \to r[u]$) in $\mathcal{R}$. After each $\vdash_{\mathrm{ext}}$ step, we apply $\vdash_{\mathrm{simp}}$ as often as possible before using $\vdash_{\mathrm{ext}}$ again.

In the resulting pair $(\mathcal{R}, \mathcal{E})$, $\mathcal{R}$ consists solely of C-rules (since otherwise, $\vdash_{\mathrm{ext}}$ would be applicable), and $\mathcal{E}$ consists of D-rules. Furthermore, $\to_{\mathcal{R}_0}$ is confluent, if and only if $\to_{\mathcal{E}^{\pm} \cup \mathcal{R}}$ is, since every deduction step preserves and reflects the confluence property (cf. Lemma 3.2). Because $\vdash_{\mathrm{simp}}$ is applied eagerly, no two left-hand sides in $\mathcal{E}$ are equal, and therefore $\mathcal{E}$ is confluent (it is orthogonal). Note that as a result, every distinct subterm occurring in $\mathcal{R}$ is represented by exactly one constant from $\mathcal{F}^{(0)} \cup K$. This is similar in spirit to the Nelson-Oppen congruence closure algorithm [8].

▶ **Lemma 3.2.** *If $(\mathcal{R}, \mathcal{E}) \vdash_{\mathrm{ext}} (\mathcal{R}', \mathcal{E}')$ or $(\mathcal{R}, \mathcal{E}) \vdash_{\mathrm{simp}} (\mathcal{R}', \mathcal{E}')$ then $\to_{\mathcal{E}^{\pm} \cup \mathcal{R}}$ is confluent if and only if $\to_{\mathcal{E}'^{\pm} \cup \mathcal{R}'}$ is.*

Proof. The rule $\vdash_{\mathrm{ext}}$ can be split into two steps, first adding the rule $u \circ v \approx w$ to $\mathcal{E}$ followed by applying $\vdash_{\mathrm{simp}}$. The first step preserves confluence since any application of the new $\mathcal{E}$-rule can be undone using the corresponding $\mathcal{E}^{-}$-rule and vice versa, and since w is fresh, no rule other than $w \approx u \circ v$ can affect a subterm containing w.

For $\vdash_{\mathrm{simp}}$, we prove that $\to_{\mathcal{E}^{\pm} \cup \mathcal{R}[t]} \subseteq \to^{*}_{\mathcal{E}^{\pm} \cup \mathcal{R}[u]}$ and $\to_{\mathcal{E}^{\pm} \cup \mathcal{R}[u]} \subseteq \to^{*}_{\mathcal{E}^{\pm} \cup \mathcal{R}[t]}$. There are two cases. (i) If a left-hand side of a rule is changed, i.e., $\ell[t] \to r \in \mathcal{R}[t]$ is changed to $\ell[u] \to r \in \mathcal{R}[u]$, then observing that $s[\ell[t]] \to_{\mathcal{E}} s[\ell[u]] \to_{\mathcal{R}[u]} s[r]$ (simulating $\to_{\ell[t] \to r}$ using rules from $\mathcal{E}^{\pm} \cup \mathcal{R}[u]$) and $s[\ell[u]] \to_{\mathcal{E}^{-}} s[\ell[t]] \to_{\mathcal{R}[t]} s[r]$ (simulating $\to_{\ell[u] \to r}$ using rules from $\mathcal{E}^{\pm} \cup \mathcal{R}[t]$) establishes the claim, since all other rules are contained in both $\mathcal{R}[u]$ and $\mathcal{R}[t]$. (ii) If a right-hand side is changed, $\ell \to r[t] \in \mathcal{R}[t]$, $\ell \to r[u] \in \mathcal{R}[u]$, then the simulations $s[\ell] \to_{\mathcal{R}[t]} s[r[t]] \to_{\mathcal{E}} s[r[u]]$ and $s[\ell] \to_{\mathcal{R}[u]} s[r[u]] \to_{\mathcal{E}^{-}} s[r[t]]$ prove the claim. ◀

Let the result of flattening be $(\mathcal{R}_1, \mathcal{E})$, over an extended signature, where $\mathcal{F}^{(0)}$ includes the fresh constants added by $\vdash_{\mathrm{ext}}$. Flattening is straightforward to implement by a bottom-up traversal of the sides of the TRS, replacing subterms of the shape $u \circ v$ by constants, and maintaining a lookup table of which such terms have been seen before. This takes time $O(\|\mathcal{R}_0\| \log(\|\mathcal{R}_0\|))$ (the $\log(\|\mathcal{R}_0\|)$ factor accounts for the lookup table operations), and we have $\|\mathcal{E}\| = O(\|\mathcal{R}_0\|)$, $\|\mathcal{R}_1\| = O(|\mathcal{R}_0|)$, i.e., the total size of the TRSs $\mathcal{R}_1$ and $\mathcal{E}$ is at most linear in that of $\mathcal{R}_0$.

▶ **Example 3.3** (continued from Example 3.1)**.** We introduce fresh constants fa, fb, ffb and fffb for $\mathsf{f} \circ \mathsf{a}$, $\mathsf{f} \circ \mathsf{b}$, $\mathsf{f} \circ \mathsf{fb}$ and $\mathsf{f} \circ \mathsf{ffb}$, respectively. The resulting TRSs are $(\mathcal{R}_1, \mathcal{E}) = (\{\mathsf{a} \to \mathsf{fa}, \mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{ffb}\}, \{\mathsf{f}{\circ}\mathsf{a} \approx \mathsf{fa}, \mathsf{f}{\circ}\mathsf{b} \approx \mathsf{fb}, \mathsf{f}{\circ}\mathsf{fb} \approx \mathsf{ffb}\})$ and $(\mathcal{R}_1', \mathcal{E}') = (\mathcal{R}_1 \cup \{\mathsf{fffb} \to \mathsf{b}\}, \mathcal{E} \cup \{\mathsf{f}{\circ}\mathsf{ffb} \approx \mathsf{fffb})$.

3.2 Rewrite Closure

In this step, we are given a pair $(\mathcal{R}_1, \mathcal{E})$, where $\mathcal{R}_1$ is a system of C-rules and $\mathcal{E}$ is a system of D-rules. We want to obtain another pair $(\mathcal{R}_2, \mathcal{E})$, where $s \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} t$ iff $s \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} t$, such that every rewrite sequence in $(\mathcal{R}_1, \mathcal{E})$ can be transformed into a rewrite sequence in $(\mathcal{R}_2, \mathcal{E})$ of a special shape (cf. Lemma 3.4). The inference rules in Figure 1 define a relation $u \rightsquigarrow v$ on constants. We will see in a moment that $u \rightsquigarrow v$ iff $u \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} v$.

$$\frac{u \to v \in \mathcal{R}_1}{u \rightsquigarrow v}\ \text{base} \qquad \frac{u \in \mathcal{F}^{(0)}}{u \rightsquigarrow u}\ \text{refl} \qquad \frac{u \rightsquigarrow v \quad v \rightsquigarrow w}{u \rightsquigarrow w}\ \text{trans}$$

$$\frac{u_1 \rightsquigarrow v_1 \quad u_2 \rightsquigarrow v_2 \quad \{u_1 \circ u_2 \approx u, v_1 \circ v_2 \approx v\} \subseteq \mathcal{E}}{u \rightsquigarrow v}\ \text{comp}$$

Figure 1 Inference Rules for Rewrite Closure

The result of the rewrite closure step is $(\mathcal{R}_2, \mathcal{E})$, where $\mathcal{R}_2 = \{u \to v \mid u \rightsquigarrow v\}$.

▶ **Lemma 3.4.** *$s \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} t$ if and only if $s \to^*_{\mathcal{E} \cup \mathcal{R}_2} \cdot \to^*_{\mathcal{E}^- \cup \mathcal{R}_2} t$.*

Proof. Because of (base), we have $\mathcal{R}_1 \subseteq \mathcal{R}_2$, so that $\to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} \subseteq \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2}$. On the other hand, all rules in Figure 1 are compatible with the requirement that $\to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} \subseteq \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1}$. Therefore, the reachability relation is preserved, i.e., $\to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} = \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2}$.

First we show that for $u, v \in \mathcal{F}^{(0)}$, $u \rightsquigarrow v$ (and therefore $u \to_{\mathcal{R}_2} v$) whenever $u \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} v$. Assume that we have $u \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} v$ but not $u \rightsquigarrow v$. Let $u = t_0 \to \cdots \to t_n = v$ be the shortest sequence of $(\mathcal{E}^\pm \cup \mathcal{R}_2)$ steps from u to v, and pick u and v such that n is minimal. If $n = 0$ then $u = v$, and $u \rightsquigarrow v$ by (refl). If $n = 1$ then $u \to v \in \mathcal{R}_2$ since $\mathcal{E}$ only contains D-rules. If $t_i \in \mathcal{F}^{(0)}$ for any $0 < i < n$, then $u \rightsquigarrow t_i \rightsquigarrow v$ by minimality of $u \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} v$, and $u \rightsquigarrow v$ by transitivity (trans). In the remaining case, we have $t_i = u_i \circ v_i$ for all $0 < i < n$, and hence $u_1 \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} u_{n-1}$ and $v_1 \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} v_{n-1}$ since any root step would have a constant from $\mathcal{F}^{(0)}$ as source or target. But these two rewrite sequences have length at most $n - 2$, and therefore $u_1 \rightsquigarrow u_{n-1}$ and $v_1 \rightsquigarrow v_{n-1}$, implying $u \rightsquigarrow v$ by the (comp) rule. In all cases we found that $u \rightsquigarrow v$, a contradiction.

Now let $s \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} t$. Then $s \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} t$. Assume that this rewrite sequence is not of the shape $s \to^*_{\mathcal{E} \cup \mathcal{R}_2} \cdot \to^*_{\mathcal{E}^- \cup \mathcal{R}_2} t$, but has a minimal number of inversions between $\mathcal{E}$ and $\mathcal{E}^-$ steps (an inversion is any pair of an $\mathcal{E}$ step following an $\mathcal{E}^-$ step, not necessarily directly). Then it has a subsequence of the shape $s' \to_{p,\mathcal{E}^-} s'' \to^*_{\mathcal{R}_2} t'' \to_{q,\mathcal{E}} t'$, starting with an $\mathcal{E}^-$ step at p and a final $\mathcal{E}$ step at q. The cases $p < q$ or $p > q$ are impossible, because $\mathcal{E}$ contains only D-rules and $\mathcal{R}_2$ only C-rules (applying C-rules does not change the set of positions of a term).

If $p = q$ then $s''|_{pi} \to^*_{\mathcal{R}_2} t''|_{qi}$ for $i \in \{1, 2\}$, collecting all $\mathcal{R}_2$ steps at positions below p from $s'' \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} t''$. By (refl) and (trans) this implies $s''|_{pi} \rightsquigarrow t''|_{qi}$ for $i \in \{1, 2\}$. Consequently, we have $s'|_p \rightsquigarrow t'|_p$ by (comp). Hence we can delete the two $\mathcal{E}^\pm$ steps and the collected $\mathcal{R}_2$ steps, and replace them by an $\mathcal{R}_2$ step using the rule $s'|_p \to t'|_p$. This decreases the number of inversions between $\mathcal{E}$ and $\mathcal{E}^-$ steps, contradicting our minimality assumption. Otherwise, if $p \parallel q$, then we can reorder the rewrite sequence $s' \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2} t'$ as $s' \to^*_{>q,\mathcal{R}_2} \cdot \to_{q,\mathcal{E}} \cdot \to^*_{\mathcal{R}_2} \cdot \to_{p,\mathcal{E}^-} \cdot \to^*_{>p,\mathcal{R}_2} t'$, commuting mutually parallel rewrite

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**rewrite-closure**$(n, \mathcal{E}, \mathcal{R})$: Compute rewrite closure.
(Assumes that $\mathcal{F}^{(0)} = \{1, \ldots, n\}$, which can be achieved as part of the flattening step).
1. By scanning $\mathcal{E}$ once, compute arrays $l$ and $r$ such that $l[u] = \{(v, w) \mid u \circ v \to w \in \mathcal{E}\}$
   and $r[v] = \{(u, w) \mid u \circ v \to w \in \mathcal{E}\}$.
2. Let $\mathcal{R}' = \varnothing \subseteq \{1, \ldots, n\}^2$ (represented by an array).
3. Process (refl): Call add$(u, u)$ for $1 \le u \le n$.
4. Process (base): Call add$(u, v)$ for $u \to v \in \mathcal{R}$.

**add**$(u, v)$: Add $u \to v$ to $\mathcal{R}'$ and process implied (trans) and (comp) rules.
1. If $u \to v \in \mathcal{R}'$, return immediately.
2. Let $\mathcal{R}' = \mathcal{R}' \cup \{u \to b\}$.
3. Process (trans): For all $w \in \{1, \ldots, n\}$,
   - if $w \to u \in \mathcal{R}'$, call add$(w, v)$.
   - if $v \to w \in \mathcal{R}'$, call add$(u, w)$.
4. Process (comp):
   - For all $(u_2, u_r) \in l[u]$ and $(v_2, v_r) \in l[v]$, if $u_2 \to v_2 \in \mathcal{R}'$, call add$(u_r, v_r)$.
   - For all $(u_1, u_r) \in r[u]$ and $(v_1, v_r) \in r[v]$, if $u_1 \to v_1 \in \mathcal{R}'$, call add$(u_r, v_r)$.
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Figure 2 Algorithm for Rewrite Closure

steps. This reduces the number of inversions between $\mathcal{E}$ and $\mathcal{E}^-$ steps, and again we reach a contradiction. ◀

The size of $\mathcal{R}_2$ is bounded by $|\mathcal{F}^{(0)}|^2 = O(\|\mathcal{R}_0\|^2)$. We can view the inference rules in Figure 1 as a system of Horn clauses with atoms of the form $u \rightsquigarrow v$ ($u, v \in \mathcal{F}^{(0)}$). This system can be solved in time proportional to the total size of the clauses [4], finding a minimal solution for the relation $\rightsquigarrow$. There are $|\mathcal{R}_1|$ instances of (base), $|\mathcal{F}^{(0)}|$ instances of (refl), $|\mathcal{F}^{(0)}|^3$ instances of (trans) and at most $|\mathcal{F}^{(0)}|^2$ instances of (comp), noting that u_1, u_2 are determined by u and v_1, v_2 are determined by v. Therefore, we can compute $\mathcal{R}_2$ in time $O(\|R_0\|^3)$.

▶ Remark. In our implementation, we do not generate these Horn clauses explicitly. Instead, whenever we make a new inference $u \rightsquigarrow v$, we check all possible rules that involve $u \rightsquigarrow v$ as a premise. The result is a neat incremental algorithm (see Figure 2). From an abstract point of view, however, this is no different than solving the Horn clauses as stated above. This remark also applies to inference rules presented later.

▶ **Example 3.5** (continued from Example 3.3)**.** We present $\mathcal{R}_2$ and $\mathcal{R}_2'$ as tables, where non-empty entries correspond to the rules contained in each TRS. For example, $\mathsf{fa} \to \mathsf{b} \in \mathcal{R}_2'$ but $\mathsf{fa} \to \mathsf{b} \notin \mathcal{R}_2$. The letters indicate the inference rule used to derive the entry, while the superscripts indicate stage numbers—each inference uses only premises that have smaller stage numbers.

$\mathcal{R}_2 =$

	f	a	fa	b	fb	ffb
f	r^0					
a		r^0	b^0	b^0	t^2	t^1
fa			r^0		c^1	c^3
b				r^0		b^0
fb					r^0	
ffb						r^0

$\mathcal{R}_2' =$

	f	a	fa	b	fb	ffb	fffb
f	r^0						
a		r^0	b^0	b^0	t^2	t^1	t^3
fa			r^0	t^3	c^1	t^3	t^2
b				r^0	t^4	b^0	t^4
fb				t^2	r^0	t^2	c^1
ffb				t^4	c^3	r^0	c^3
fffb				b^0	t^4	t^1	r^0

3.3 Congruence Closure

We are also interested in the congruence closure of $(\mathcal{R}_1, \mathcal{E})$, because it allows us to decide when two terms are convertible. We calculate the congruence closure as the rewrite closure of $(\mathcal{R}_1^{\pm}, \mathcal{E})$ and call it $(\mathcal{C}, \mathcal{E})$. This step also takes $O(\|\mathcal{R}_0\|^3)$ time. By Lemma 3.4 we have

$$s \leftrightarrow^*_{\mathcal{E}^{\pm} \cup \mathcal{R}_1} t \iff s \to^*_{\mathcal{E}^{\pm} \cup \mathcal{R}_1^{\pm}} t \iff s \to^*_{\mathcal{E} \cup \mathcal{C}} \cdot \to^*_{\mathcal{E}^- \cup \mathcal{C}} t \iff s \downarrow_{\mathcal{E} \cup \mathcal{C}} t.$$

Note that $\mathcal{C}$ is symmetric and therefore, $\to_{\mathcal{E}^- \cup \mathcal{C}} = {}_{\mathcal{E} \cup \mathcal{C}}\!\leftarrow$.

▶ Remark. There are far more efficient methods for calculating the congruence closure (an almost linear time algorithm can be found in [2]), but the simple reduction to the rewrite closure is sufficient for our purposes, since the total asymptotic running time is unchanged.

▶ **Example 3.6** (continued from Example 3.5). Since $\mathcal{C}$ is an equivalence relation, we just give its equivalence classes: $[\mathsf{f}]_{\mathcal{C}} = \{\mathsf{f}\}$ and $[\mathsf{a}]_{\mathcal{C}} = \{\mathsf{a}, \mathsf{fa}, \mathsf{b}, \mathsf{fb}, \mathsf{ffb}\}$. For $\mathcal{C}'$, we obtain $[\mathsf{f}]_{\mathcal{C}'} = [\mathsf{f}]_{\mathcal{C}}$ and $[\mathsf{a}]_{\mathcal{C}'} = [\mathsf{a}]_{\mathcal{C}} \cup \{\mathsf{fffb}\}$. Note that $\mathcal{C}$ and $\mathcal{C}'$ are the symmetric, transitive closures of $\mathcal{R}_2$ and $\mathcal{R}_2'$, respectively. This holds in general.

3.4 Confluence Conditions

So far we have flattened the TRS $\mathcal{R}_0$ and computed its rewrite and congruence closures, enabling us to check reachability and convertibility of any given terms efficiently. In this section we use these tools to decide confluence of $\mathcal{R}_0$.

We closely follow the approach in [10] and [6], which is based on the analysis of two convertible terms s, t and their normal forms with respect to a system of so-called *forward rules* of a rewrite closure. In our approach, these correspond to the system $\mathcal{E} \cup \mathcal{R}_2$. However, $\to_{\mathcal{E} \cup \mathcal{R}_2}$ is typically non-terminating, and we cannot use this idea directly. This problem is easy to overcome though. We define $\mathcal{A}; \mathcal{B} = \{\ell \to r \mid \ell \to m \in \mathcal{A} \text{ and } m \to r \in \mathcal{B}\}$ and $\to_{\mathcal{A}/\mathcal{B}} = \to^*_{\mathcal{B}} \cdot \to_{\mathcal{A}} \cdot \to^*_{\mathcal{B}}$. Note that $\to_{\mathcal{E}/\mathcal{R}_2}$ is terminating. We will use $\to_{\mathcal{E}/\mathcal{R}_2}$ in place of the forward reduction. This choice is justified by Lemma 3.8 below. We will abuse notation slightly and speak of $\mathcal{E}/\mathcal{R}_2$ normal forms.

▶ **Lemma 3.7.** *Let $\mathcal{S}$ be a transitive, reflexive (as a relation) set of C-rules and $\mathcal{E}$ a set of D-rules. Then $\to^*_{\mathcal{E} \cup \mathcal{S}} = \to^*_{\mathcal{S}} \cdot \to^*_{\mathcal{E};\mathcal{S}}$ and $\to^*_{\mathcal{E}^- \cup \mathcal{S}} = \to^*_{\mathcal{S};\mathcal{E}^-} \cdot \to^*_{\mathcal{S}}$.*

Proof. We first show that $\to^*_{\mathcal{E} \cup \mathcal{S}} = \to^*_{\mathcal{S}} \cdot \to^*_{\mathcal{E};\mathcal{S}}$. Start with a rewrite sequence $s \to^*_{\mathcal{E} \cup \mathcal{S}} t$. Whenever an $\mathcal{S}$ step is followed by another $\mathcal{S}$ step at the same position, we can combine them using transitivity of $\mathcal{S}$. Note that since $\mathcal{E}$ only contains D-rules, no intermediate $\mathcal{E}$ step can overlap with either of the $\mathcal{S}$ steps. Once there are no more $\mathcal{S}$ steps that can be combined this way, we replace each $\mathcal{E}$ step that is followed by an $\mathcal{S}$ step at the same position by the corresponding $\mathcal{E}; \mathcal{S}$ step. If there is no following $\mathcal{S}$ step, we add an identity $\mathcal{S}$ step (which exists by reflexivity of $\mathcal{S}$) first. It is easy to verify that the final rewrite sequence is of the desired shape.

For $\to^*_{\mathcal{E}^- \cup \mathcal{S}} = \to^*_{\mathcal{S};\mathcal{E}^-} \cdot \to^*_{\mathcal{S}}$ it suffices to note that by reversing the rewrite sequences this is equivalent to $\to^*_{\mathcal{E} \cup \mathcal{S}^-} = \to^*_{\mathcal{S}^-} \cdot \to^*_{\mathcal{E};\mathcal{S}^-}$. Since $\mathcal{S}^-$ is transitive and reflexive if $\mathcal{S}$ is, the claim reduces to the previous one. ◀

▶ **Lemma 3.8.**

1. *If $s \to^*_{\mathcal{E}^{\pm} \cup \mathcal{R}_1} t$ then $s \to^*_{\mathcal{E}/\mathcal{R}_2} \cdot \to^*_{\mathcal{R}_2;\mathcal{E}^-} \cdot \to^*_{\mathcal{R}_2} t$.*
2. *If $s \leftrightarrow^*_{\mathcal{E}^{\pm} \cup \mathcal{R}_1} t$ then $s \to^*_{\mathcal{C}} \cdot \to^*_{\mathcal{E};\mathcal{C}} \cdot {}_{\mathcal{E};\mathcal{C}}\!\overset{*}{\leftarrow} \cdot {}_{\mathcal{C}}\!\overset{*}{\leftarrow} t$.*

Proof. 1. Assume that $s \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} t$. By Lemma 3.4, this is equivalent to $s \to^*_{\mathcal{E} \cup \mathcal{R}_2} \cdot \to^*_{\mathcal{E}^- \cup \mathcal{R}_2} t$, or $s \to^*_{\mathcal{E}/\mathcal{R}_2} \cdot \to^*_{\mathcal{E}^- \cup \mathcal{R}_2} t$, which according to Lemma 3.7 is equivalent to $s \to^*_{\mathcal{E}/\mathcal{R}_2} \cdot \to^*_{\mathcal{R}_2;\mathcal{E}^-} \cdot \to^*_{\mathcal{R}_2} t$, noting that $\mathcal{R}_2^-$ is both reflexive and transitive by construction.

2. Assume that $s \leftrightarrow^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} t$, i.e., $s \to_{\mathcal{E}^\pm \cup \mathcal{R}_1^\pm} t$. Again by Lemma 3.4, this is equivalent to $s \to^*_{\mathcal{E} \cup \mathcal{C}} \cdot {}_{\mathcal{E} \cup \mathcal{C}}^{\;*}{\leftarrow}\, t$. Since $\mathcal{C}$ is reflexive and transitive, the claim follows from Lemma 3.7. ◀

Let us assume that $\mathcal{R}_1 \cup \mathcal{E}^\pm$ is confluent, and that we have two convertible terms s and t. There are corresponding $\mathcal{E}/\mathcal{R}_2$ normal forms s' and t' for s and t, respectively. Now s' and t' are convertible, so that by Lemma 3.8(2), for some term r,

$$s' \to^*_{\mathcal{C}} \cdot \to^*_{\mathcal{E};\mathcal{C}} r \;{}_{\mathcal{E};\mathcal{C}}^{\;*}{\leftarrow} \cdot {}_{\mathcal{C}}^{*}{\leftarrow}\, t'. \tag{1}$$

Furthermore, by confluence and Lemma 3.8(1), noting that the choice of s' and t' forces the $\to^*_{\mathcal{E}/\mathcal{R}_2}$ sequences to be empty, it follows that for their common reduct r',

$$s' \to^*_{\mathcal{R}_2;\mathcal{E}^-} \cdot \to^*_{\mathcal{R}_2} r' \;{}_{\mathcal{R}_2}^{\;*}{\leftarrow} \cdot {}_{\mathcal{R}_2;\mathcal{E}^-}^{\quad *}{\leftarrow}\, t'. \tag{2}$$

To capture the conditions on s' and t' (which are $\mathcal{E}/\mathcal{R}_2$ normal forms), we adapt the notion of *top-stabilizable* terms and constants from [6] to our purposes.

▶ **Definition 3.9.** A term $u \circ v$ with $u, v \in \mathcal{F}^{(0)}$ is *top-stabilizable* if there exists an $\mathcal{E}/\mathcal{R}_2$ normal form s such that $s \to^*_{\mathcal{C}} \cdot \to^*_{\mathcal{E};\mathcal{C}} u \circ v$. A constant $u \in \mathcal{F}^{(0)}$ is *top-stabilizable* if there exist $v, w \in \mathcal{F}^{(0)}$ such that $u \to_{\mathcal{C};\mathcal{E}^-} v \circ w$ and $v \circ w$ is top-stabilizable.

The equations (1,2) define two rewrite sequences from r to r' that consist solely of C- and inverse D-steps (note that we consider the rewrite sequences from (1) in reverse). This means that no rewrite step occurs below a preceding rewrite step. In fact all rewrite steps modify a leaf of a term. Therefore we may assume without loss of generality that $r \in \mathcal{F}^{(0)}$. Looking at the surrounding rewrite steps in equation (1), we distinguish three cases depending on whether the sequence of $\mathcal{E};\mathcal{C}$ steps is empty or not.

1. $s' \to^*_{\mathcal{C} \cup \mathcal{E}} s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} r \;{}_{\mathcal{E};\mathcal{C}}{\leftarrow}\, t_1 \circ t_2 \;{}_{\mathcal{C} \cup \mathcal{E}}^{\;*}{\leftarrow}\, t'$. In this case $s_1 \circ s_2$, $t_1 \circ t_2$ must be top-stabilizable. Furthermore, for $i \in \{1, 2\}$, the terms s_i and t_i are convertible via r_i, so that $s_i \downarrow_{\mathcal{C} \cup \mathcal{E}} t_i$ by Lemma 3.8.
2. $s' \to^*_{\mathcal{C} \cup \mathcal{E}} s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} t' \in \mathcal{F}^{(0)}$. (Note that we use the fact that $\mathcal{C}$ is an equivalence relation: $s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} \cdot {}_{\mathcal{C}}^{*}{\leftarrow}\, t'$ implies $s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} t'$ if $t' \in \mathcal{F}^{(0)}$.) Then there must be $t_1, t_2 \in \mathcal{F}^{(0)}$ such that $t' \to_{\mathcal{R}_2;\mathcal{E}^-} t_1 \circ t_2$, and $s_i \downarrow_{\mathcal{C} \cup \mathcal{E}} t_i$ for $i \in \{1, 2\}$. This case also covers $s' \;{}_{\mathcal{E};\mathcal{C}}{\leftarrow}\, t_1 \circ t_2 \;{}_{\mathcal{C} \cup \mathcal{E}}^{\;*}{\leftarrow}\, t'$ by symmetry.
3. $\mathcal{F}^{(0)} \ni s' \to_{\mathcal{C}} t' \in \mathcal{F}^{(0)}$. Then $s' \downarrow_{\mathcal{E}^- \cup \mathcal{R}_2} t'$, with common reduct r'.

Hence we have found the following necessary conditions for confluence of $\mathcal{R}_1 \cup \mathcal{E}^\pm$:

▶ **Definition 3.10.** The *confluence conditions* for confluence of $\mathcal{R}_2 \cup \mathcal{E}^\pm$ are as follows.

1. If $s_1 \circ s_2$ and $t_1 \circ t_2$ are top-stabilizable for constants $s_1, s_2, t_1, t_2 \in \mathcal{F}^{(0)}$ such that $s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} r \;{}_{\mathcal{E};\mathcal{C}}{\leftarrow}\, t_1 \circ t_2$ then $s_i \downarrow_{\mathcal{C} \cup \mathcal{E}} t_i$ for $i \in \{1, 2\}$.
2. If $s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} t'$ for $s_1, s_2, t' \in \mathcal{F}^{(0)}$ and top-stabilizable $s_1 \circ s_2$, then there must be $t_1, t_2 \in \mathcal{F}^{(0)}$ such that $t' \to_{\mathcal{R}_2;\mathcal{E}^-} t_1 \circ t_2$, and $s_i \downarrow_{\mathcal{C} \cup \mathcal{E}} t_i$ for $i \in \{1, 2\}$.
3. If $\mathcal{F}^{(0)} \ni s' \to_{\mathcal{C}} t' \in \mathcal{F}^{(0)}$ then $s' \downarrow_{\mathcal{E}^- \cup \mathcal{R}_2} t'$.

▶ **Lemma 3.11.** *The confluence conditions are necessary and sufficient for confluence of* $\mathcal{R}_1 \cup \mathcal{E}^\pm$.

Proof. Necessity has already been shown above. For sufficiency, assume that the confluence conditions are satisfied and there are convertible terms s and t with no common reduct. Then any corresponding $\mathcal{E}/\mathcal{R}_2$ normal forms do not have a common reduct either. Let s' and t' be convertible $\mathcal{E}/\mathcal{R}_2$ normal forms with no common reduct such that $|s'| + |t'|$ is minimal. Recall that $\to^*_{\mathcal{E}^\pm \cup \mathcal{R}_1} = \to^*_{\mathcal{E}^\pm \cup \mathcal{R}_2}$ so that $\mathcal{R}_1 \cup \mathcal{E}^\pm$ joinability and $\mathcal{R}_2 \cup \mathcal{E}^\pm$ joinability coincide. The same holds for convertibility. We will simply use the terms "joinable" and "convertible" for both $\mathcal{R}_1 \cup \mathcal{E}^\pm$ and $\mathcal{R}_2 \cup \mathcal{E}^\pm$. We distinguish three cases.

1. If $s', t' \in \mathcal{F}^{(0)}$, then by Lemma 3.8(2), $s' \to_{\mathcal{C}} t'$ (since s', t' are $\mathcal{E};\mathcal{C}$ normal forms and $\mathcal{C}$ is an equivalence relation) and we obtain a joining sequence from the third confluence condition, contradicting the non-joinability of s' and t'.
2. If $s' = s'_1 \circ s'_2 \notin \mathcal{F}^{(0)}$ and $t' \in \mathcal{F}^{(0)}$, then by Lemma 3.8(2) there is a rewrite sequence $s' \to^*_{\mathcal{C} \cup \mathcal{E}} s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} t'$ (again using that t' is an $\mathcal{E};\mathcal{C}$ normal form and that $\mathcal{C}$ is an equivalence relation). By the second confluence condition we obtain a term $t_1 \circ t_2$ such that $t' \to_{\mathcal{R}_2;\mathcal{E}^-} t_1 \circ t_2$, and t_i and s_i are convertible for $i \in \{1,2\}$. Therefore, t_1 and s'_1 are convertible. Furthermore, since $|t_1| + |s'_1| < |t'| + |s'|$, this implies that t_1 and s'_1 are joinable. Analogously, t_2 and s'_2 are also joinable, and therefore s' is joinable with $t_1 \circ t_2 \ {}_{\mathcal{R}_2;\mathcal{E}^-}\!\!\leftarrow t'$, contradicting our assumptions.
   The case that $s' \in \mathcal{F}^{(0)}$ and $t' \notin \mathcal{F}^{(0)}$ is handled symmetrically.
3. If $s' = s'_1 \circ s'_2 \notin \mathcal{F}^{(0)}$ and $t' = t'_1 \circ t'_2 \notin \mathcal{F}^{(0)}$, then by Lemma 3.8(2), $s' \to^*_{\mathcal{E} \cup \mathcal{C}} r \ {}_{\mathcal{E} \cup \mathcal{C}}^{\ \ *}\!\!\leftarrow t'$. If $r = r_1 \circ r_2$ is not a constant, then s'_1 and t'_1 are convertible via r_1 and likewise s'_2 and t'_2 are convertible via r_2. However, one of these pairs cannot be joinable, and we obtain a smaller counterexample to confluence, a contradiction. Therefore, r must be a constant. Using Lemma 3.8(2) we obtain a rewrite proof $s' \to^*_{\mathcal{C} \cup \mathcal{E}} s_1 \circ s_2 \to_{\mathcal{E};\mathcal{C}} r \ {}_{\mathcal{E};\mathcal{C}}\!\!\leftarrow t_1 \circ t_2 \ {}_{\mathcal{C} \cup \mathcal{E}}^{\ \ *}\!\!\leftarrow t'$. From the first confluence condition, we conclude that s_1 and t_1 are convertible and therefore also s'_1 and t'_1. By minimality of $|s'| + |t'|$, s'_1 and t'_1 must be joinable. Likewise, s'_2 and t'_2 must also be joinable, from which we conclude that $s' = s'_1 \circ s'_2$ and $t' = t'_1 \circ t'_2$ are joinable as well, a contradiction.

This completes the proof. ◀

3.5 Computation of Confluence Conditions

The computation consists of two major steps: First we compute all top-stabilizable constants and terms of the form $u \circ v$. Then we check the three confluence conditions.

In order to compute the top-stabilizable constants and terms, we first need to find the $\mathcal{E}/\mathcal{R}_2$ normal forms of the shape $u \circ v$—denoted by $\mathsf{NF}(u \circ v)$. We can compute the complement of that set, i.e., the $\mathcal{E}/\mathcal{R}_2$ reducible terms of that shape using the following inference rules.

$$\frac{u \circ v \approx w \in \mathcal{E}}{\neg\mathsf{NF}(u \circ v)}\ \text{base} \qquad \frac{\{u_1 \to v_1, u_2 \to v_2\} \subseteq \mathcal{R}_2 \quad \neg\mathsf{NF}(v_1 \circ v_2)}{\neg\mathsf{NF}(u_1 \circ u_2)}\ \text{comp}$$

To obtain a cubic time algorithm, note that thanks to transitivity of $\mathcal{R}_2$, inferences made by (comp) need not be processed—if $\neg\mathsf{NF}(w_1 \circ w_2)$ implies $\neg\mathsf{NF}(v_1 \circ v_2)$ by (comp) and $\neg\mathsf{NF}(v_1 \circ v_2)$ implies $\neg\mathsf{NF}(u_1 \circ u_2)$ by (comp) then $\neg\mathsf{NF}(w_1 \circ w_2)$ implies $\neg\mathsf{NF}(u_1 \circ u_2)$ by (comp) as well. Therefore we simply consider each $\mathcal{E}$ rule (there are $O(\|\mathcal{R}_0\|)$ of these) in turn, and then make the corresponding inferences by the (comp) rule in $O(\|\mathcal{R}_0\|^2)$ time, for a total of $O(\|\mathcal{R}_0\|^3)$.

Let us turn to top-stabilizable terms and constants now. First note that any constant is an $\mathcal{E}/\mathcal{R}_2$ normal form. The top-stabilizable constants and terms can be found using another incremental computation. Every $\mathcal{E}/\mathcal{R}_2$ normal form is top-stabilizable. If $u \circ v$ is top-stabilizable and $u \circ v \to_{\mathcal{E}/\mathcal{C}} w$, then w is a top-stabilizable constant, and $u' \circ v$ and $u \circ v'$ are top-stabilizable terms whenever $u \to_{\mathcal{C}} u'$, $v \to_{\mathcal{C}} v'$. For any top-stabilizable constant w, $w \circ v$, $u \circ w$ for constants u, v are also top-stabilizable. Consequently, we obtain the following inference rules, where $\mathsf{TS}(u)$ and $\mathsf{TS}(v \circ w)$ assert that u and $v \circ w$ are top-stabilizable, respectively, and $(i, i') \in \{(1,2),(2,1)\}$.

$$\frac{u_1 \circ u_2 \in \mathsf{NF}(\mathcal{E}/\mathcal{R}_2)}{\mathsf{TS}(u_1 \circ u_2)}\ \mathrm{nf} \qquad \frac{u_1 \circ u_2 \approx u \in \mathcal{E} \quad \mathsf{TS}(u_1 \circ u_2)}{\mathsf{TS}(u)}\ \mathrm{ts}_0 \qquad \frac{\mathsf{TS}(u_i)}{\mathsf{TS}(u_1 \circ u_2)}\ \mathrm{ts}_i$$

$$\frac{u \to v \in \mathcal{C} \quad \mathsf{TS}(v)}{\mathsf{TS}(u)}\ \mathrm{comp}_0 \qquad \frac{u_i \to v_i \in \mathcal{C} \quad u_{i'} = v_{i'} \quad \mathsf{TS}(v_1 \circ v_2)}{\mathsf{TS}(u_1 \circ u_2)}\ \mathrm{comp}_i$$

There are $O(\|\mathcal{R}_0\|^2)$ instances of (nf), ($\mathrm{ts}_{\{1,2\}}$) and (comp_0), and $O(\|\mathcal{R}_0\|^3)$ instances of (ts_0) and ($\mathrm{comp}_{\{1,2\}}$). Again these inference rules have the shape of Horn clauses and can be processed in time proportional to their total size, which is $O(\|\mathcal{R}_0\|^3)$.

▶ **Example 3.12** (continued from Example 3.6)**.** For $\mathcal{R}$ we have $\neg\mathsf{NF} = \{\mathsf{f} \circ \mathsf{b}, \mathsf{f} \circ \mathsf{fa}, \mathsf{f} \circ \mathsf{fb}, \mathsf{f} \circ \mathsf{a}\}$. Indeed $\mathsf{f} \circ \mathsf{ffb}$ is an $\mathcal{E}/\mathcal{R}_2$ normal form since using $\mathcal{R}_2$ it can only be rewritten to itself and it is not the left-hand side of any $\mathcal{E}$ rule. On the other hand, for $\mathcal{R}'$ we obtain $\neg\mathsf{NF}' = \neg\mathsf{NF} \cup \{\mathsf{f} \circ \mathsf{ffb}, \mathsf{f} \circ \mathsf{fffb}\}$. Note that normal forms also include terms like $\mathsf{f} \circ \mathsf{f}$ or $\mathsf{fa} \circ \mathsf{a}$ that have no correspondence in the original TRS.

In the $\mathcal{R}$ case, all terms of the form $u \circ v$ are top-stabilizable and so are all constants except for f. For $\mathcal{R}'$, only the normal forms are top-stabilizable.

With this pre-computation done, checking the confluence conditions becomes a straightforward matter. The only tricky part is checking joinability of constants in the third condition. This relation can be computed in a way strikingly similar to the rewrite closure from Section 3.2, using the following inference rules for computing $\downarrow = \downarrow_{\mathcal{E}^- \cup \mathcal{R}_2}$ on constants:

$$\frac{u \in \mathcal{F}^{(0)}}{u \downarrow u}\ \mathrm{refl} \qquad \frac{\{u_1 \circ u_2 \approx u, v_1 \circ v_2 \approx v\} \subseteq \mathcal{E} \quad u_1 \downarrow v_1 \quad u_2 \downarrow v_2}{u \downarrow v}\ \mathrm{comp}$$

$$\frac{u \to v \in \mathcal{R}_2 \quad v \downarrow w}{u \downarrow w}\ \mathrm{trans}_l \qquad \frac{u \downarrow v \quad w \to v \in \mathcal{R}_2}{u \downarrow w}\ \mathrm{trans}_r$$

As with the previous inference rules, this is a system of Horn clauses. There are $O(\|\mathcal{E}\|^2) = O(\|\mathcal{R}_0\|^2)$ instances of (comp), $O(\|\mathcal{R}_0\|)$ instances of (refl) and $O(\|\mathcal{R}_0\|^3)$ instances of ($\mathrm{trans}_{l,r}$). Therefore, computing $\downarrow$ can be done in $O(\|\mathcal{R}_0\|^3)$ time.

▶ **Example 3.13** (continued from Example 3.12)**.** The joinability relations $\mathcal{R}$ and $\mathcal{R}'$ are given below. As in Example 3.5, the letters and superscripts indicate the rule being used to derive the entry and computation stage.

$$\downarrow = \begin{array}{r|cccccc} & \mathsf{f} & \mathsf{a} & \mathsf{fa} & \mathsf{b} & \mathsf{fb} & \mathsf{ffb} \\ \hline \mathsf{f} & r^0 & & & & & \\ \mathsf{a} & & r^0 & t_l^1 & t_l^1 & t_l^1 & t_l^1 \\ \mathsf{fa} & & t_r^1 & r^0 & t_l^1 & t_l^1 & t_l^1 \\ \mathsf{b} & & t_r^1 & t_r^1 & r^0 & & t_l^1 \\ \mathsf{fb} & & t_r^1 & t_r^1 & & r^0 & \\ \mathsf{ffb} & & t_r^1 & t_r^1 & t_r^1 & & r^0 \end{array} \qquad \downarrow' = \begin{array}{r|ccccccc} & \mathsf{f} & \mathsf{a} & \mathsf{fa} & \mathsf{b} & \mathsf{fb} & \mathsf{ffb} & \mathsf{fffb} \\ \hline \mathsf{f} & r^0 & & & & & & \\ \mathsf{a} & & r^0 & t_l^1 & t_l^1 & t_l^1 & t_l^1 & t_l^1 \\ \mathsf{fa} & & t_r^1 & r^0 & t_l^1 & c^2 & c^2 & c^2 \\ \mathsf{b} & & t_r^1 & t_r^1 & r^0 & t_r^1 & c^3 & c^3 \\ \mathsf{fb} & & t_r^1 & c^2 & t_l^1 & r^0 & c^2 & c^4 \\ \mathsf{ffb} & & t_r^1 & c^2 & c^3 & c^2 & r^0 & c^3 \\ \mathsf{fffb} & & t_r^1 & c^2 & c^3 & c^4 & c^3 & r^0 \end{array}$$

It is now easy to verify that $\mathcal{R}$ violates the third confluence condition ($\mathsf{fb} \to_{\mathcal{C}} \mathsf{ffb}$ but not $\mathsf{fb} \downarrow \mathsf{ffb}$), and therefore is not confluent. The other two confluence conditions are satisfied. $\mathcal{R}'$, on the other hand, satisfies all confluence conditions and is, therefore, confluent.

Table 1 Confluence of Ground Cops.

Cop	21	33	34	38	39	40	80	81	84	114	115	116
CR	×	✓	✓	×	×	✓	×	✓	✓	✓	✓	✓

Table 2 Runtimes for $\mathcal{R}_n$.

system	$\mathcal{R}_{100}$	$\mathcal{R}_{200}$	$\mathcal{R}_{400}$	$\mathcal{R}_{800}$	$\mathcal{R}_{1600}$	$\mathcal{R}_{3200}$
time (s)	0.2 (×)	0.2 (×)	1.3 (×)	19.2 (×)	254.3 (×)	2321 (×)
system	$\mathcal{R}_{101}$	$\mathcal{R}_{201}$	$\mathcal{R}_{401}$	$\mathcal{R}_{801}$	$\mathcal{R}_{1601}$	$\mathcal{R}_{3201}$
time (s)	0.2 (✓)	0.2 (✓)	2.3 (✓)	30.1 (✓)	427.4 (✓)	3919 (✓)

Putting everything together, we obtain the following theorem.

▶ **Theorem 3.14.** *The confluence of a ground TRS $\mathcal{R}$ can be decided in cubic time.*

Proof. Let $n = \|\mathcal{R}\|$. We follow the process outlined above. First we curry $\mathcal{R}$ in linear time, obtaining $\mathcal{R}_0$ with $\|\mathcal{R}_0\| = O(n)$. Then we flatten $\mathcal{R}_0$, obtaining $(\mathcal{R}_1, \mathcal{E})$ with $\|\mathcal{E}\| = O(n)$ and $\|\mathcal{R}_1\| = O(n)$ in time $O(n \log(n))$. In the next step we compute the rewrite and congruence closures $(\mathcal{R}_2, \mathcal{E})$ and $(\mathcal{C}, \mathcal{E})$ of $(\mathcal{R}_1, \mathcal{E})$ in $O(n^3)$ time. Afterwards, we compute the $\mathcal{E}/\mathcal{R}_2$ normal forms $\mathsf{NF}(- \circ -)$, which as seen above takes $O(n^3)$ time. We then compute $\mathsf{TS}(-)$, $\mathsf{TS}(-,-)$ and $\downarrow_{\mathcal{E}^- \cup \mathcal{R}_2}$ in $O(n^3)$ time. Finally we check the three confluence conditions. For the first condition, we check each of the $O(n^2)$ pairs of rules $s_1 \circ s_2 \to_{\mathcal{E}} u, t_1 \circ t_2 \to_{\mathcal{E}} v$ with $u \to_{\mathcal{C}} v$. For the second condition, we consider the $O(n^3)$ triples such that $s_1 \circ s_2 \to_{\mathcal{E}} u \to_{\mathcal{C}} t' \to_{\mathcal{R}_2} v \; {}_{\mathcal{E}}\!\leftarrow t_1 \circ t_2$. For the third condition we check all $O(n^2)$ pairs $s' \to_{\mathcal{C}} t'$. All these steps can be accomplished in $O(n^3)$ time. ◀

4 Experiments

We have implemented the above algorithm in the confluence tool CSI[2] [12], and tested it on the ground confluence problems from the Cops database.[3] The results are displayed in Table 1. There are no runtimes given because they are all negligible. Note though that even before implementing ground confluence in CSI, the tool could handle all these problems. The runtime improved from 14s to 3s for checking all the TRSs. In order to obtain runtime measurements, we considered the family of TRSs $\mathcal{R}_n = \mathcal{R} \cup \{\mathsf{f}^n(\mathsf{b}) \to \mathsf{b}\}$ extending $\mathcal{R}$ from Example 3.1. One can easily argue that the system $\mathcal{R}_n$ is confluent if and only if n is odd. (Since $\mathcal{R}$ is a subsystem, all terms are convertible. However, $\mathsf{f}(\mathsf{b})$ and b are only joinable if n is odd—otherwise the parity of k in the reducts $\mathsf{f}^k(\mathsf{b})$ is invariant.) The runtimes for various n are given in Table 2. ACP[4] [1] and Saigawa[5] fail on all these systems. The numbers from Table 2 do not agree well with the proven complexity bound. This is due to cache effects—as the input size increases, the intermediate arrays outgrow the first and second level caches. Note that for the last two columns, the factor is very close to 8, finally meeting expectations. The difference between odd and even n can be explained by the different size of the rewrite closures. All measurements were done on a 2.67GHz Intel i7-620M computer with 4GB RAM using a single core.

[2] http://cl-informatik.uibk.ac.at/software/csi/

[3] http://coco.nue.riec.tohoku.ac.jp/cops/

[4] version 0.20, http://www.nue.riec.tohoku.ac.jp/tools/acp/

[5] version 1.2, http://www.jaist.ac.jp/project/saigawa/

5 Conclusion

We have described an efficient algorithm for deciding the confluence of ground TRSs. In our opinion, this is a worthwhile addition to an automated confluence checker, since other methods fail on relatively simple ground TRSs. In fact, ACP can not handle either TRS from Example 3.1, and neither can Saigawa. Before adding the ground TRS code, CSI could not disprove confluence of $\mathcal{R}$, but it was able to prove confluence of $\mathcal{R}'$. It still failed on a close relative of $\mathcal{R}'$, namely the confluent ground TRS $\mathcal{R}_5 = \mathcal{R} \cup \{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))))) \to \mathsf{b}\}$.

A natural question is whether we can improve the bounds for the other known classes of TRSs with fixed maximum arity that have a known polynomial complexity for deciding complexity, foremost the class of shallow, left-linear TRSs. Our main improvement over [10] is the limitation to C-rules in the rewrite closure, effectively constraining the considered rules to relations between subterms of the original curried TRS. This does no longer work once we have variables in rules. Therefore, at present, we do not know how to improve the other results.

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