Rule Labeling for Confluence of Left-Linear Term Rewrite Systems*

Bertram Felgenhauer

Institute of Computer Science, University of Innsbruck, Austria
bertram.felgenhauer@uibk.ac.at

Abstract

Rule labeling is a heuristic, suggested by van Oostrom, for applying decreasing diagrams to linear TRSs. The heuristic also works for left-linear TRSs under certain relative termination conditions. In this note we apply the rule labeling heuristic to arbitrary left-linear TRSs, based on considering parallel reduction relations and parallel critical peaks.

1 Introduction

The decreasing diagrams technique by van Oostrom [6] is a powerful criterion for showing confluence of abstract rewrite systems. In [7], van Oostrom suggested the rule labeling heuristic for establishing confluence of linear term rewrite systems. The heuristic consists of labeling each rewrite step with the used rule. Confluence can be established by showing that all critical pairs can be joined decreasingly. Rule labeling can also be applied to certain duplicating left-linear TRSs, by combining it lexicographically with other labelings [7, 3, 9].

In this note we revisit the rule labeling heuristic and show how it can be applied to arbitrary left-linear TRSs, if parallel reduction and parallel critical pairs are considered.

The paper is organized as follows. Section 2 is devoted to preliminaries. Then, in Section 3, we present a confluence criterion for left-linear systems based on rule labeling. In Section 4 we sketch how this idea extends to weak ll-labelings from [9]. Finally, we conclude in Section 5.

2 Preliminaries

We assume that the reader is familiar with standard term rewriting terminology [1, 5].

Given a rewrite relation $\to$, we use $\xrightarrow{=}$ and $\xrightarrow{*}$ to denote its reflexive closure and reflexive, transitive closure, respectively. The space below the arrows is reserved for labels: $\xrightarrow[\alpha]{}$ denotes a rewrite step labeled with α. Given a set of labels $\mathcal{L}$ with well-founded precedence $\succ$, and a family $(\xrightarrow[\alpha]{})_{\alpha\in\mathcal{L}}$, we let $\xrightarrow[\curlyvee\alpha]{} = \bigcup_{\alpha\succ\beta}(\xrightarrow[\beta]{})$ and $\xrightarrow[\curlyvee\alpha\beta]{} = \xrightarrow[\curlyvee\alpha]{} \cup \xrightarrow[\curlyvee\beta]{}$. A local peak $s \xleftarrow[\alpha]{} \cdot \xrightarrow[\beta]{} t$ is said to be joined decreasingly if $s \xleftrightarrow[\curlyvee\alpha]{*} \cdot \xrightarrow[\beta]{=} \cdot \xleftrightarrow[\curlyvee\alpha\beta]{*} \cdot \xleftarrow[\alpha]{=} \cdot \xleftrightarrow[\curlyvee\beta]{*} t$. If all local peaks can be joined decreasingly, then $\to = \bigcup_{\alpha\in\mathcal{L}}(\xrightarrow[\alpha]{})$ is confluent [7].

We denote parallel rewrite steps by $\mathrel{+\!\!\!\!\!\!\to}$. If we wish to indicate the set of positions involved in a parallel rewrite step, we write $\overset{P}{\mathrel{+\!\!\!\!\!\!\to}}$. For a set of positions P and term t we let $t|_P = \{t|_p \mid p \in P\}$. Parallel critical pairs [2] arise similarly to critical pairs: whereas we obtain a critical pair $l[r']_p\sigma \leftarrow\!\rtimes\!\rightarrow r\sigma$ whenever $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(l)$ and σ is a most general unifier of $l|_p$ and l', where $l \to r$ and $l' \to r'$ are variants of rules in $\mathcal{R}$ with no common variables, a parallel critical pair is conceived as $l[r^p]_{p\in P}\sigma \mathrel{\leftarrow\!\!\!\!\!\!+}\!\rtimes\!\rightarrow r\sigma$ if $P \subseteq \mathcal{P}\mathsf{os}_\mathcal{F}(l)$ is a set of mutually parallel positions, $l \to r$,

*This research was supported by the Austrian Science Fund (FWF) project P22467-N23.

$l^p \to r^p$ are variants of rules in $\mathcal{R}$ with no common variables, and σ is a most general solution to the set of equations $l|_p\sigma = l^p\sigma$ for $p \in P$. If $\mathcal{R}$ is finite, then the set of its parallel critical pairs is finite as well.

Proposition 1. *Let $\mathcal{R}$ be a left-linear TRS and $t \overset{P}{\leftarrow\!\!\!\shortmid\!\shortmid} s \xrightarrow{\epsilon} u$. Then there are substitutions $\sigma \twoheadrightarrow \sigma'$ and a parallel critical pair $t' \leftarrow\!\!\shortmid\!\shortmid\!\rtimes\!\!\to u'$ such that $t = t'\sigma' \xleftarrow{P\setminus P'} t'\sigma \overset{P'}{\leftarrow\!\!\!\shortmid\!\shortmid} s \to u'\sigma = u$, where $P' \subseteq P$. (Note that left-linearity is essential for the substitutions σ and σ' to exist.)*

3 Confluence by Rule Labeling

Throughout this section we assume a given left-linear TRS $\mathcal{R}$, and a set of labels $\mathcal{L}$ equipped with a well-founded order $\succ$. Furthermore, let $\hat{\ell} : \mathcal{R} \to \mathcal{L}$ map rewrite rules to labels. The rule labeling heuristic labels rewrite steps according to the used rule, $\ell(t \to_{l\to r} t') = \hat{\ell}(l \to r)$.

Definition 2. Consider a parallel step $t \overset{P}{\twoheadrightarrow} t'$. For each $p \in P$, we have a rewrite step $t \to t[t'|_p]_p$ at position p. The set $\Gamma \subseteq \mathcal{L}$ is a valid label for $t \twoheadrightarrow t'$ if $\ell^{\|}(t \twoheadrightarrow t') \subseteq \Gamma$, where

$$\ell^{\|}(t \overset{P}{\twoheadrightarrow} t') = \{\ell(t \to t[t'|_p]_p) \mid p \in P\}$$

This means that a parallel rewrite step is labeled—at least—by the set of the labels of the rules used in the step. We indicate labels along with the step, writing $t \underset{\Gamma}{\twoheadrightarrow} t'$. A step $t \underset{\Gamma}{\twoheadrightarrow} t'$ is *homogeneous* if all its labels are the same, i.e., $\#\Gamma \leqslant 1$, and *heterogeneous* otherwise.

Parallel labels are ordered by the (multi-)set extension of $\succ$, which we also denote by $\succ$. The interest in homogeneous and heterogeneous steps stems from the following proposition:

Proposition 3. *If $s \overset{P}{\underset{\Gamma}{\twoheadrightarrow}} t$ is a heterogeneous step, i.e., $\#\Gamma > 1$, then we can split it into a sequence of individual steps whose labels are smaller than Γ w.r.t. $\succ$.*

We are now ready to state and prove the main theorem of this section.

Theorem 4. *A left-linear TRS $\mathcal{R}$ is confluent if all its parallel critical peaks $t \overset{P}{\underset{\Gamma}{\twoheadleftarrow}} s \underset{\Delta}{\longrightarrow} u$ with homogeneous parallel step, i.e., $\#\Gamma = \#\Delta = 1$ can be joined decreasingly as*

$$t \xrightarrow[\curlyvee\Gamma]{*} \cdot \underset{\Delta}{\twoheadrightarrow} \cdot \xrightarrow[\curlyvee\Gamma\Delta]{*} \cdot \xleftarrow[\curlyvee\Gamma\Delta]{*} v \overset{Q}{\underset{\Gamma}{\twoheadleftarrow}} \cdot \xleftarrow[\curlyvee\Delta]{*} u$$

such that $\mathcal{V}\mathsf{ar}(v|_Q) \subseteq \mathcal{V}\mathsf{ar}(s|_P)$.

Proof. We show that $\twoheadrightarrow$ is locally decreasing, which implies confluence of $\mathcal{R}$. Let $t \overset{P}{\underset{\Gamma}{\twoheadleftarrow}} s \overset{Q}{\underset{\Delta}{\twoheadrightarrow}} u$, where $\Gamma, \Delta \subseteq \mathcal{L}$. If $\#\Gamma > 1$ then we can join t and u by replacing $s \twoheadrightarrow t$ by a sequence of smaller steps using Proposition 3. We obtain a conversion $s \xleftarrow[\curlyvee\Gamma]{*} \cdot \underset{\Delta}{\twoheadrightarrow} u$, resulting in a locally decreasing diagram. The case $\#\Delta > 1$ is handled symmetrically. If $\Gamma = \varnothing$ or $\Delta = \varnothing$, then t and u can be joined by the other rewrite step, also resulting in a decreasing diagram.

In the remaining case, $\#\Gamma = 1$ and $\#\Delta = 1$, that is, we have a peak between homogeneous parallel steps steps. Let $\alpha \in \Gamma$ and $\beta \in \Delta$. Then $t \overset{P}{\underset{\Gamma}{\twoheadleftarrow}} s \overset{Q}{\underset{\Delta}{\twoheadrightarrow}} u$. It suffices to show that

$$t \xrightarrow[\curlyvee\Gamma]{*} \cdot \underset{\Delta}{\twoheadrightarrow} \cdot \xrightarrow[\curlyvee\Gamma\Delta]{*} \cdot \xleftarrow[\curlyvee\Gamma\Delta]{*} \cdot \underset{\Gamma}{\twoheadleftarrow} \cdot \xleftarrow[\curlyvee\Delta]{*} u \qquad (1)$$

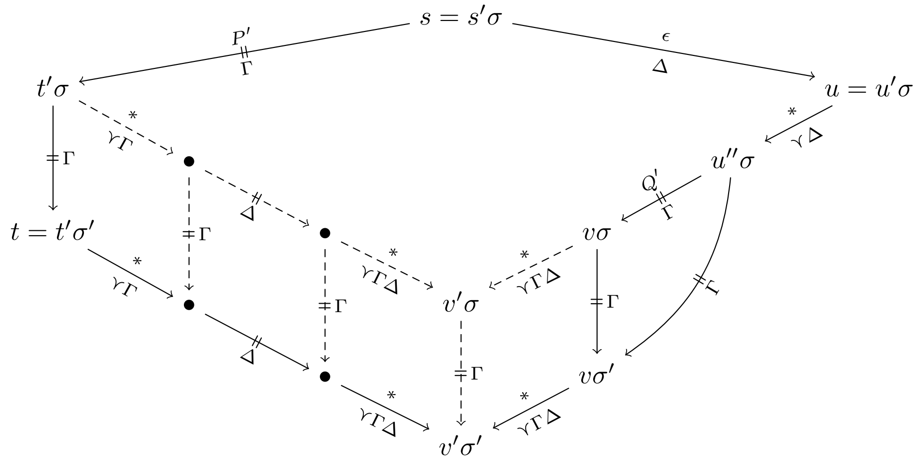

Figure 1: Proof of Theorem 4

We claim that (1) holds whenever $P = \{\epsilon\}$ or $Q = \{\epsilon\}$. Then for all $p \in \min(P \cup Q)$, $t \mathrel{\overset{P}{\underset{\Gamma}{\leftarrow\!\!\#}}} s \mathrel{\overset{Q}{\underset{\Delta}{\#\!\!\rightarrow}}} u$ induces a peak $t|_p \mathrel{\overset{P'}{\underset{\Gamma}{\leftarrow\!\!\#}}} s|_p \mathrel{\overset{Q'}{\underset{\Delta}{\#\!\!\rightarrow}}} u|_p$, where $P' = \{\epsilon\}$ or $Q' = \{\epsilon\}$. So for each p, we obtain a joining sequence for $t|_p$ and $u|_p$ of the shape (1). Since the positions in $\min(P \cup Q)$ are mutually parallel, these sequences for each $p \in \min(P \cup Q)$ can be combined into a single sequence of the same shape without any difficulties. In particular, the $\mathrel{\underset{\Delta}{\#\!\!\rightarrow}}$ steps combine into a single $\mathrel{\underset{\Delta}{\#\!\!\rightarrow}}$ step and likewise, the $\mathrel{\underset{\Gamma}{\leftarrow\!\!\#}}$ steps can be combined into a single $\mathrel{\underset{\Gamma}{\leftarrow\!\!\#}}$ step.

In order to show (1) for $P = \{\epsilon\}$ or $Q = \{\epsilon\}$, assume w.l.o.g. that $Q = \{\epsilon\}$. By Proposition 1, there are a parallel critical peak $t' \mathrel{\overset{P'}{\leftarrow\!\!\#}} s' \rightarrow u'$ and substitutions σ, σ' such that $\sigma \mathrel{\underset{\Gamma}{\#\!\!\rightarrow}} \sigma'$ and $t = t'\sigma' \mathrel{\overset{P \setminus P'}{\underset{\Gamma}{\longleftarrow\!\!\#}}} t'\sigma \mathrel{\overset{P'}{\underset{\Gamma}{\leftarrow\!\!\#}}} s'\sigma = s \xrightarrow[\Delta]{\epsilon} u'\sigma = u$, where $P' \subseteq P$. By assumption we can join t' and u' decreasingly, and consequently there are v, u'' and v' such that

$$t' \xrightarrow[\curlyvee\Gamma]{*} \cdot \mathrel{\underset{\Delta}{\#\!\!\rightarrow}} \cdot \xrightarrow[\curlyvee\Gamma\Delta]{*} v' \xleftarrow[\curlyvee\Gamma\Delta]{*} v \mathrel{\overset{Q'}{\underset{\Gamma}{\leftarrow\!\!\#}}} u'' \xleftarrow[\curlyvee\Delta]{*} u'$$

with $\mathcal{V}\mathsf{ar}(v|_{Q'}) \subseteq \mathcal{V}\mathsf{ar}(s|_{P'})$. Consequently,

$$t = t'\sigma' \xrightarrow[\curlyvee\Gamma]{*} \cdot \mathrel{\underset{\Delta}{\#\!\!\rightarrow}} \cdot \xrightarrow[\curlyvee\Gamma\Delta]{*} v'\sigma' \xleftarrow[\curlyvee\Gamma\Delta]{*} v\sigma' \mathrel{\underset{\Gamma}{\leftarrow\!\!\#}} v\sigma \mathrel{\overset{Q'}{\underset{\Gamma}{\leftarrow\!\!\#}}} u'' \xleftarrow[\curlyvee\Delta]{*} u'\sigma = u$$

Since $\sigma(x) = \sigma'(x)$ for $x \in \mathcal{V}\mathsf{ar}(s|_{P'})$ (otherwise $s \mathrel{\underset{\Gamma}{\#\!\!\rightarrow}} t$ would not be a parallel step), and because $\mathcal{V}\mathsf{ar}(v|_{Q'}) \subseteq \mathcal{V}\mathsf{ar}(s|_{P'})$, the two parallel steps in the leftward sequence can be combined into a single one. Thus we can join t and u decreasingly with common reduct $v'\sigma'$, completing the proof. See also Figure 1. ☐

To conclude the section we demonstrate Theorem 4 on two examples.

Example 1. Consider the TRS $\mathcal{R}$ consisting of the following five rules with labels $2 \succ 1 \succ 0$:

$$\mathsf{a} \xrightarrow[1]{} \mathsf{b} \qquad \mathsf{b} \xrightarrow[0]{} \mathsf{a} \qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \xrightarrow[1]{} \mathsf{c} \qquad \mathsf{f}(\mathsf{b},\mathsf{b}) \xrightarrow[2]{} \mathsf{c} \qquad \mathsf{h}(x) \xrightarrow[0]{} \mathsf{h}(\mathsf{f}(x,x))$$

There are six parallel critical pairs that can all be joined decreasingly as required by Theorem 4:

$$\begin{array}{ll}
\mathsf{f}(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \underset{\{1\}}{\leftarrow\!\!\!\!\!\shortparallel\;} \mathsf{f}(\mathsf{a},\mathsf{a}) \xrightarrow[\{1\}]{} \mathsf{c}: & \mathsf{f}(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \underset{\{0\}}{\;\shortparallel\!\!\!\!\!\rightarrow} \mathsf{f}(\mathsf{a},\mathsf{a}) \xrightarrow[\{1\}]{} \mathsf{c} \\
\mathsf{f}(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \underset{\{0\}}{\leftarrow\!\!\!\!\!\shortparallel\;} \mathsf{f}(\mathsf{b},\mathsf{b}) \xrightarrow[\{2\}]{} \mathsf{c}: & \mathsf{f}(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \underset{\{0\}}{\;\shortparallel\!\!\!\!\!\rightarrow} \mathsf{f}(\mathsf{a},\mathsf{a}) \xrightarrow[\{1\}]{} \mathsf{c}
\end{array}$$

Therefore, $\mathcal{R}$ is confluent.

Example 2. Let $\mathcal{O}$ be the TRS consisting of the rules (confluence of $\mathcal{O}$ was shown in [4])

$$\begin{array}{llll}
x - 0 \xrightarrow[0]{} x & 0 - x \xrightarrow[0]{} 0 & \mathsf{s}(x) - \mathsf{s}(y) \xrightarrow[0]{} x - y & \mathsf{if}(\mathsf{true}, x, y) \xrightarrow[0]{} x \\
0 < \mathsf{s}(x) \xrightarrow[0]{} \mathsf{true} & x < 0 \xrightarrow[0]{} \mathsf{false} & \mathsf{s}(x) < \mathsf{s}(y) \xrightarrow[0]{} x < y & \mathsf{if}(\mathsf{false}, x, y) \xrightarrow[0]{} y \\
\mathsf{mod}(x, 0) \xrightarrow[0]{} x & \mathsf{mod}(0, x) \xrightarrow[0]{} 0 & \multicolumn{2}{l}{\mathsf{mod}(x, \mathsf{s}(y)) \xrightarrow[1]{} \mathsf{if}(x < \mathsf{s}(y), x, \mathsf{mod}(x - \mathsf{s}(y), \mathsf{s}(y)))} \\
\mathsf{gcd}(x, 0) \xrightarrow[0]{} x & \mathsf{gcd}(0, x) \xrightarrow[0]{} x & \multicolumn{2}{l}{\mathsf{gcd}(x, y) \xrightarrow[1]{} \mathsf{gcd}(y, \mathsf{mod}(x, y))}
\end{array}$$

There are 12 critical pairs, of which 6 are trivial, and the remaining 6 come in 3 symmetrical pairs. They can all be joined decreasingly, using the precedence $1 \succ 0$:

$$\begin{array}{rl}
\mathsf{if}(0 < \mathsf{s}(\mathsf{y}), 0, \mathsf{mod}(0 - \mathsf{s}(y), \mathsf{s}(y))) \underset{\{1\}}{\leftarrow\!\!\!\!\!\shortparallel\;} \mathsf{mod}(0, \mathsf{s}(y)) \xrightarrow[\{0\}]{} 0: & \ldots \xrightarrow[\{0\}]{} \mathsf{if}(\mathsf{true}, 0, \ldots) \xrightarrow[\{0\}]{} 0 \\
\mathsf{gcd}(0, \mathsf{mod}(x, 0)) \underset{\{1\}}{\leftarrow\!\!\!\!\!\shortparallel\;} \mathsf{gcd}(x, 0) \xrightarrow[\{0\}]{} x: & \ldots \xrightarrow[\{0\}]{} \mathsf{gcd}(x, 0) \xrightarrow[\{0\}]{} x \\
\mathsf{gcd}(x, \mathsf{mod}(0, x)) \underset{\{1\}}{\leftarrow\!\!\!\!\!\shortparallel\;} \mathsf{gcd}(0, x) \xrightarrow[\{0\}]{} x: & \ldots \xrightarrow[\{0\}]{} \mathsf{gcd}(0, x) \xrightarrow[\{0\}]{} x
\end{array}$$

There are no inner critical pairs, so this accounts for all parallel critical pairs, and we can conclude that $\mathcal{O}$ is confluent by Theorem 4.

4 Generalized Labelings

We implemented rule-labeling for left-linear TRSs in the confluence tool CSI [8]. However, the implementation does not use Theorem 4. Instead, we use the framework of labelings from [9], in particular weak ll-labelings, which we recall below. Let $\mathcal{L}$ be a set of labels equipped with a well-founded order $\succ$, and a quasi-order $\succeq$ on $\mathcal{L}$ that is compatible with $\succ$, i.e., $\succeq \cdot \succ \cdot \succeq \subseteq \succ$.[1]

Definition 5. A *labeling* is a function ℓ mapping rewrite steps to labels such that comparing labels is closed under contexts and substitutions, that is, for all contexts $C[\cdot]$, substitutions σ and rewrite steps $s \to t$, $s' \to t'$, we have

$$\begin{array}{lll}
\ell(C[s\sigma] \to C[t\sigma]) \succ \ell(C[s'\sigma] \to C[t'\sigma]) & \text{if} & \ell(s \to t) \succ \ell(s' \to t') \\
\ell(C[s\sigma] \to C[t\sigma]) \succeq \ell(C[s'\sigma] \to C[t'\sigma]) & \text{if} & \ell(s \to t) \succeq \ell(s' \to t')
\end{array}$$

A labeling is a *weak ll-labeling* if for all $s \xrightarrow{p} t$ and $s \xrightarrow{p'} t'$ with $p \parallel p'$ (parallel overlap), we have

$$\ell(s \to t) \succeq \ell(t \to t'[t|_p]_p) \tag{2}$$

and whenever $s \xrightarrow{\epsilon}_{l \to r} t$, $s \xrightarrow{pq} t'$ with $p \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(l)$ (variable overlap), if $l|_p = r|_{p'}$, we have[2]

$$\ell(s \to t') \succeq \ell(t \to t[t'|_p]_{p'}) \tag{3}$$

[1] Another common definition is $\succ \cdot \succeq \subseteq \succ$, in which case $\succ' = \succeq \cdot \succ$ satisfies $\succeq \cdot \succ' \cdot \succeq \subseteq \succ'$.

[2] Condition (3) is stronger than the left-linear variable overlap condition in [9], where the parallel step of the joining valley is a rewrite sequence, but the weak ll-labelings presented there satisfy our new condition as well.

Given a weak ll-labeling ℓ, we can define $\ell^{\parallel}$ as before in Definition 2. However, it appears that we cannot use the multiset extension of $(\succ, \succeq)$ for comparing labels. Instead, we use the Hoare order on sets,[3]

$$\Gamma \succ \Delta \iff \Gamma \neq \varnothing \wedge \forall \beta \in \Delta\, \exists \alpha \in \Gamma\, (\alpha \succ \beta) \quad \text{and} \quad \Gamma \succeq \Delta \iff \forall \beta \in \Delta\, \exists \alpha \in \Gamma\, (\alpha \succeq \beta)$$

The implementation is based on the following theorem.

Theorem 6. *A left-linear TRS $\mathcal{R}$ is confluent if all its parallel critical peaks $t \overset{P}{\underset{\Gamma}{\leftarrow\!\!\!\!+\!\!\!+}} s \underset{\Delta}{\longrightarrow} u$ can be joined decreasingly (with respect to some ll-labeling) as $t \xrightarrow[\curlyvee\Gamma]{*} \cdot \underset{\Delta'}{-\!\!\!+\!\!\!+\!\!\!\to} \cdot \xrightarrow[\curlyvee\Gamma\Delta]{*} \cdot \xleftarrow[\curlyvee\Gamma\Delta]{*} v \overset{Q}{\underset{\Gamma'}{\leftarrow\!\!\!\!+\!\!\!+}} \cdot \xleftarrow[\curlyvee\Delta]{*} u$ such that $\mathcal{V}\mathsf{ar}(v|_Q) \subseteq \mathcal{V}\mathsf{ar}(s|_P)$ and $\Delta \succeq \Delta'$, $\Gamma \succeq \Gamma'$.*

The proof of Theorem 6 is similar to that of Theorem 4. In order to find suitable labelings, we use the techniques of [9] on joining sequences for critical pairs, and finally check whether the resulting labeling allows joining parallel critical pairs decreasingly as well.

5 Conclusion

We have demonstrated a rule labeling technique based on parallel reductions and parallel critical pairs that works for general left-linear term rewrite systems. A variant of the approach is implemented in CSI, which found the proofs for Examples 1,2. As future work we plan to flesh out the generalization from Section 4, explore the different orders on sets. There is also room for better heuristics for finding suitable ll-labelings. We would also like to know whether the variable condition $\mathcal{V}\mathsf{ar}(v|_Q) \subseteq \mathcal{V}\mathsf{ar}(s|_P)$ in Theorems 4, 6 can be relaxed.

It would also be interesting to devise a conversion version of Theorem 6. This should be straightforward in the pure rule labeling setting (Theorem 4), but it is unclear whether the generalization to weak ll-labelings would work out.

References

[1] F. Baader and T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1998.

[2] B. Gramlich. Confluence without termination via parallel critical pairs. In *Proc. 21st CAAP*, volume 1059 of *LNCS*, pages 211–225, 1996.

[3] N. Hirokawa and A. Middeldorp. Decreasing diagrams and relative termination. In *Proc. 5th IJCAR*, volume 6173 of *LNCS (LNAI)*, pages 487–501, 2010.

[4] M. Oyamaguchi and Y. Ohta. On the Church-Rosser property of left-linear term rewriting systems. *IEICE TIS*, E86-D(1):131–135, 2003.

[5] Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, 2003.

[6] V. van Oostrom. Confluence by decreasing diagrams. *TCS*, 126(2):259–280, 1994.

[7] V. van Oostrom. Confluence by decreasing diagrams – converted. In *Proc. 19th RTA*, volume 5117 of *LNCS*, pages 306–320, 2008.

[8] H. Zankl, B. Felgenhauer, and A. Middeldorp. CSI – A confluence tool. In *Proc. 23rd CADE*, volume 6803 of *LNCS (LNAI)*, pages 499–505, 2011.

[9] H. Zankl, B. Felgenhauer, and A. Middeldorp. Labelings for decreasing diagrams. In *Proc. 22nd RTA*, volume 10 of *LIPIcs*, pages 377–392, 2011.

[3] This order was introduced for power domains in domain theory.