Labeling Multi-Steps for Confluence of Left-Linear Term Rewrite Systems*

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Abstract

We show how to use the commutation version of van Oostrom's decreasing diagrams for labeling left-linear term rewriting systems, based on Zankl et al.'s labeling framework. The resulting confluence criterion requires joining simultaneous critical pairs decreasingly, subsuming the criterion by Okui.

1 Introduction

This note is concerned with the confluence of term rewrite systems. Okui introduced simultaneous critical pairs in [3], which are critical pairs between a multi-step (to the left) and a single rewrite step (to the right). He showed that any left-linear term rewrite system (TRS) is confluent if all its simultaneous critical pairs are joinable using a rewrite sequence from the left and a single multi-step from the right. In this note we show how to combine this idea with van Oostrom's decreasing diagrams [4] by labeling the rewrite steps in a suitable way. We base our work on [6], where such ideas have already been used for single rewrite steps and critical pairs as well as parallel rewrite steps and parallel critical pairs.

This work is motivated by the fact that the parallel critical pair criterion of [6] comes with an awkward restriction on the variables involved in the parallel step in the joining peak, and furthermore by the hope that the criterion will become applicable to higher order rewrite systems.

2 Preliminaries

We assume familiarity with term rewriting. For an introduction see [1].

Redex patterns. Let $\mathcal{R}$ be a left-linear TRS. A *redex pattern* is a pair $\pi = \langle p_\pi, l_\pi \to r_\pi \rangle$ consisting of a position p_π and a rewrite rule $l_\pi \to r_\pi \in \mathcal{R}$. A redex pattern π *matches* a term t if $t|_{p_\pi}$ is an instance of l_π. If π matches t, then π and t uniquely determine a term t^π such that $t \to_{p_\pi, l_\pi \to r_\pi} t^\pi$. We denote this rewrite step as $t \to^\pi t^\pi$. For a position q, $q\pi$ denotes the redex pattern $\langle qp_\pi, l_\pi \to r_\pi \rangle$. Let π_1 and π_2 be redex patterns that match a common term. They are called *parallel* ($\pi_1 \parallel \pi_2$) if $p_{\pi_1} \parallel p_{\pi_2}$. If $p_{\pi_2} \leqslant p_{\pi_1}$ and $p_{\pi_1} \backslash p_{\pi_2} \in \mathcal{P}os_{\mathcal{F}}(l_{\pi_2})$ or $p_{\pi_1} \leqslant p_{\pi_2}$ and $p_{\pi_1} \backslash p_{\pi_1} \in \mathcal{P}os_{\mathcal{F}}(l_{\pi_1})$ then π_1 and π_2 *overlap*, otherwise they are *orthogonal* ($\pi_1 \perp \pi_2$). Note that $\pi_1 \parallel \pi_2$ implies $\pi_1 \perp \pi_2$. We write $P \perp Q$ if $\pi \perp \pi'$ for all $\pi \in P$ and $\pi' \in Q$ and similarly $P \parallel Q$ if $\pi \parallel \pi'$ for all $\pi \in P$ and $\pi' \in Q$. We say that a set of redex patterns is *compatible* if P is a set of pairwise orthogonal redex patterns and there is a term t such that all $\pi \in P$ match t. Given a compatible set of redex patterns P matching a term t there is a multi-step $t \multimap^P t^P$.

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Residuals. We refer to [5] for a formal treatment of residuals. Recall the parallel moves lemma: If $P \cup Q$ is a compatible set of redex patterns and we have co-initial multi-steps $t \mathrel{\circ\!\!\!\rightarrow}^P t^P$, $t \mathrel{\circ\!\!\!\rightarrow}^Q t^Q$, then there are multi-steps $t^P \mathrel{\circ\!\!\!\rightarrow}^{Q/P} t^{P \cup Q}$ and $t^Q \mathrel{\circ\!\!\!\rightarrow}^{P/Q} t^{P \cup Q}$, where P/Q is the set of residuals of P after Q. Another important property of residuals for left-linear term rewrite systems is that residuals of orthogonal redex patterns remain orthogonal: If $P \cup Q \cup R$ is a compatible set of redex patterns and Q and R are disjoint (which implies that $Q \perp R$), then $Q/P \perp R/P$.

Simultaneous Critical Pairs. Let $\mathcal{R}$ be a left-linear TRS, π be a redex pattern and P be a non-empty set of pairwise orthogonal redex patterns that overlap with π. By choosing variants of rules in $\mathcal{R}$, we can ensure that l_π and $l_{\pi'}$ ($\pi' \in P$) have no variables in common. Furthermore assume that $\epsilon = p_\pi$ or $\epsilon \in \{p_{\pi'} \mid \pi' \in P\}$. Let π_ϵ be one of these redex patterns at the root position. We set up a unification problem as follows. Let $p \mathbin{\dot{-}} q = p \setminus q$ if $p \geqslant q$ and $p \mathbin{\dot{-}} q = \epsilon$ if $p < q$. For each $\pi' \in P$ we consider the equation $l_\pi|_{p_{\pi'} \dot{-} p_\pi} \sigma =^? l_{\pi'}|_{p_\pi \dot{-} p_{\pi'}} \sigma$. If the unification problem consisting of all these equations has an mgu σ, then there is a peak $t \;{}^P\!\!\mathrel{\leftarrow\!\!\!\circ} l_{\pi_\epsilon}\sigma \rightarrow^{\pi \dot{-} \pi_\epsilon} u$, and we call $t \mathrel{\leftarrow\!\!\!\circ} \rtimes \rightarrow u$ a *simultaneous critical pair*.

Lemma 1. *If $t \;{}^P\!\!\mathrel{\leftarrow\!\!\!\circ} s \rightarrow^\pi u$ then $P \perp \pi$ or there are a simultaneous critical pair $t' \;{}^Q\!\!\mathrel{\leftarrow\!\!\!\circ} \rtimes \rightarrow u'$, a context C with hole at position p and a substitution σ such that $t \;{}^{P/pQ}\!\!\mathrel{\leftarrow\!\!\!\circ} C[t'\sigma] \;{}^{pQ}\!\!\mathrel{\leftarrow\!\!\!\circ} s = C[s'\sigma] \rightarrow C[u'\sigma] = u$, where $pQ = \{p\pi \mid \pi \in Q\}$.*

Finally we recall van Oostrom's decreasing diagrams [4]. We will use a commutation version of extended decreasingness (cf. [2]). Let L be a set of labels equipped with a well-founded order $>$ and a compatible quasi-order $\geqslant$ (i.e., $\geqslant \cdot > \subseteq >$.)

Theorem 2. *Let $(\rightharpoonup_\alpha)_{\alpha \in L}$ and $(\rightharpoondown_\alpha)_{\alpha \in L}$ be labeled ARSs. Then $\rightharpoonup$ and $\rightharpoondown$ commute if for all $\alpha, \beta \in L$,*

$$\underset{\alpha}{\leftharpoonup} \cdot \underset{\beta}{\rightharpoondown} \;\subseteq\; \overset{*}{\underset{\vee\alpha}{\leftrightharpoons}} \cdot \overset{=}{\underset{\vee\!\!/\beta}{\rightharpoondown}} \cdot \overset{*}{\underset{\vee\alpha\beta}{\leftrightharpoons}} \cdot \overset{=}{\underset{\vee\!\!/\alpha}{\leftharpoonup}} \cdot \overset{*}{\underset{\vee\beta}{\leftrightharpoons}}$$

Sets of labels are ordered by the Hoare preorder of $(\geqslant, >)$, which we denote by $(\geqslant_H, >_H)$ and is defined by

$$\begin{aligned} \Gamma >_H \Delta &\iff \Gamma \neq \varnothing \wedge \forall \beta \in \Delta.\, \exists\, \alpha \in \Gamma.\, \alpha > \beta \\ \Gamma \geqslant_H \Delta &\iff \forall \beta \in \Delta.\, \exists\, \alpha \in \Gamma.\, \alpha \geqslant \beta \end{aligned}$$

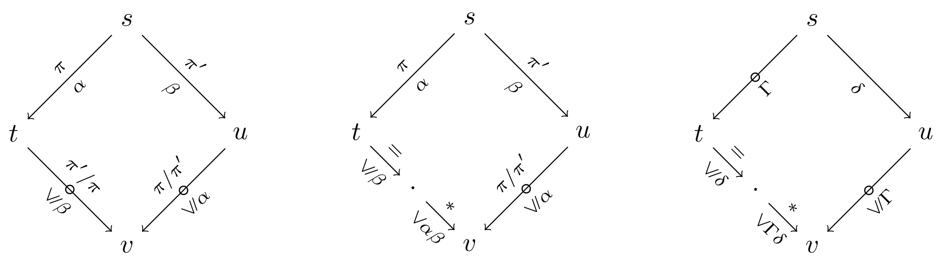
If $(\geqslant, >)$ is a pair of a well-founded order $>$ and a compatible quasi-order $\geqslant$, then so is $(\geqslant_H, >_H)$. For readability we drop the subscript H when attaching labels to rewrite steps as in $\mathrel{\circ\!\!\!\rightarrow}_{\vee\Gamma}$.

3 Labeling Development Steps

In this section we show that the weak LL-labelings of [6] can be fruitfully applied to development steps, leading to a criterion based on simultaneous critical pairs [3]. Our result subsumes Okui's main result [3]. The key idea for establishing confluence in [3] is to show that $\mathrel{\leftarrow\!\!\!\circ}$ and $\rightarrow$ commute. We do the same, using the commutation version of extended decreasing diagrams (Theorem 2).

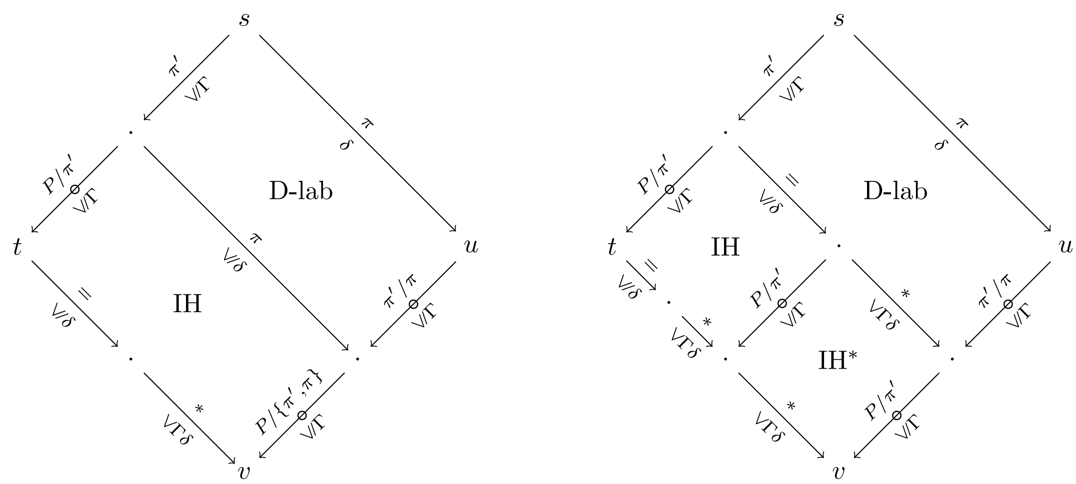
Definition 3. Let L be a set of labels equipped with a well-founded order $>$ and compatible quasi-order $\geqslant$. A function ℓ that maps rewrite steps $t \rightarrow^\pi t^\pi$ is a labeling function if for all contexts C with hole position p and substitutions σ,

1. $\ell(t \rightarrow^\pi t^\pi) > \ell(u \rightarrow^{\pi'} u^{\pi'})$ implies $\ell(C[t\sigma] \rightarrow^{p\pi} C[t^\pi\sigma]) > \ell(C[u\sigma] \rightarrow^{p\pi'} C[u^{\pi'}\sigma])$, and



(a) weak LL-labeling (b) D-labeling (c) Lemma 6

Figure 1: Labelings



(a) $p_{\pi'} \not< p_\pi$ (b) $p_{\pi'} < p_\pi$

Figure 2: Proof of Lemma 6

2. $\ell(t \to^{\pi} t^{\pi}) \geqslant \ell(u \to^{\pi'} u^{\pi'})$ implies $\ell(C[t\sigma] \to^{p\pi} C[t^{\pi}\sigma]) \geqslant \ell(C[u\sigma] \to^{p\pi'} C[u^{\pi'}\sigma])$.

We need to label development steps. We do so in essentially the same way as we labeled parallel steps in [6], i.e., we collect the labels of the constituent rewrite steps in a set.

Definition 4. Consider a development step $t \mathrel{\circ\!\!\!\!\to}^{P} t'$. For each $\pi \in P$, there is a rewrite step $t \to_{\pi} t^{\pi}$. We label $t \mathrel{\circ\!\!\!\!\to}^{P} t'$ by $\ell^{\circ}(t \mathrel{\circ\!\!\!\!\to}^{P} t')$, where

$$\ell^{\circ}(t \overset{P}{\mathrel{\circ\!\!\!\!\to}} t') = \{\ell(t \to^{\pi} t^{\pi}) \mid \pi \in P\}$$

This means that a development rewrite step is labeled by the set of the labels of the constituent steps. We indicate labels along with the step, writing $t \mathrel{\circ\!\!\!\!\to}_{\Gamma} t'$ if $\Gamma = \ell^{\circ}(t \mathrel{\circ\!\!\!\!\to} t')$.

Definition 5. A labeling ℓ is a *weak LL-labeling* if any orthogonal peak $t \ {}_{\alpha}\!\leftarrow s \to_{\beta} u$ is joined as in Figure 1(a), where we $\vee\!\!\!/\,\gamma$ stands for $\vee\!\!\!/\,\{\gamma\}$.

A pair of weak LL-labelings $\langle \ell', \ell \rangle$ is a *D-labeling* if any orthogonal peak $t \ {}_{\alpha}\!\leftarrow s \to_{\beta} u$ can be joined as in Figure 1(b), where leftward steps are labeled using ℓ' and rightward steps are labeled using ℓ, and the $t \to^{=}_{\vee\!\!\!/\,\beta} \cdot \to^{*}_{\vee\alpha\beta} v$ sequence is a complete development of $t \mathrel{\circ\!\!\!\!\to}_{\pi'/\pi} v$.

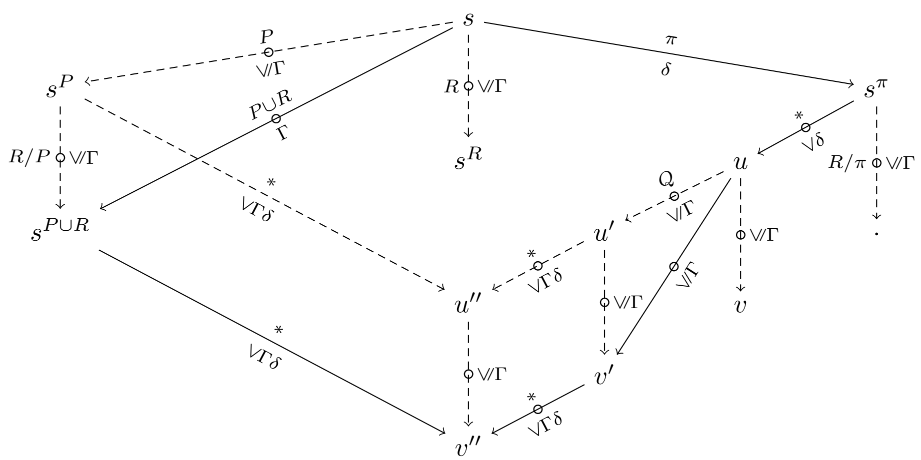


Figure 3: Proof of Theorem 7.

The D-labeling property ensures that orthogonal peaks can be joined decreasingly:

Lemma 6. *If $P \perp \pi$ and $t\ {}^{P}_{\Gamma}\!\!\leftarrow\!\!\!\!\circ\!\!-\ s \to_{\delta}^{\pi} u$ then $t \to_{\vee\!\!\!/\delta}^{=} \cdot \to_{\vee\Gamma\delta}^{*} \cdot {}_{\vee\!\!\!/\Gamma}\!\!\leftarrow\!\!\!\!\circ\!\!-\ u$. See also Figure 1(c).*

Proof. The proof is by induction on $\#P$. If P is empty, then there is nothing to prove; otherwise, let $\pi' \in P$ be an innermost redex pattern in P. The remainder of the proof is sketched in Figure 2. Note that if π/π' is not a singleton set then $p_{\pi'} < p_\pi$ and therefore by the choice of π', $P/\pi = P$. Therefore all applications of the induction hypothesis in the $\cdot\ {}_{\vee\!\!\!/\Gamma}\!\!\leftarrow\!\!\!\!\circ\!\!-\ \cdot \to_{\vee\!\!\!/\delta}^{=} \cdot$ and $\cdot\ {}_{\vee\!\!\!/\Gamma}\!\!\leftarrow\!\!\!\!\circ\!\!-\ \cdot \to_{\vee\Gamma\delta}^{*} \cdot$ peaks use the same set $P/\pi' = P \setminus \{\pi'\}$ (because π' is an innermost redex pattern of P) which satisifies $\#(P/\pi') < \#P$. □

Theorem 7. *Let $\langle \ell', \ell \rangle$ be a D-labeling for a left-linear TRS $\mathcal{R}$. Then $\mathcal{R}$ is confluent if every simultaneous critical pair $t\ {}_{\Gamma}\!\!\leftarrow\!\!\!\!\circ\!\!-\ s \to_{\delta} u$ can be joined decreasingly as*

$$t \xrightarrow[\vee\Gamma\delta]{*} \cdot \xleftarrow[\vee\Gamma\delta]{*\,\circ} \cdot \xleftarrow[\vee\!\!\!/\Gamma]{\circ} \cdot \xleftarrow[\vee\delta]{*\,\circ} u$$

Proof. Consider a peak $t\ {}^{P_0}_{\Gamma}\!\!\leftarrow\!\!\!\!\circ\!\!-\ s \to_{\delta}^{\pi} u$. We claim that this peak can be joined decreasingly. If $P_0 \perp \pi$ then the claim follows from Lemma 6. Otherwise, we can apply Lemma 1 to obtain a simultaneous critical pair $t'\ {}^{P_0'}\!\!\leftarrow\!\!\!\!\circ\!\!-\ s' \to^{\pi'} u'$, a context C with hole at position p, and a substitution σ such that with $P = pP_0'$ and $R = P_0 - P$, $P \perp R$, $\pi = p\pi'$, and

$$s^P = C[t'\sigma] \xleftarrow[\vee\!\!\!/\Gamma]{P\,\circ} s = C[s'\sigma] \xrightarrow[\delta]{\pi} s^\pi = C[u'\sigma] \tag{1}$$

By the assumption and the definition of weak LL-labelings, there are terms u, u', u'' such that

$$s^P \xrightarrow[\vee\Gamma\delta]{*} u'' \xleftarrow[\vee\Gamma\delta]{*} u' \xleftarrow[\vee\!\!\!/\Gamma]{Q\,\circ} u \xleftarrow[\vee\delta]{\pi_n} \cdots \xleftarrow[\vee\delta]{\pi_1} s^\pi$$

where $\pi_1, \ldots, \pi_n$ is a sequence of redex patterns. Furthermore the step $s \multimap_{\vee\!\!\!/\Gamma}^{R} s^R$ has residuals as shown in Figure 3: $s^P \multimap_{\vee\!\!\!/\Gamma}^{R/P} s^{P\cup R} = t$, $u \multimap_{\vee\!\!\!/\Gamma}^{R/(\pi;\overline{\pi_i})} v$, $u' \multimap_{\vee\!\!\!/\Gamma} v'$ and $u'' \multimap_{\vee\!\!\!/\Gamma} v''$. Since R is orthogonal to the whole instance of the simultaneous critical pair (1), its residual $u \multimap v$ is orthogonal to Q, and therefore we obtain a single development step $u \multimap_{\vee\!\!\!/\Gamma}^{Q\cup R/(\pi;\pi_1;\ldots;\pi_n)} v'$, completing the decreasing diagram. □

Corollary 8 (Okui's confluence criterion)**.** *If all simultaneous critical pairs of a left-linear TRS* $t \mathrel{\leftarrow\!\!\!\!\circ}\, s \to u$ *are joinable as* $t \to^* v \mathrel{\leftarrow\!\!\!\!\circ}\, u$ *then* $\mathcal{R}$ *is confluent.*

Proof. Using $L = \{\bot, \top\}$ with $\bot < \top$, and the D-labeling $\langle \ell', \ell \rangle$ defined by $\ell'(\cdot) = \top$ and $\ell(\cdot) = \bot$, we see that the requirements of Theorem 7 are fulfilled. Therefore, confluence of $\mathcal{R}$ follows. □

Example 9. The TRS consisiting of the following rules demonstrates that Theorem 7 strictly subsumes Okui's criterion.

$$1\colon \mathsf{g}(\mathsf{b}, x) \to \mathsf{f}(x, x) \qquad 2\colon \mathsf{c} \to \mathsf{a} \qquad 3\colon \mathsf{c} \to \mathsf{b} \qquad 4\colon \mathsf{a} \to \mathsf{b} \qquad 5\colon \mathsf{f}(\mathsf{a}, \mathsf{a}) \to \mathsf{g}(\mathsf{c}, \mathsf{c})$$

We let $\ell'(s \to^{\pi} s^{\pi})$ be the index of the used rule $l_\pi \to r_\pi$ and $\ell(\cdot) = 0$; this results in a D-labeling with the standard order on natural numbers. There are 5 simultaneous critical pairs, $\{\mathsf{f}(\mathsf{a},\mathsf{b}), \mathsf{f}(\mathsf{b},\mathsf{a}), \mathsf{f}(\mathsf{b},\mathsf{b})\} \mathrel{\leftarrow\!\!\!\!\circ}\rtimes\!\!\to \mathsf{g}(\mathsf{c},\mathsf{c})$ and $\mathsf{g}(\mathsf{c},\mathsf{c}) \mathrel{\leftarrow\!\!\!\!\circ}\rtimes\!\!\to \{\mathsf{f}(\mathsf{a},\mathsf{b}), \mathsf{f}(\mathsf{b},\mathsf{a})\}$. They are joinable with steps below $\ell'(\mathsf{a} \to \mathsf{b}) = 4$, because $\mathsf{g}(\mathsf{c},\mathsf{c}) \mathrel{\circ\!\!\!\!\to}_{\vee\!/3} \mathsf{g}(\mathsf{b},\mathsf{c}) \mathrel{\circ\!\!\!\!\to}_{\vee\!/3} \mathsf{f}(\mathsf{c},\mathsf{c}) \mathrel{\circ\!\!\!\!\to}_{\vee\!/3} \{\mathsf{f}(\mathsf{a},\mathsf{b}), \mathsf{f}(\mathsf{b},\mathsf{a}), \mathsf{f}(\mathsf{b},\mathsf{b})\}$ (and we can use the same rewrite sequence to the left, with all labels equal to 0). Note that a single development step does not suffice, so Okui's criterion fails.

4 Conclusion

We have derived a new application of decreasing diagrams to left-linear term rewrite systems, based on the commutation of single steps and development steps, and simultaneous critical pairs. Our criterion subsumes Okui's criterion. It should be noted that commutation is essential for obtaining a finite criterion: If one were to consider peaks composed of two development steps, one would end up with an infinite set of critical peaks in general. For example, the single rule TRS $\{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x)\}$ has *critical* overlaps of arbitrary size, e.g., $\mathsf{f}^{n+1}(x) \mathrel{\leftarrow\!\!\!\!\circ}\, \mathsf{f}^{2n+1}(x) \mathrel{\circ\!\!\!\!\to} \mathsf{f}^{n+1}(x)$, where the left multi-step has redexes at positions of even length and the right multi-step has redexes at positions of odd length.

As future work, we plan to implement this criterion in CSI. We also hope to apply the criterion to higher-order systems, in particular pattern rewrite systems. In order to do so, we need to generalise simultaneous critical pairs to that setting.

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