Linear Termination is Undecidable

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ABSTRACT

By means of a simple reduction from Hilbert's 10th problem we prove the somewhat surprising result that termination of *one-rule* rewrite systems by a *linear* interpretation in the natural numbers is undecidable. The very same reduction also shows the undecidability of termination of one-rule rewrite systems using the Knuth–Bendix order with *subterm coefficients*. The linear termination problem remains undecidable for one-rule rewrite systems that can be shown terminating by a (non-linear) polynomial interpretation. We further show the undecidability of the problem whether a one-rule rewrite system can be shown terminating by a polynomial interpretation with *rational* or *real* coefficients. Several of our results have been formally verified in the Isabelle/HOL proof assistant.

CCS CONCEPTS

• **Theory of computation** → **Equational logic and rewriting**; **Computability**.

KEYWORDS

term rewriting, polynomial termination, undecidability

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1 INTRODUCTION

In this paper we consider the (uniform) termination problem for first-order term rewrite systems. Using a reduction from the halting problem for Turing machines, Huet and Lankford [18] were the first to show the undecidability of the termination problem. Dauchet [9] proved that the termination problem remains undecidable for *one-rule* term rewrite systems. For one-rule string rewrite systems (also known as semi-Thue systems) it is unknown whether termination is decidable. Matiyasevich and Sénizergues [28, 29] proved that the termination problem is undecidable for string rewrite systems having three rules.

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Decades of research have been devoted to develop powerful sufficient conditions that are amenable to automation. In this paper we are concerned with one of the earliest termination methods: using a polynomial interpretation over the natural numbers, which goes back to Lankford [23]. Two problems need to be addressed when using polynomial interpretations for proving termination, whether by hand or by a tool: (1) finding suitable polynomials for the function symbols, and (2) showing that the induced order constraints on polynomials are valid. Heuristics for the former problem are presented in [7, 41]. The latter problem amounts to proving $P(x_1, \ldots, x_n) > 0$ for all natural numbers $x_1, \ldots, x_n \in \mathbb{N}$, for polynomials $P \in \mathbb{Z}[x_1, \ldots, x_n]$. This is known to be undecidable, as an easy consequence of the undecidability of Hilbert's 10th problem, see e.g., Zantema [41, Proposition 6.2.11]. In a recent paper [31] the first two authors proved by a non-trivial reduction from Hilbert's 10th problem that (1) is undecidable. In this paper we prove the somewhat surprising result that (1) remains undecidable if we restrict the allowed interpretation functions to linear ones, even for one-rule rewrite systems that can be shown terminating by a polynomial interpretation. This contrasts with the fact that (2) is decidable for linear interpretations. We further show that the problem is undecidable when strict monotonicity of the interpretation functions is weakened to weak monotonicity. This is relevant for automated termination proving, e.g. when dependency pairs [1] are used.

Polynomial termination over the natural numbers is the strongest property in the following hierarchy of termination [41]

$$\mathrm{PT} \implies \omega\mathrm{T} \implies \mathrm{TT} \implies \mathrm{ST} \implies \mathrm{SN} \qquad (\star)$$

where the acronyms from left to right stand for the following properties: Polynomial termination over $\mathbb{N}$ (PT), ω-termination (ωT), total termination (TT), simple termination (ST) and termination (SN). All properties in the hierarchy are known to be undecidable [14, 18, 30, 31, 40]. Moreover these properties remain undecidable for term rewrite systems consisting of a single rewrite rule. For polynomial termination this is a new result proved in this paper. For the other properties this was known [15, 30]. In [15] it is further shown that for all implications $P_1 \implies P_2$ in the termination hierarchy with the exception of PT $\implies$ ωT, P_1 is undecidable for TRSs having the property P_2, even for one-rule TRSs. For PT $\implies$ ωT this has been conjectured. We present a proof of this conjecture.

The Knuth–Bendix order (KBO) [19] is a popular termination method. Rewrite systems that can be oriented by KBO are simply terminating and the orientation problem is known to be decidable in polynomial time (Korovin and Voronkov [21]). An important limitation of KBO is that duplicating rewrite rules like $\mathsf{f}(x) \to \mathsf{g}(x,x)$ cannot be oriented. Ludwig and Waldmann [26] generalized KBO with *subterm coefficients* to address this issue. Reusing the reduction for the undecidability of linear termination we show the undecidability of the orientation problem of one-rule rewrite systems with respect to this extension of KBO.

Polynomial interpretations over the reals and rationals have also been considered in the termination literature [10, 25, 32]. Using a different reduction from Hilbert's 10th problem, we prove the undecidability of the problem whether a one-rule rewrite system can be shown terminating by a polynomial interpretation with rational or real coefficients.

We also formally verify our new results in the proof assistant Isabelle/HOL [33]. To be more precise, each reduction proof within this paper is formally verified, and this is indicated by a small Isabelle symbol () throughout the text.

The remainder of this paper is organized as follows. After recalling some definitions and Hilbert's 10th problem in the next section, we prove the undecidability of linear termination for one-rule rewrite systems in Section 3. In Section 4 we show the undecidability of termination of one-rule rewrite systems using the Knuth–Bendix order with subterm coefficients. In Section 5 we prove the undecidability of polynomial termination for one-rule ω-terminating rewrite systems. Our results on polynomial termination with rational and real coefficients are presented in Section 6. Some details of the formalization are presented in Section 7. We conclude in Section 8.

2 PRELIMINARIES

We assume familiarity with term rewriting [2]. Before presenting a variation of Hilbert's 10th problem, we recall some basic termination methods.

Polynomial Termination over $\mathbb{N}$

Given a signature $\mathcal{F}$, a well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of a non-empty $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ and a well-founded order $>$ on the carrier A of $\mathcal{A}$ such that every algebra operation is strictly monotone in all its arguments, i.e., if $f \in \mathcal{F}$ has arity $n \geqslant 1$ then $f_{\mathcal{A}}(a_1,\dots,a_i,\dots,a_n) > f_{\mathcal{A}}(a_1,\dots,b,\dots,a_n)$ for all $a_1,\dots,a_n,b \in A$ and $i \in \{1,\dots,n\}$ with $a_i > b$. The induced order $>_{\mathcal{A}}$ on terms is a reduction order that ensures the termination of any compatible (i.e., $\ell >_{\mathcal{A}} r$ for all rewrite rules $\ell \to r$) term rewrite system (TRS for short) $\mathcal{R}$. We call $\mathcal{R}$ *polynomially terminating over* $\mathbb{N}$ if compatibility holds when the underlying algebra $\mathcal{A}$ is restricted to the set of natural numbers $\mathbb{N}$ with standard order $>_{\mathbb{N}}$ such that every n-ary function symbol f is interpreted as a monotone polynomial $f_{\mathbb{N}}$ in $\mathbb{Z}[x_1,\dots,x_n]$. We use $\mathbb{N}_+$ to denote $\mathbb{N} \setminus \{0\}$.

Whereas well-founded monotone algebras are complete for termination [39], polynomial termination gives rise to a much more restricted class of TRSs. For instance, Hofbauer and Lautemann [17] proved that polynomially terminating TRSs induce a double-exponential upper bound on the derivational complexity.

ω-Termination

The second class of TRSs in the termination hierarchy ($\star$) which is important in this paper is *ω-termination.* A TRS $\mathcal{R}$ is ω-terminating if its termination can be shown using a well-founded monotone algebra over $\mathbb{N}$ equipped with the standard order $>_{\mathbb{N}}$. In contrast to polynomial termination, the interpretation functions are not restricted and can be any monotone function over the naturals. It is therefore trivial to see that any polynomial terminating system over $\mathbb{N}$ is also ω-terminating.

Polynomial Termination over $\mathbb{Q}$ and $\mathbb{R}$

Next we define polynomial termination over $\mathbb{Q}$ and $\mathbb{R}$. Let $D \in \{\mathbb{Q}, \mathbb{R}\}$, $D_{\geqslant 0} = \{x \in D \mid x \geqslant 0\}$ and $D_{>0} = \{x \in D \mid x > 0\}$. Given a fixed $\delta > 0$ in D, the non-total order $>_\delta$ is defined as $x >_\delta y$ if $x - y \geqslant \delta$ for all $x, y \in D$ [16, 24]. A TRS $\mathcal{R}$ is *polynomially terminating over* D if it is compatible with a well-founded monotone algebra $\mathcal{D} = (D_{\geqslant 0}, \{f_{\mathcal{D}}\}_{f \in \mathcal{F}})$ that is equipped with $>_\delta$ and such that every $f_{\mathcal{D}}$ is a strictly monotone polynomial in $D[x_1,\dots,x_n]$. Strictly monotone here means that $a_i >_\delta b$ implies $f_{\mathcal{D}}(a_1,\dots,a_i,\dots,a_n) >_\delta f_{\mathcal{D}}(a_1,\dots,b,\dots,a_n)$ for all n-ary symbols $f \in \mathcal{F}$ and all $i \in \{1,\dots,n\}$. Polynomially terminating TRSs over $\mathbb{Q}$ are also polynomially terminating over $\mathbb{R}$, but polynomially terminating TRSs over $\mathbb{N}$ need not be polynomially terminating over $\mathbb{R}$ [32].

KBO with Subterm Coefficients

A *weight function* for a signature $\mathcal{F}$ is a pair (w, w_0) consisting of a mapping $w\colon \mathcal{F} \to \mathbb{N}$ and a constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for every constant $c \in \mathcal{F}$. Let $>$ be a precedence on $\mathcal{F}$. A weight function (w, w_0) is *admissible* for $>$ if $f > g$ for all function symbols g different from f, whenever f is a unary function symbol with $w(f) = 0$. A *subterm coefficient function* is a partial mapping $\mathrm{sc}\colon \mathcal{F} \times \mathbb{N} \to \mathbb{N}$ such that for each $f \in \mathcal{F}$ with arity n we have $\mathrm{sc}(f, i) > 0$ for all $i \in \{1,\dots,n\}$. Given a weight function (w, w_0) and a subterm coefficient function sc, the weight of a term is inductively defined as follows:

$$w_{\mathrm{sc}}(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \displaystyle\sum_{1 \leqslant i \leqslant n} \mathrm{sc}(f,i) \cdot w_{\mathrm{sc}}(t_i) & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

Here $\mathcal{V}$ denotes the set of variables that is used to construct terms. Given a subterm coefficient function sc, the *variable coefficient* of a variable x in the term t is defined as

$$\mathrm{vc}(x,t) = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{if } t \in \mathcal{V} \setminus \{x\} \\ \displaystyle\sum_{i=1}^{n} \mathrm{sc}(f,i) \cdot \mathrm{vc}(x,t_i) & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

For a precedence $>$, weight function (w, w_0) and subterm coefficient function sc we define $>_{\mathrm{kbo}}$ as follows: $s >_{\mathrm{kbo}} t$ if $\mathrm{vc}(x,s) \geqslant \mathrm{vc}(x,t)$ for all $x \in \mathcal{V}$, and either $w_{\mathrm{sc}}(s) > w_{\mathrm{sc}}(t)$ or both $w_{\mathrm{sc}}(s) = w_{\mathrm{sc}}(t)$ and one of the following conditions holds:

(1) $s = f^n(t)$ and $t \in \mathcal{V}$ for some unary f and $n > 0$, or
(2) $s = f(s_1,\dots,s_n)$ and $t = g(t_1,\dots,t_n)$ and $f > g$, or

(3) $s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$ and there exists an $i \in \{1, \ldots, n\}$ such that $s_i >_{\mathrm{kbo}} t_i$ and, for all $1 \leqslant j < i$, $s_j = t_j$.

In case (1), $f^1(t) = t$ and $f^{n+1}(t) = f(f^n(t))$ for $n > 0$. When the subterm coefficient function sc satisfies $\mathrm{sc}(f, i) = 1$ for every n-ary $f \in \mathcal{F}$ and $i \in \{1, \ldots, n\}$, the original Knuth–Bendix order [19] is obtained. The following result is due to Ludwig and Waldmann [26].

Theorem 2.1. *Every* TRS *that is compatible with* $>_{kbo}$ *for some well-founded precedence* $>$*, admissible weight function* (w, w_0) *and subterm coefficient function* sc *is terminating.* □

In [22, 26] ordinal weights are considered as well, resulting in the *transfinite* KBO. In [38] it is shown that ordinal weights do not add power when orienting finite TRSs.

Hilbert's 10th Problem

All undecidability results in this paper are obtained by reduction from a variant of Hilbert's 10th problem, which was shown to be undecidable by Matiyasevich [27]. Hilbert's 10th problem is a natural fit for our reduction proofs, since polynomial termination proofs (over $\mathbb{N}$) also use integer polynomials, allowing us to relatively directly encode one problem as the other. However, while Hilbert's 10th problem allows arbitrary integer polynomials as its instance, polynomial interpretations are limited to strictly monotone functions over $\mathbb{N}$. To simplify the encoding of Hilbert's 10th problem in the remainder of this paper, we first reduce it to a slightly modified decision problem. Instead of using an arbitrary integer polynomial, we consider two polynomials P and Q with only positive coefficients and ask if $P(x_1, \ldots, x_n) \geqslant Q(x_1, \ldots, x_n)$ for some arguments $x_1, \ldots, x_n \in \mathbb{N}_+$. This is also undecidable and is more easily applicable in the proofs related to polynomial termination.

Lemma 2.2 (🔘). *The following decision problem is undecidable:*
instance: polynomials P *and* Q *with positive integer coefficients*
question: $P(x_1, \ldots, x_n) \geqslant Q(x_1, \ldots, x_n)$
for some $x_1, \ldots, x_n \in \mathbb{N}_+$ *?*

Proof. We proceed by a reduction from Hilbert's 10th problem. Assume the decision problem is decidable and let $R \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial. We can modify Hilbert's 10th problem for R as follows:

$$\begin{aligned} & \exists\, x_1, \ldots, x_n \in \mathbb{Z} \quad R(x_1, \ldots, x_n) = 0 \\ \iff & \exists\, x_1, \ldots, x_n \in \mathbb{Z} \quad R(x_1, \ldots, x_n)^2 \leqslant 0 \\ \iff & \exists\, a_1, \ldots, a_n \in \{-1, 0, 1\}\ \exists\, x_1, \ldots, x_n \in \mathbb{N}_+ \\ & \quad R(a_1 x_1, \ldots, a_n x_n)^2 \leqslant 0 \end{aligned} \tag{1}$$

For each $\vec{a} = a_1, \ldots, a_n$ we can now split $R(a_1 x_1, \ldots, a_n x_n)^2$ into two polynomials $P_{\vec{a}}$ and $Q_{\vec{a}}$ containing only positive coefficients and $R(a_1 x_1, \ldots, a_n x_n)^2 = Q_{\vec{a}}(x_1, \ldots, x_n) - P_{\vec{a}}(x_1, \ldots, x_n)$. Hence (1) is equivalent to

$$\begin{aligned} & \exists\, a_1, \ldots, a_n \in \{-1, 0, 1\}\ \exists\, x_1, \ldots, x_n \in \mathbb{N}_+ \\ & P_{\vec{a}}(x_1, \ldots, x_n) \geqslant Q_{\vec{a}}(x_1, \ldots, x_n) \end{aligned}$$

The final problem is decidable by our assumption, since it consists of 3^n instances of the decision problem. This contradicts the undecidability of Hilbert's 10th problem, thereby proving the lemma. □

3 LINEAR POLYNOMIAL TERMINATION

Before showing undecidability of the general case, we limit termination proofs to polynomial interpretations using only linear interpretation functions. This not only has a simpler proof which we build upon later, but is also interesting for practical reasons. Many tools automating the search for polynomial termination proofs, use mostly linear interpretations. This is due to the fact that generating monotone functions is simple for linear interpretations, linearity is preserved by composition and checking if $[\ell]_{\mathbb{N}} > [r]_{\mathbb{N}}$ can be reduced to comparing coefficients. For example checking if $ax + by + c > dx + ey + f$ for all $x, y \in \mathbb{N}$ is equivalent to checking $a \geqslant d$, $b \geqslant e$ and $c > f$. This idea of only comparing coefficients of matching variables is central to the following results where we will be encoding the polynomials of Lemma 2.2 in the coefficients of the interpretations of terms. In general, given a term t with variables $x_1, \ldots, x_n$, $[t]_{\mathbb{N}}$ denotes the polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ that is the interpretation of t.

To prove undecidability of linear polynomial termination we define a TRS $\mathcal{R}$ which is parameterized by two polynomials P and Q containing only positive coefficients. We then prove that $\mathcal{R}$ can be shown terminating using a linear polynomial interpretation if and only if $P(x_1, \ldots, x_n) \geqslant Q(x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n \in \mathbb{N}_+$. For polynomials containing the indeterminates $v_1, \ldots, v_n$, the signature of $\mathcal{R}$ is $\mathcal{F} = \{\mathsf{z}, \mathsf{o}, \mathsf{a}, \mathsf{f}, \mathsf{v}_1, \ldots, \mathsf{v}_n\}$, where z and o are constants, $\mathsf{v}_1, \ldots, \mathsf{v}_n$ are unary symbols, a is a binary symbol and f has arity four.

In order to define $\mathcal{R}$, we first define an encoding $\ulcorner \cdot \urcorner^x$, which maps polynomials with positive coefficients to terms containing the variable x.

Definition 3.1 (🔘). Let P be a polynomial containing only positive coefficients, and the indeterminates $v_1, \ldots, v_n$. We can then encode natural numbers as

$$\ulcorner 0 \urcorner^x = \mathsf{z} \qquad\qquad \ulcorner m + 1 \urcorner^x = \mathsf{a}(x, \ulcorner m \urcorner^x)$$

A monomial $M = c \cdot v_1^{m_1} \cdot v_2^{m_2} \cdots v_n^{m_n}$ with $c \in \mathbb{N}_+$ and $m_1, \ldots, m_n \in \mathbb{N}$ is encoded as

$$\ulcorner M \urcorner^x = \mathsf{v}_1^{m_1}(\mathsf{v}_2^{m_2}(\cdots(\mathsf{v}_n^{m_n}(\ulcorner c \urcorner^x))\cdots))$$

where $v^0(t) = t$ and $v^{i+1}(t) = v(v^i(t))$ for $v \in \{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$. Recall that $\mathsf{v}_1, \ldots, \mathsf{v}_n$ are unary function symbols. Finally the polynomial $P = M_1 + M_2 + \cdots + M_k$ is encoded as

$$\ulcorner P \urcorner^x = \mathsf{a}(\ulcorner M_1 \urcorner^x, \mathsf{a}(\ulcorner M_2 \urcorner^x, \cdots \mathsf{a}(\ulcorner M_k \urcorner^x, \mathsf{z}) \cdots))$$

Example 3.2. For the polynomial $P = X^3 + 2X + 2$ we obtain the term

$$\ulcorner P \urcorner^y = \mathsf{a}(\ulcorner X^3 \urcorner^y, \mathsf{a}(\ulcorner 2X \urcorner^y, \mathsf{a}(\ulcorner 2 \urcorner^y, \mathsf{z})))$$

where

$$\begin{aligned} \ulcorner X^3 \urcorner^y &= \mathsf{X}(\mathsf{X}(\mathsf{X}(\mathsf{a}(y, \mathsf{z})))) \\ \ulcorner 2X \urcorner^y &= \mathsf{X}(\mathsf{a}(y, \mathsf{a}(y, \mathsf{z}))) \\ \ulcorner 2 \urcorner^y &= \mathsf{a}(y, \mathsf{a}(y, \mathsf{z})). \end{aligned}$$

The TRS $\mathcal{R}$ can then be defined via this encoding.

Definition 3.3 (). For polynomials P and Q containing only positive coefficients we obtain the TRS $\mathcal{R}$ consisting of the single rule

$$\mathsf{f}(y_1, y_2, \mathsf{a}(\ulcorner P \urcorner^{y_3}, y_3), \mathsf{o}) \longrightarrow \mathsf{f}(\mathsf{a}(y_1, \mathsf{z}), \mathsf{a}(\mathsf{z}, y_2), \mathsf{a}(\ulcorner Q \urcorner^{y_3}, y_3), \mathsf{z})$$

The rule serves two purposes. First it constrains any linear polynomial interpretation proving its termination to conform to a very limited shape. Secondly it uses these restricted shapes to encode the inequality $P \geqslant Q$ in the orientation of the rule $[\ell]_\mathbb{N} > [r]_\mathbb{N}$. This leads to the following result.

Theorem 3.4 (). *Termination of $\mathcal{R}$ can be shown by a linear polynomial interpretation if and only if $P(v_1, \dots, v_n) \geqslant Q(v_1, \dots, v_n)$ for some $v_1, \dots, v_n \in \mathbb{N}_+$.*

Proof. For the if direction assume $P(v_1, \dots, v_n) \geqslant Q(v_1, \dots, v_n)$ for some $v_1, \dots, v_n \in \mathbb{N}_+$. We then choose the monotone interpretations

$$\mathsf{z}_\mathbb{N} = 0 \qquad \mathsf{a}_\mathbb{N}(x_1, x_2) = x_1 + x_2$$
$$\mathsf{o}_\mathbb{N} = 1 \qquad \mathsf{f}_\mathbb{N}(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$$
$$\mathsf{v}_{i\,\mathbb{N}}(x) = v_i \cdot x \quad \text{for all } i \in \{1, \dots, n\}$$

Note that $[\ulcorner P \urcorner^{y_3}]_\mathbb{N} = P(v_1, \dots, v_n) \cdot y_3$ using this interpretation, and the same holds for Q. Hence we orient the rule in $\mathcal{R}$, as seen in

$$\begin{aligned}[\ell]_\mathbb{N} &= y_1 + y_2 + (P(v_1, \dots, v_n) + 1)y_3 + 1 \\ &> y_1 + y_2 + (Q(v_1, \dots, v_n) + 1)y_3 = [r]_\mathbb{N}\end{aligned}$$

For the only-if direction we assume a linear interpretation for all $f \in \mathcal{F}$, such that $[\ell]_\mathbb{N} > [r]_\mathbb{N}$. Hence we know that all interpretations have the shape $f_\mathbb{N}(x_1, \dots, x_k) = f_0 + f_1 x_1 + \cdots + f_k x_k$ where $f_0 \in \mathbb{N}$ and $f_1, \dots, f_k \in \mathbb{N}_+$ due to monotonicity. For any term t we write $[t]^{y_i}_\mathbb{N}$ for the coefficient of the indeterminate y_i of the linear polynomial $[t]_\mathbb{N}$. Using this notation, $[\ell]_\mathbb{N} > [r]_\mathbb{N}$ implies $[\ell]^{y_i}_\mathbb{N} \geqslant [r]^{y_i}_\mathbb{N}$ for $i \in \{1, 2, 3\}$. By the shape of the rule we deduce $\mathsf{f}_1 = [\ell]^{y_1}_\mathbb{N} \geqslant [r]^{y_1}_\mathbb{N} = \mathsf{f}_1\mathsf{a}_1$ and in combination with $\mathsf{f}_1 > 0$ and $\mathsf{a}_1 > 0$ we conclude $\mathsf{a}_1 = 1$. Similarly, from $[\ell]^{y_2}_\mathbb{N} \geqslant [r]^{y_2}_\mathbb{N}$ we infer $\mathsf{a}_2 = 1$, and in turn $\mathsf{a}_\mathbb{N}(x_1, x_2) = x_1 + x_2 + \mathsf{a}_0$ for some $\mathsf{a}_0 \in \mathbb{N}$. The shape of the interpretations of v_i is $\mathsf{v}_{i\,\mathbb{N}}(x) = \hat{v}_i x + \hat{w}_i$ for some coefficients $\hat{v}_i \in \mathbb{N}_+$ and $\hat{w}_i \in \mathbb{N}$. Due to the shape of $\mathsf{a}_\mathbb{N}$ it is easy to see that $[\ulcorner m \urcorner^{y_3}]^{y_3}_\mathbb{N} = m$ for any $m \in \mathbb{N}$, $[\ulcorner c \cdot v_1^{m_1} \cdots v_n^{m_n} \urcorner^{y_3}]^{y_3}_\mathbb{N} = c \cdot \hat{v}_1^{m_1} \cdots \hat{v}_n^{m_n}$ and further $[\ulcorner P \urcorner^{y_3}]^{y_3}_\mathbb{N} = P(\hat{v}_1, \dots, \hat{v}_n)$ for any polynomial P. Hence

$$\mathsf{f}_3 \cdot (P(\hat{v}_1, \dots, \hat{v}_n) + 1) = [\ell]^{y_3}_\mathbb{N} \geqslant [r]^{y_3}_\mathbb{N} = \mathsf{f}_3 \cdot (Q(\hat{v}_1, \dots, \hat{v}_n) + 1)$$

Since $\mathsf{f}_3 > 0$, division by f_3 is possible, resulting in the desired inequality $P(\hat{v}_1, \dots, \hat{v}_n) \geqslant Q(\hat{v}_1, \dots, \hat{v}_n)$. □

Corollary 3.5. *Linear polynomial termination is undecidable, even for one-rule* TRSs*.*

Proof. This follows directly from Theorem 3.4 and Lemma 2.2. □

Interestingly the TRS $\mathcal{R}$ is always terminating, independent of the polynomials P and Q. This can be shown by using a non-linear polynomial interpretation.

Lemma 3.6 (). *The* TRS *$\mathcal{R}$ is polynomially terminating.*

Proof. Use the following monotone interpretation over $\mathbb{N}$

$$\mathsf{o}_\mathbb{N} = Q(1, \dots, 1) + 1 \qquad \mathsf{a}_\mathbb{N}(x, y) = x + y$$
$$\mathsf{z}_\mathbb{N} = 0 \qquad \mathsf{f}_\mathbb{N}(x_1, x_2, x_3, x_4) = x_3 x_4 + x_1 + x_2 + x_3 + x_4$$
$$\mathsf{v}_{i\,\mathbb{N}}(x) = x \quad \text{for all } i \in \{1, \dots, n\}$$

Note that due to $[\mathsf{v}_i(x)]^x_\mathbb{N} = 1$ we have $[\ulcorner P \urcorner^{y_3}]^{y_3}_\mathbb{N} = P(1, \dots, 1)$ and $[\ulcorner Q \urcorner^{y_3}]^{y_3}_\mathbb{N} = Q(1, \dots, 1)$. Hence, we can orient the rule as seen in

$$\begin{aligned}[\ell]_\mathbb{N} &= (Q(1, \dots, 1) + 1)(P(1, \dots, 1) + 1)y_3 + \\ &\quad y_1 + y_2 + (P(1, \dots, 1) + 1)y_3 + (Q(1, \dots, 1) + 1) \\ &> y_1 + y_2 + (Q(1, \dots, 1) + 1)y_3 = [r]_\mathbb{N}\end{aligned}$$
□

Corollary 3.7. *Linear polynomial termination is undecidable even for polynomially terminating one-rule* TRSs*.* □

Polynomial interpretations are often also utilized in the weakly monotone setting, e.g., when termination proofs are performed using dependency pairs [1]. Here, the strict monotonicity requirement is weakened to weak monotonicity, i.e., $f_\mathcal{A}(a_1, \dots, a_i, \dots, a_n) \geqslant f_\mathcal{A}(a_1, \dots, b, \dots, a_n)$ must be satisfied for all $a_1, \dots, a_n, b \in A$ and $i \in \{1, \dots, n\}$ whenever $a_i \geqslant b$. In the linear case, this means that weakly monotone interpretations $f_\mathbb{N}(x_1, \dots, x_n) = f_0 + f_1 x_1 + \cdots + f_n x_n$ have to satisfy $f_1, \dots, f_n \in \mathbb{N}$ instead of $f_1, \dots, f_n \in \mathbb{N}_+$ as in the strictly monotone case.

The one-rule TRS $\mathcal{R}'$ in the upcoming definition will be used to show that the main results of this section are also satisfied in the weakly monotone setting.

Definition 3.8. Let L be a list of all function symbols of positive arity paired with their argument positions, i.e.,

$$L = [(\mathsf{f}, 1), \dots, (\mathsf{f}, 4), (\mathsf{a}, 1), (\mathsf{a}, 2), (\mathsf{v}_1, 1), \dots, (\mathsf{v}_n, 1)].$$

Define the z-context of a pair (f, i) as the function $\mathsf{z}C_{f,i}$ which wraps a context around a term, i.e.,

$$\mathsf{z}C_{f,i}(t) = f(\mathsf{z}, \dots, \mathsf{z}, t, \mathsf{z}, \dots, \mathsf{z})$$

where the term t is at the i-th position of f. We extend $\mathsf{z}C$ to lists of pairs as follows:

$$\mathsf{z}C_{[(f_1,i_1),\dots,(f_m,i_m)]}(t) = \mathsf{z}C_{f_1,i_1}(\dots(\mathsf{z}C_{f_m,i_m}(t)\dots)$$

The TRS $\mathcal{R}'$ is defined as $\mathcal{R}' = \{\mathsf{z}C_L(\ell) \to \mathsf{z}C_L(r)\}$ where $\ell \to r$ is the rule of $\mathcal{R}$.

The main purpose of the additional context is that the potential decrease of $\mathcal{R}$ is simulated in $\mathcal{R}'$ below all argument positions of the signature. Hence, the interpretation for $\mathcal{R}'$ cannot ignore a single argument of some function symbol. Therefore, weak monotonicity implies strict monotonicity, and we are able to reuse the results for $\mathcal{R}$.

Theorem 3.9 (). *The* TRS *$\mathcal{R}'$ enjoys the following properties.*

- *There is a linear weakly monotone polynomial interpretation over $\mathbb{N}$ that strictly orients the rule of $\mathcal{R}'$ if and only if $P(v_1, \dots, v_n) \geqslant Q(v_1, \dots, v_n)$ for some $v_1, \dots, v_n \in \mathbb{N}_+$.*
- *The monotone non-linear polynomial interpretation in the proof of Lemma 3.6 strictly orients $\mathcal{R}'$.*

4 KBO WITH SUBTERM COEFFICIENTS

In this section we show that the one-rule TRS $\mathcal{R}$ of Definition 3.3 can also be used to show the undecidability of the orientability problem for KBO with subterm coefficients. The proofs reveal that the precedence and the recursive structure of $s >_{\text{kbo}} t$ can completely be ignored for the undecidability result. Mainly the condition $\mathsf{vc}(x,s) \geqslant \mathsf{vc}(x,t)$ is utilized, and for one direction we obtain a decrease by choosing the weight function appropriately.

Lemma 4.1. *Suppose* $\mathsf{sc}(\mathsf{a},1) = \mathsf{sc}(\mathsf{a},2) = 1$ *and assume that* $P \in \mathbb{Z}[v_1,\ldots,v_n]$ *is a polynomial with positive coefficients. Then* $\mathsf{vc}(x, \ulcorner P \urcorner^{x}) = P(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1))$.

Proof. We use induction on the structure of P. If $P = 0$ then $\mathsf{vc}(x,\ulcorner P \urcorner^{x}) = \mathsf{vc}(x,\mathsf{z}) = 0$. For $P = n+1$ we have $\mathsf{vc}(x,\ulcorner P \urcorner^{x}) = \mathsf{vc}(x,\mathsf{a}(x,\ulcorner n \urcorner^{x})) = 1 + \mathsf{vc}(x,\ulcorner n \urcorner^{x}) = n+1$. For a monomial $P = c \cdot v_1^{m_1}\cdots v_k^{m_k}$ with $c \in \mathbb{N}_+$ we obtain

$$\begin{aligned}\mathsf{vc}(x,\ulcorner c\cdot v_1^{m_1}\cdots v_k^{m_k}\urcorner^{x}) &= \mathsf{vc}(x,\mathsf{v}_1^{m_1}(\cdots(\mathsf{v}_k^{m_k}(\ulcorner c\urcorner^{x}))\cdots))\\ &= c\cdot \mathsf{sc}(\mathsf{v}_1,1)^{m_1}\cdots \mathsf{sc}(\mathsf{v}_k,1)^{m_k}\\ &= P(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1))\end{aligned}$$

Finally for a polynomial $P = M_1 + M_2 + \cdots + M_k$ we have

$$\begin{aligned}&\mathsf{vc}(x,\ulcorner M_1 + M_2 + \cdots + M_k\urcorner^{x})\\ &= \mathsf{vc}(x,\mathsf{a}(\ulcorner M_1\urcorner^{x},\mathsf{a}(\ulcorner M_2\urcorner^{x},\cdots \mathsf{a}(\ulcorner M_k\urcorner^{x},\mathsf{z}))))\\ &= M_1(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1)) + \cdots + M_k(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1))\\ &= P(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1))\end{aligned}$$

where we use $\mathsf{sc}(\mathsf{a},1) = \mathsf{sc}(\mathsf{a},2) = 1$ in the second identity. □

Theorem 4.2. *Let* $\mathcal{R}$ *be the* TRS *from Definition 3.3. If* $\mathcal{R}$ *can be shown terminating using KBO with subterm coefficients then* $P(v_1,\ldots,v_n) \geqslant Q(v_1,\ldots,v_n)$ *for some* $v_1,\ldots,v_n \in \mathbb{N}_+$.

Proof. Assume $\mathcal{R} = \{\ell \to r\}$ can be shown terminating using KBO with subterm coefficients. Then there exists a subterm coefficient function sc such that $\mathsf{vc}(y,\ell) \geqslant \mathsf{vc}(y,r)$ for all $y \in \{y_1,y_2,y_3\}$. Hence

$$\mathsf{vc}(y_1,\ell) = \mathsf{sc}(\mathsf{f},1) \geqslant \mathsf{sc}(\mathsf{f},1)\cdot \mathsf{sc}(\mathsf{a},1) = \mathsf{vc}(y_1,r)$$

which only holds for $\mathsf{sc}(\mathsf{a},1) = 1$. Moreover

$$\mathsf{vc}(y_2,\ell) = \mathsf{sc}(\mathsf{f},2) \geqslant \mathsf{sc}(\mathsf{f},2)\cdot \mathsf{sc}(\mathsf{a},2) = \mathsf{vc}(y_2,r)$$

leading to $\mathsf{sc}(\mathsf{a},2) = 1$. Hence, using Lemma 4.1,

$$\begin{aligned}\mathsf{vc}(y_3,\ell) &= \mathsf{sc}(\mathsf{f},3)\cdot \mathsf{vc}(y_3,\mathsf{a}(\ulcorner P\urcorner^{y_3},y_3))\\ &= \mathsf{sc}(\mathsf{f},3)\cdot (P(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1)) + 1)\end{aligned}$$

and

$$\begin{aligned}\mathsf{vc}(y_3,r) &= \mathsf{sc}(\mathsf{f},3)\cdot \mathsf{vc}(y_3,\mathsf{a}(\ulcorner Q\urcorner^{y_3},y_3))\\ &= \mathsf{sc}(\mathsf{f},3)\cdot (Q(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1)) + 1)\end{aligned}$$

and thus

$$P(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1)) \geqslant Q(\mathsf{sc}(\mathsf{v}_1,1),\ldots,\mathsf{sc}(\mathsf{v}_n,1)) \qquad \square$$

Theorem 4.3. *If* $P(v_1,\ldots,v_n) \geqslant Q(v_1,\ldots,v_n)$ *for some numbers* $v_1,\ldots,v_n \in \mathbb{N}_+$ *then* $\mathcal{R}$ *can be shown terminating using KBO with subterm coefficients.*

Proof. Assume $P(v_1,\ldots,v_n) \geqslant Q(v_1,\ldots,v_n)$. Let $\mathcal{R} = \{\ell \to r\}$. Take an arbitrary precedence, e.g., the empty one. Take the weight function (w,w_0) where $w(\mathsf{a}) = w(\mathsf{f}) = 0$, $w_0 = w(\mathsf{z}) = w(\mathsf{v}_1) = \cdots = w(\mathsf{v}_n) = 1$, and $w(\mathsf{o}) = w_{\mathsf{sc}}(r)$. Since o does not appear in r and $w_{\mathsf{sc}}(r) > 0$ this is well-defined. And since the unary functions $\mathsf{v}_1,\ldots,\mathsf{v}_n$ have positive weight, (w,w_0) is admissible for any precedence. Additionally we choose the subterm coefficient function where $\mathsf{sc}(\mathsf{a},1) = \mathsf{sc}(\mathsf{a},2) = \mathsf{sc}(\mathsf{f},1) = \cdots = \mathsf{sc}(\mathsf{f},4) = 1$ and $\mathsf{sc}(\mathsf{v}_i,1) = v_i$. Applying Lemma 4.1 to P and Q results in $\mathsf{vc}(y_3,\ulcorner P\urcorner^{y_3}) = P(v_1,\ldots,v_n)$ and $\mathsf{vc}(y_3,\ulcorner Q\urcorner^{y_3}) = Q(v_1,\ldots,v_n)$. We have $\mathsf{vc}(y_i,\ell) \geqslant \mathsf{vc}(y_i,r)$ for all $i \in \{1,2,3\}$, as seen by

$$\begin{aligned}\mathsf{vc}(y_1,\ell) &= 1 \geqslant 1 = \mathsf{vc}(y_1,r)\\ \mathsf{vc}(y_2,\ell) &= 1 \geqslant 1 = \mathsf{vc}(y_2,r)\\ \mathsf{vc}(y_3,\ell) &= P(v_1,\ldots,v_n) + 1 \geqslant Q(v_1,\ldots,v_n) + 1 = \mathsf{vc}(y_3,r)\end{aligned}$$

Finally the rule $\ell \to r$ is oriented by KBO since

$$w_{\mathsf{sc}}(\ell) = w(o) + w_{\mathsf{sc}}(\ulcorner P\urcorner^{x}) + 3 > w_{\mathsf{sc}}(r) \qquad \square$$

Corollary 4.4. *It is undecidable whether a* TRS *can be shown terminating using KBO with subterm coefficients.*

Proof. This follows directly from Lemma 2.2 and Theorems 4.2 and 4.3. □

5 POLYNOMIAL TERMINATION OVER $\mathbb{N}$

Now that we have shown undecidability for linear polynomial termination, we move on to general polynomial termination. The previous proofs rely on linearity of $\mathsf{a}_{\mathbb{N}}$ and $\mathsf{v}_{i\mathbb{N}}$ to correctly encode the polynomial. They also require linearity of $\mathsf{f}_{\mathbb{N}}$ to be able to compare the arguments of f independently, and therefore allowing us to encode the order on the polynomials. In this section we extend the signature and rule of $\mathcal{R}$, such that the new parts of the rule still induce a linearity constraint on the previously mentioned interpretations. The encoding of the polynomials as terms does not change.

Definition 5.1. For a pair of polynomials P and Q containing only positive coefficients and indeterminates $v_1,\ldots,v_n$, the TRS $\mathcal{S}$ is defined over the signature $\mathcal{F} = \{\mathsf{z},\mathsf{o},\mathsf{a},\mathsf{q},\mathsf{h},\mathsf{g},\mathsf{f}\} \cup \{\mathsf{v}_i \mid 1 \leqslant i \leqslant n\}$ and consists of the single rule

$$\mathsf{f}(\ell_1(y_1),\ell_2(y_2),\ldots,\ell_7(y_7)) \to \mathsf{f}(r_1(y_1),r_2(y_2),\ldots,r_7(y_7))$$

where the subterms $\ell_i(y_i)$ and $r_i(y_i)$ for $i \in \{1,\ldots,7\}$ are defined as follows:

$$\begin{aligned}\ell_1(y_1) &= y_1 & r_1(y_1) &= \mathsf{a}(y_1,\mathsf{z})\\ \ell_2(y_2) &= y_2 & r_2(y_2) &= \mathsf{a}(\mathsf{z},y_2)\\ \ell_3(y_3) &= \mathsf{a}(\ulcorner P\urcorner^{y_3},y_3) & r_3(y_3) &= \mathsf{a}(\ulcorner Q\urcorner^{y_3},y_3)\\ \ell_4(y_4) &= \mathsf{q}(\mathsf{h}(y_4)) & r_4(y_4) &= \mathsf{h}(\mathsf{h}(\mathsf{q}(y_4)))\\ \ell_5(y_5) &= \mathsf{h}(y_5) & r_5(y_5) &= \mathsf{v}_1(\cdots(\mathsf{v}_n(\mathsf{a}(y_5,y_5)))\cdots)\\ \ell_6(y_6) &= \mathsf{h}(y_6) & r_6(y_6) &= \mathsf{f}(y_6,\ldots,y_6)\\ \ell_7(y_7) &= \mathsf{g}(y_7,\mathsf{o}) & r_7(y_7) &= \mathsf{g}(y_7,\mathsf{z})\end{aligned}$$

We drop the argument of $\ell_i(y_i)$ or $r_i(y_i)$ if it is clear from the context and write ℓ_i or r_i for brevity.

Note that the subterm pairs are similar to the ones from $\mathcal{R}$, except we add the pairs (ℓ_4, r_4), (ℓ_5, r_5) and (ℓ_6, r_6), and modify the final arguments of f slightly. These changes are necessary since we are no longer limited to linear interpretation functions. So we use these pairs to limit any valid interpretations to ensure that $\mathsf{a}_\mathbb{N}$, $\mathsf{f}_\mathbb{N}$ and $\mathsf{v}_{i\mathbb{N}}$ must be linear. Then the reasoning follows the proof of Theorem 3.4.

But first we introduce the following two lemmata to help reason about the degree of polynomials.

Lemma 5.2. *Let $p(x_1, \ldots, x_n)$ be a monotone polynomial. For all $1 \leqslant i \leqslant n$ there exist $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1} \in \mathbb{N}$ such that for all $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n-1} \in \mathbb{N}$, where $a_j \leqslant b_j$ for $j \neq i$, we have*

$$\begin{aligned}&\deg(p(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots a_n))\\ &= \deg(p(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n))\end{aligned}$$

Proof. From monotonicity of p we obtain

$$\begin{aligned}&\deg(p(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots a_n))\\ &\leqslant \deg(p(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n))\end{aligned}$$

for any $a_j \leqslant b_j$ with $j \neq i$. Moreover we know that the maximal degree

$$\begin{aligned}\max\{\ &\deg(p(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots a_n)) \mid\\ &a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in \mathbb{N}\,\}\end{aligned}$$

is bounded by the largest degree of p. Hence the lemma holds. □

Later the pair of arguments ℓ_6 and r_6 will be used to bound the degree of the polynomial $\mathsf{f}_\mathbb{N}$ from above, i.e., $\deg(\mathsf{h}_\mathbb{N}(y_6)) \geqslant \deg(\mathsf{f}_\mathbb{N}(y_6, \ldots, y_6))$. In general the degree of a univariate polynomial $q(x) := p(x, x)$ can be lower than the degree of the corresponding multivariate polynomial $p(x, y)$, and therefore an upper bound on the degree of q does not bound the degree of p. Take for example $p(x, y) = x^2 - xy + x$ where $\deg(p) = 2$ and $\deg(q) = \deg(x^2 - x^2 + x) = \deg(x) = 1$. In our setting however we consider monotone polynomials (the interpretation functions), where this cancellation of high degree monomials cannot happen.

Lemma 5.3. *Let $p(x_1, \ldots, x_n)$ be a monotone polynomial. For $q(x) := p(x, \ldots, x)$ we have $\deg(q) = \deg(p)$.*

Proof. Let $d := \deg(q)$ and $m := \deg(p)$. The result is trivial if $m = 0$, since then p is a constant and hence $p = q$. In the other case we have $m > 0$. From $q(x) = p(x, \ldots, x)$ it follows that p contains at least one monomial of degree d, implying $m \geqslant d$. We can split p into the polynomials $p_1(x_1, \ldots, x_n)$ which contains all monomials of p of degree m and $p_2(x_1, \ldots, x_n)$ consisting of the remaining monomials of degree strictly less than m. Therefore, $p(x_1, \ldots, x_n) = p_1(x_1, \ldots, x_n) + p_2(x_1, \ldots, x_n)$, where $p_1 \neq 0$ since $m > 0$. So, there are $b_1, \ldots, b_n \geqslant 0$ with $p_1(b_1, \ldots, b_n) \neq 0$. Define $c = \max\{b_1, \ldots, b_n\}$. We conclude

$$q(ca) = p(ca, \ldots, ca) \geqslant p(b_1 a, \ldots, b_n a) \geqslant p(0, \ldots, 0)$$

for every $a \geqslant 0$ by using monotonicity of p. Therefore,

$$\begin{aligned}d = \deg(q(x)) &= \deg(q(cx)) \geqslant \deg(p(b_1 x, \ldots, b_n x))\\ &= \deg(p_1(b_1 x, \ldots, b_n x) + p_2(b_1 x, \ldots, b_n x))\\ &= \deg(p_1(b_1, \ldots, b_n) \cdot x^m + p_2(b_1 x, \ldots, b_n x)) = m\end{aligned}$$

where the last equality is a consequence of $p_1(b_1, \ldots, b_n) \neq 0$ and $\deg(p_2(b_1 x, \ldots, b_n x)) < m$. Since $d \leqslant m$ and $d \geqslant m$ we arrive at $\deg(q) = d = m = \deg(p)$. □

Theorem 5.4. *Termination of $\mathcal{S}$ can be shown by a polynomial interpretation if and only if $P(v_1, \ldots, v_n) \geqslant Q(v_1, \ldots, v_n)$ for some $v_1, \ldots, v_n \in \mathbb{N}_+$.*

Proof. For the *if* direction assume that there are $v_1, \ldots, v_n \in \mathbb{N}_+$ such that $P(v_1, \ldots, v_n) \geqslant Q(v_1, \ldots, v_n)$. For the interpretations choose

$$\begin{aligned}&\mathsf{z}_\mathbb{N} = 0 \quad \mathsf{o}_\mathbb{N} = 1 && \mathsf{a}_\mathbb{N}(x, y) = \mathsf{g}_\mathbb{N}(x, y) = x + y\\ &\mathsf{f}_\mathbb{N}(x_1, \ldots, x_7) = x_1 + \cdots + x_7 && \mathsf{h}_\mathbb{N}(x) = c \cdot x\\ &\mathsf{q}(\mathsf{x})_\mathbb{N} = c \cdot x^2 && \mathsf{v}_{i\mathbb{N}}(x) = v_i \cdot x \quad \text{for } 1 \leqslant i \leqslant n\end{aligned}$$

where $c = \max(7, 2 \cdot v_1 \cdot v_2 \cdot \ldots \cdot v_n)$. We have

$$\begin{aligned}[\ell_1]_\mathbb{N} &= y_1 \geqslant y_1 = [r_1]_\mathbb{N}\\ [\ell_2]_\mathbb{N} &= y_2 \geqslant y_2 = [r_2]_\mathbb{N}\\ [\ell_3]_\mathbb{N} &= (P(v_1, \ldots, v_n) + 1) \cdot y_3 \geqslant (Q(v_1, \ldots, v_n) + 1) \cdot y_3 = [r_3]_\mathbb{N}\\ [\ell_4]_\mathbb{N} &= c^3 \cdot y_4^2 \geqslant c^3 \cdot y_4^2 = [r_4]_\mathbb{N}\\ [\ell_5]_\mathbb{N} &= c \cdot y_5 \geqslant 2 \cdot v_1 \cdot \ldots \cdot v_n \cdot y_5 = [r_5]_\mathbb{N}\\ [\ell_6]_\mathbb{N} &= c \cdot y_6 \geqslant 7 \cdot y_6 = [r_6]_\mathbb{N}\\ [\ell_7]_\mathbb{N} &= y_7 + 1 > y_7 = [r_7]_\mathbb{N}\end{aligned}$$

and thus

$$[\ell]_\mathbb{N} = [\ell_1]_\mathbb{N} + \cdots + [\ell_7]_\mathbb{N} > [r_1]_\mathbb{N} + \cdots + [r_7]_\mathbb{N} = [r]_\mathbb{N}$$

Hence $\mathcal{S}$ is polynomially terminating.

For the *only-if* direction we assume $\mathcal{S}$ can be oriented using a polynomial interpretation. Let $f(x_1, \ldots, x_7) = [\mathsf{f}(x_1, \ldots, x_7)]_\mathbb{N}$. Abusing notation, we write $\ell_{i\mathbb{N}}$ for $[\ell_i(x)]_\mathbb{N}$ and $r_{i\mathbb{N}}$ for $[r_i(x)]_\mathbb{N}$. Applying Lemma 5.2 to the polynomial f at position i, we obtain numbers $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_7 \in \mathbb{N}$ such that

$$\begin{aligned}&\deg(f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_7))\\ =\ &\deg(f(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_7))\end{aligned}$$

for all $a_j \leqslant b_j \in \mathbb{N}$ with $j \neq i$. Since $\ell_{i\mathbb{N}}(y_i) > r_{i\mathbb{N}}(y_i)$ for all $y_1, \ldots, y_7 \in \mathbb{N}$, we also have

$$\begin{aligned}&f(c_1, \ldots, c_{i-1}, \ell_{i\mathbb{N}}(x), c_{i+1}, \ldots, c_7)\\ >\ &f(e_1, \ldots, e_{i-1}, r_{i\mathbb{N}}(x), e_{i+1}, \ldots, e_7)\end{aligned} \tag{1}$$

for all $x \in \mathbb{N}$, where $c_j = \ell_{j\mathbb{N}}(a_j)$ and $e_j = r_{i\mathbb{N}}(a_j)$ for $j \neq i$. With the help of monotonicity we have $c_j \geqslant a_j$ and $e_j \geqslant a_j$ for all $j \neq i$. Therefore

$$\begin{aligned}&\deg(f(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_7))\\ = d =\ &\deg(f(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_7))\end{aligned}$$

for some $d \in \mathbb{N}_+$. From (1) we further obtain

$$\begin{aligned}&\deg(f(c_1, \ldots, c_{i-1}, \ell_{i\mathbb{N}}(x), c_{i+1}, \ldots, c_7))\\ \geqslant\ &\deg(f(e_1, \ldots, e_{i-1}, r_{i\mathbb{N}}(x), e_{i+1}, \ldots, e_7))\end{aligned}$$

which simplifies to $d \cdot \deg(\ell_{i\mathbb{N}}) \geqslant d \cdot \deg(r_{i\mathbb{N}})$. Thus $\deg(\ell_{i\mathbb{N}}) \geqslant \deg(r_{i\mathbb{N}})$ for all $1 \leqslant i \leqslant 7$. From $\deg(\ell_{4\mathbb{N}}) \geqslant \deg(r_{4\mathbb{N}})$ we obtain

$\deg(\mathsf{h}_{\mathbb{N}})\cdot\deg(\mathsf{q}_{\mathbb{N}}) \geqslant \deg(\mathsf{h}_{\mathbb{N}})^2\cdot\deg(\mathsf{q}_{\mathbb{N}})$, so $\deg(\mathsf{h}_{\mathbb{N}}) = 1$. Applying this to the argument positions 5 and 6, we obtain

$$\deg([\mathsf{v}_1(x)]_{\mathbb{N}}) = \cdots = \deg([\mathsf{v}_n(x)]_{\mathbb{N}}) = 1$$
$$\deg([\mathsf{f}(x,\ldots,x)]_{\mathbb{N}}) = \deg([\mathsf{a}(x,x)]_{\mathbb{N}}) = 1$$

which by Lemma 5.3 also shows

$$\deg([\mathsf{f}(x_1,\ldots,x_7)]_{\mathbb{N}}) = \deg([\mathsf{a}(x,y)]_{\mathbb{N}}) = 1$$

Hence, the interpretation must have the shapes

$$[\mathsf{v}_i(x)]_{\mathbb{N}} = v_{0,i} + v_{1,i}\cdot x$$
$$[\mathsf{f}(x_1,\ldots,x_7)]_{\mathbb{N}} = f_0 + f_1\cdot x_1 + \cdots + f_7\cdot x_7$$
$$[\mathsf{a}(x,y)]_{\mathbb{N}} = a_0 + a_1\cdot x + a_2\cdot y$$

Consequently, we can complete the proof like in the one of Theorem 3.4. □

Corollary 5.5. *Polynomial termination is undecidable, even for one-rule* TRS*s.*

Proof. This follows directly from Theorem 5.4 and Lemma 2.2. □

Just like with $\mathcal{R}$ we can prove termination of $\mathcal{S}$ independent of the polynomials P and Q. Moreover the proof uses a monotone interpretation in the natural numbers.

Lemma 5.6. *The* TRS $\mathcal{S}$ *is* ω*-terminating.*

Proof. Consider the following monotone interpretations over $\mathbb{N}$

$$\mathsf{o}_{\mathbb{N}} = Q(1,\ldots,1) \qquad \mathsf{z}_{\mathbb{N}} = 0$$
$$\mathsf{a}_{\mathbb{N}}(x,y) = x+y \qquad \mathsf{g}_{\mathbb{N}}(x,y) = (y+1)x+y$$
$$\mathsf{h}_{\mathbb{N}}(x) = x^2+7x+4 \qquad \mathsf{f}_{\mathbb{N}}(x_1,\ldots,x_7) = x_3\cdot x_7 + x_1+\cdots+x_7$$
$$\mathsf{q}_{\mathbb{N}}(x) = 5^x \qquad \mathsf{v_i}_{\mathbb{N}}(x) = x \quad \text{for all } i\in\{1,\ldots,n\}$$

We have

$$[\ell_1]_{\mathbb{N}} = y_1 \geqslant y_1 = [r_1]_{\mathbb{N}}$$
$$[\ell_2]_{\mathbb{N}} = y_2 \geqslant y_2 = [r_2]_{\mathbb{N}}$$
$$[\ell_5]_{\mathbb{N}} = y_5^2+7y_5+4 \geqslant 2y_5 = [r_5]_{\mathbb{N}}$$
$$[\ell_6]_{\mathbb{N}} = y_6^2+7y_6+4 \geqslant y_6^2+7y_6 = [r_6]_{\mathbb{N}}$$
$$[\ell_7]_{\mathbb{N}} = (Q(1,\ldots,1)+1)y_7 + Q(1,\ldots,1) > y_7 = [r_7]_{\mathbb{N}}$$

Moreover

$$[\ell_4]_{\mathbb{N}} = 5^{y_4^2+7y_4+4} = 5\cdot(5^3\cdot 5^{y_4^2+7y_4})$$
$$> 5^{4y_4} + 14\cdot 5^{3y_4} + 64\cdot 5^{2y_4} + 105\cdot 5^{y_4} + 48 = [r_4]_{\mathbb{N}}$$

Furthermore,

$$[\ell_3]_{\mathbb{N}} = (P(1,\ldots,1)+1)y_3$$
$$[\ell_7]_{\mathbb{N}} = (Q(1,\ldots,1)+1)y_7 + Q(1,\ldots,1)$$
$$[r_3]_{\mathbb{N}} = (Q(1,\ldots,1)+1)y_3$$
$$[r_7]_{\mathbb{N}} = y_7$$

We therefore obtain

$$[\ell_3]_{\mathbb{N}}\cdot[\ell_7]_{\mathbb{N}} + [\ell_3]_{\mathbb{N}}$$
$$= ([\ell_7]_{\mathbb{N}}+1)\cdot[\ell_3]_{\mathbb{N}}$$
$$= ((Q(1,\ldots,1)+1)y_7 + Q(1,\ldots,1)+1)(P(1,\ldots,1)+1)y_3$$
$$\geqslant ((Q(1,\ldots,1)+1)y_7 + Q(1,\ldots,1)+1)y_3$$
$$= (Q(1,\ldots,1)+1)y_3y_7 + (Q(1,\ldots,1)+1)y_3$$
$$= [r_3]_{\mathbb{N}}\cdot[r_7]_{\mathbb{N}} + [r_3]_{\mathbb{N}}$$

for all $y_3, y_7 \in \mathbb{N}$. It follows that the rewrite rule of $\mathcal{S}$ is oriented:

$$[\ell]_{\mathbb{N}} = [\ell_3]_{\mathbb{N}}\cdot[\ell_7]_{\mathbb{N}} + \sum_{i=1}^{7}[\ell_i]_{\mathbb{N}}$$
$$> [r_3]_{\mathbb{N}}\cdot[r_7]_{\mathbb{N}} + \sum_{i=1}^{7}[r_i]_{\mathbb{N}} = [r]_{\mathbb{N}}$$

Hence $\mathcal{S}$ is ω-terminating. □

Corollary 5.7. *Polynomial termination is undecidable, even for* ω*-terminating one-rule* TRS*s.* □

This proves the conjecture from [15, p. 129].

6 POLYNOMIAL TERMINATION OVER $\mathbb{Q}/\mathbb{R}$

In this section we consider polynomial termination over $\mathbb{Q}$ and $\mathbb{R}$ which we uniformly handle by fixing a domain $D \in \{\mathbb{Q},\mathbb{R}\}$ arbitrarily.

In the previous sections we encoded polynomials as terms such that indeterminates of the polynomials correspond to coefficients of some interpretation. When dealing with polynomial termination over D a new approach for proving undecidability is required, since both coefficients and variables can be values in D. However, what does not change is that the exponents of our interpretations must still be natural numbers. We can make use of this by encoding the polynomials and the order on polynomials in the degrees of our interpretations. As long as we can represent multiplication in the interpretations we can use basic arithmetic to encode the polynomials in the degrees.

To this end the definition of the set $D[x]_{\geqslant 0} \subseteq D[x]$ of non-negative univariate polynomials will be useful. A polynomial p is in $D[x]_{\geqslant 0}$ if and only if $p(x) \geqslant 0$ for all $x \geqslant 0$. We can simulate arithmetic operations in the degrees of polynomials in $D[x]_{\geqslant 0}$ as illustrated in the following lemma.

Lemma 6.1. *Assume* $p, q \in D[x]_{\geqslant 0}$ *and* $a_3, a_2, a_1 \in D_{>0}$ *and* $a_0 \in D$*. Then the following properties are satisfied.*

1. $p+q, p\cdot q, p\circ q \in D[x]_{\geqslant 0}$,
2. $\deg(p\circ q) = \deg(p)\cdot\deg(q)$,
3. $\deg(p\cdot(a_3\cdot x + a_2) + (a_1\cdot x + a_0)) = 1 + \deg(p)$*, and*
4. $\deg(a_3\cdot p\cdot q + a_2\cdot p + a_1\cdot q + a_0) = \deg(p)+\deg(q)$. □

For encoding polynomials with positive coefficients as terms we use Definition 3.1, so using function symbols from $\{\mathsf{z},\mathsf{a}\}\cup\{\mathsf{v}_i \mid 1 \leqslant i \leqslant n\}$. Moreover, we write $\ulcorner p\urcorner$ for $\ulcorner p\urcorner^x$ with some fixed variable x.

Lemma 6.2. *Suppose* $\mathsf{z}_{\mathcal{D}} = z_0$ *and* $\mathsf{a}_{\mathcal{D}} = a_3xy + a_2x + a_1y + a_0$ *for some* $z_0, a_0 \in D_{\geqslant 0}$ *and* $a_3, a_2, a_1 \in D_{>0}$*. If* $p \in \mathbb{Z}[x_1,\ldots,x_n]$

with positive coefficients then $p(\deg([\mathsf{v}_1]_{\mathcal{D}}), \ldots, \deg([\mathsf{v}_n]_{\mathcal{D}})) = \deg([\ulcorner p \urcorner]_{\mathcal{D}})$ *and* $[\ulcorner p \urcorner]_{\mathcal{D}} \in D[x]_{\geqslant 0}$.

Proof. Using Lemma 6.1(1) we show $[\ulcorner p \urcorner]_{\mathcal{D}} \in D[x]_{\geqslant 0}$ by induction on $\ulcorner p \urcorner$ for arbitrary polynomials p. Hence, by Lemma 6.1(4) we conclude

$$\deg([\mathsf{a}(\ulcorner p \urcorner, \ulcorner q \urcorner)]_{\mathcal{D}}) = \deg([\ulcorner p \urcorner]_{\mathcal{D}}) + \deg([\ulcorner q \urcorner]_{\mathcal{D}})$$

for arbitrary p and q, where we require the preconditions on a_1, a_2, a_3. Similarly, we know

$$\deg([\mathsf{a}(x, \ulcorner p \urcorner)]_{\mathcal{D}}) = 1 + \deg([\ulcorner p \urcorner]_{\mathcal{D}})$$

and

$$\deg([\mathsf{v}_i(\ulcorner p \urcorner)]_{\mathcal{D}}) = \deg([\mathsf{v}_i]_{\mathcal{D}}) \cdot \deg([\ulcorner p \urcorner]_{\mathcal{D}})$$

by Lemma 6.1(3) and (2), respectively.

These equations are now used in a straightforward induction on $\ulcorner P \urcorner$ to show $\deg([\ulcorner P \urcorner]_{\mathcal{D}}) = P(\deg([\mathsf{v}_1]_{\mathcal{D}}), \ldots, \deg([\mathsf{v}_n]_{\mathcal{D}}))$. □

Definition 6.3. For a pair of polynomials P and Q containing only positive coefficients with indeterminates $v_1, \ldots, v_n$ the TRS $\mathcal{Q}$ is defined over the signature $\mathcal{F} = \{\mathsf{f}, \mathsf{z}, \mathsf{a}, \mathsf{h}, \mathsf{q}, \mathsf{g}\} \cup \{\mathsf{v}_i \mid i \in \{1, \ldots, n\}\}$ and consists of the single rule

$$\mathsf{f}(\ell_1(y_1), \ldots, \ell_9(y_9)) \to \mathsf{f}(r_1(y_1), \ldots, r_9(y_9))$$

where

$$\begin{aligned}
\ell_1(y_1) &= \mathsf{q}(\mathsf{h}(y_1)) & r_1(y_1) &= \mathsf{h}(\mathsf{h}(\mathsf{q}(y_1))) \\
\ell_2(y_2) &= \mathsf{h}(y_2) & r_2(y_2) &= \mathsf{g}(y_2, y_2) \\
\ell_3(y_3) &= \mathsf{h}(y_3) & r_3(y_3) &= \mathsf{f}(y_3, \ldots, y_3) \\
\ell_4(y_4) &= \mathsf{g}(\mathsf{q}(y_4), \mathsf{h}(\mathsf{h}(\mathsf{h}(y_4)))) & r_4(y_4) &= \mathsf{q}(\mathsf{g}(y_4, t)) \\
\ell_5(y_5) &= \mathsf{q}(y_5) & r_5(y_5) &= \mathsf{a}(y_5, y_5) \\
\ell_6(y_6) &= \mathsf{a}(y_6, y_6) & r_6(y_6) &= \mathsf{q}(y_6) \\
\ell_7(y_7) &= y_7 & r_7(y_7) &= \mathsf{a}(\mathsf{z}, y_7) \\
\ell_8(y_8) &= y_8 & r_8(y_8) &= \mathsf{a}(y_8, \mathsf{z}) \\
\ell_9(y_9) &= \mathsf{h}(\mathsf{a}(\ulcorner P \urcorner^{y_9}, y_9)) & r_9(y_9) &= \mathsf{a}(\ulcorner Q \urcorner^{y_9}, y_9)
\end{aligned}$$

where t in $r_4(y_4)$ is the ground term defined as

$$g\text{-}list([\mathsf{z}, \mathsf{f}(\mathsf{z}, \ldots, \mathsf{z}), \mathsf{q}(\mathsf{z}), \mathsf{h}(\mathsf{z}), \mathsf{a}(\mathsf{z}, \mathsf{z}), \mathsf{v}_1(\mathsf{z}), \ldots, \mathsf{v}_n(\mathsf{z})])$$

and *g-list* is a function that converts a list of terms into a term via the binary symbol g.

$$\begin{aligned}
g\text{-}list([\,]) &= \mathsf{z} \\
g\text{-}list([s_1, s_2, \ldots, s_k]) &= \mathsf{g}(s_1, g\text{-}list([s_2, \ldots, s_k]))
\end{aligned}$$

Using Lemma 6.2 we are ready to prove one direction of the soundness of the reduction from $P(v_1, \ldots, v_n) \geqslant Q(v_1, \ldots, v_n)$ to polynomial termination of $\mathcal{Q}$ over D.

Theorem 6.4. *If* $P(v_1, \ldots, v_n) \geqslant Q(v_1, \ldots, v_n)$ *for some numbers* $v_1, \ldots, v_n \in \mathbb{N}_+$ *then* $\mathcal{Q}$ *is polynomially terminating over* D.

Proof. Assume $P(v_1, \ldots, v_n) \geqslant Q(v_1, \ldots, v_n)$. Take the algebra $\mathcal{D}$ with interpretations

$$\begin{aligned}
\mathsf{z}_{\mathcal{D}} &= 0 & \mathsf{g}_{\mathcal{D}}(x, y) &= x + y \\
\mathsf{a}_{\mathcal{D}}(x, y) &= xy + x + y & \mathsf{q}_{\mathcal{D}}(x) &= x^2 + 2x \\
\mathsf{h}_{\mathcal{D}}(x) &= hx + h & \mathsf{f}_{\mathcal{D}}(x_1, \ldots, x_9) &= x_1 + x_2 + \cdots + x_9 \\
\mathsf{v}_{i\mathcal{D}}(x) &= x^{v_i} \quad \text{for all } i \in \{1, \ldots, n\}
\end{aligned}$$

where $h > 9$ is some natural number. Comparing the first eight pairs of subterms we have $[\ell_i]_{\mathcal{D}} \geqslant [r_i]_{\mathcal{D}}$ for $i \in \{1, \ldots, 8\}$. In order this can be seen by:

$$\begin{aligned}
(hy_1 + h)^2 + 2(hy_1 + h) &\geqslant h^2(y_1^2 + 2y_1) + h^2 + h \\
hy_2 + h &\geqslant 2y_2 \\
hy_3 + h &\geqslant 9y_3 \\
y_4^2 + 2y_4 + h^3 y_4 + h^3 + h^2 + h &\geqslant (y_4 + h)^2 + 2(y_4 + h) \\
y_5^2 + 2y_5 &\geqslant y_5^2 + 2y_5 \\
y_6^2 + 2y_6 &\geqslant y_6^2 + 2y_6 \\
y_7 &\geqslant y_7 \\
y_8 &\geqslant y_8
\end{aligned}$$

To orient the remaining subterms we first use Lemma 6.2 to obtain both $\deg([\ulcorner P \urcorner^{y_9}]_{\mathcal{D}}) = P(v_1, \ldots, v_n)$ and $\deg([\ulcorner Q \urcorner^{y_9}]_{\mathcal{D}}) = Q(v_1, \ldots, v_n)$. Together with our initial assumption this implies

$$\begin{aligned}
\deg([\mathsf{a}(\ulcorner P \urcorner^{y_9}, y_9)]_{\mathcal{D}}) &= P(v_1, \ldots, v_n) + 1 \\
&\geqslant Q(v_1, \ldots, v_n) + 1 \\
&= \deg([\mathsf{a}(\ulcorner Q \urcorner^{y_9}, y_9)]_{\mathcal{D}})
\end{aligned}$$

It follows that by choosing the coefficient h large enough, we have $[\ell_9]_{\mathcal{D}} >_\delta [r_9]_{\mathcal{D}}$ for $\delta = 1$, since

$$h \cdot [\mathsf{a}(\ulcorner P \urcorner^{y_9}, y_9)]_{\mathcal{D}} + h >_\delta [\mathsf{a}(\ulcorner Q \urcorner^{y_9}, y_9)]_{\mathcal{D}}$$

Finally we can orient the TRS $\mathcal{Q}$, since

$$[\ell_1]_{\mathcal{D}} + \cdots + [\ell_9]_{\mathcal{D}} >_\delta [r_1]_{\mathcal{D}} + \cdots + [r_9]_{\mathcal{D}}$$

proving the polynomial termination of $\mathcal{Q}$ over D. □

For the other direction we first restate the Lemmata 5.2 and 5.3 for polynomials over D that are monotone with respect to $>_\delta$.

Lemma 6.5. *Let* $p(x_1, \ldots, x_n)$ *be a polynomial over* D *that is monotone with respect to* $>_\delta$ *for some* $\delta > 0$. *For all* $1 \leqslant i \leqslant n$ *there exist* $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1} \in D_{\geqslant 0}$ *such that for all* $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n-1} \in D$, *where* $a_j <_\delta b_j$ *for* $j \neq i$, *we have*

$$\begin{aligned}
&\deg(p(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots a_n)) \\
&= \deg(p(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n))
\end{aligned}$$

Proof. This follows by the same proof as in Lemma 5.2, if we replace $a_i \leqslant b_i$ by $a_i <_\delta b_i$. □

Lemma 6.6. *Let* $p(x_1, \ldots, x_n)$ *be a polynomial over* D *that is monotone with respect to* $>_\delta$. *For* $q(x) := p(x, \ldots, x)$, *we have* $\deg(q) = \deg(p)$.

Proof. Let $d := \deg(q)$ and $m := \deg(p)$. As in Lemma 5.3 we only consider the case $m > 0$, we deduce $m \geqslant d$, and we split p such that $p(x_1, \ldots, x_n) = p_1(x_1, \ldots, x_n) + p_2(x_1, \ldots, x_n)$, where p_1 contains only monomials of degree m and $\deg(p_2) < m$. Since p_1 is not the 0-polynomial we have that $p_1(b_1, \ldots, b_n) \neq 0$ for some $b_1, \ldots, b_n$ where each $b_i \geqslant 1$. We define $c = \max\{b_1, \ldots, b_n\} + \delta$. Hence $cx \geqslant b_i x + \delta x$ which implies $cx >_\delta b_i x$ for all $x \geqslant 1$. From the monotonicity with respect to $>_\delta$ we then conclude

$$q(cx) = p(cx, \ldots, cx) >_\delta p(b_1 x, \ldots, b_n x)$$

for all $x \geqslant 1$. Note that since $b_i \geqslant 1$ we have $b_i x >_\delta b_i y$ for all $x >_\delta y$. Hence the polynomial $p(b_1 x, \ldots, b_n x)$ is monotone with respect to $>_\delta$. This in turn implies that $p_1(b_1, \ldots, b_n) > 0$, since $p(b_1x, \ldots, b_nx) = p_1(b_1, \ldots, b_n)x^m + p_2(b_1x, \ldots, b_nx)$. Therefore

$$\begin{aligned} d = \deg(q(x)) &= \deg(q(cx)) \geqslant \deg(p(b_1x, \ldots, b_nx)) \\ &= \deg(p_1(b_1x, \ldots, b_nx) + p_2(b_1x, \ldots, b_nx)) \\ &= \deg(p_1(b_1, \ldots, b_n)x^m + p_2(b_1x, \ldots, b_nx)) = m \end{aligned}$$

Hence $d \geqslant m$ and $m \geqslant d$, and we have shown $\deg(q) = \deg(p)$. □

We need one further technical lemma on polynomials to establish the soundness of the reduction.

Lemma 6.7. *Let p and q be univariate polynomials over D where p is linear, $\deg(q) \geqslant 2$ and the leading coefficient of q is positive. Let $a, c \in D$ with $a \geqslant 1$ and $c > 0$. If $q(ax + c) - aq(x) \leqslant p(x)$ for all $x \geqslant 0$ then $\deg(q) = 2$ and $a = 1$.*

Proof. We first show that $a = 1$ by a proof by contradiction. So assume $a \neq 1$. From the assumption $a \geqslant 1$ we conclude $a > 1$ and thus the leading coefficient of $q(ax + c)$ is strictly larger than the one of $aq(x)$. In that case the leading coefficient of $q(ax+c) - aq(x)$ is positive and

$$2 \leqslant \deg(q(x)) = \deg(q(ax + c) - aq(x))$$

These two facts are in contradiction to the assumption that $q(ax + c) - aq(x)$ is bounded by the linear polynomial p. This completes the proof of $a = 1$.

Let $m = \deg(q)$. So $q(x) = \sum_{i=0}^{m} q_i x^i$ with $q_m > 0$. Hence

$$\begin{aligned} &q(ax + c) - aq(x) \\ &= q(x + c) - q(x) \\ &= \sum_{i=0}^{m} q_i (x + c)^i - \sum_{i=0}^{m} q_i x^i \\ &= \sum_{i=0}^{m} q_i \sum_{k=0}^{i} \binom{i}{k} x^k c^{i-k} - \sum_{i=0}^{m} q_i x^i \\ &= \sum_{i=0}^{m} q_i \left(x^i + \sum_{k=0}^{i-1} \binom{i}{k} x^k c^{i-k} \right) - \sum_{i=0}^{m} q_i x^i \\ &= \underbrace{\sum_{i=0}^{m} q_i \sum_{k=0}^{i-1} \binom{i}{k} x^k c^{i-k}}_{=:\, p'(x)} \end{aligned}$$

Since $q_m > 0$ and $c > 0$ we have $\deg(p') = m - 1$ and thus $1 \geqslant \deg(q(ax+c) - aq(x)) = m - 1$. This is only possible if $\deg(q) = m = 2$. □

Theorem 6.8. *If the TRS $\mathcal{Q}$ is polynomially terminating over D then $P(v_1, \ldots, v_n) \geqslant Q(v_1, \ldots, v_n)$ for some $v_1, \ldots, v_n \in \mathbb{N}_+$.*

Proof. We start similarly to the corresponding direction in the proof of Theorem 5.4. Assuming that $\mathcal{Q}$ is polynomially terminating over D we obtain interpretation functions in an algebra $\mathcal{D}$ that orient the rule. For convenience we write $\ell_{i\,\mathcal{D}}$ and $r_{i\,\mathcal{D}}$ for the polynomials $[\ell_i]_\mathcal{D}$ and $[r_i]_\mathcal{D}$ respectively. Applying Lemma 6.5 to the arguments of $\mathsf{f}_\mathcal{D}$, we obtain numbers $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_9 \in D_{\geqslant 0}$ such that

$$\begin{aligned} d &= \deg(\mathsf{f}_\mathcal{D}(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_9)) \\ &= \deg(\mathsf{f}_\mathcal{D}(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_9)) \end{aligned}$$

for some $d \in \mathbb{N}_+$ and all $a_j <_\delta b_j \in D$ with $j \neq i$. Therefore we have

$$\begin{aligned} d &= \deg(\mathsf{f}_\mathcal{D}(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_9)) \\ &= \deg(\mathsf{f}_\mathcal{D}(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_9)) \end{aligned}$$

for some $d \in \mathbb{N}_+$, where the numbers c_j and e_j are obtained by picking y_j large enough such that $c_j = \ell_{j\,\mathcal{D}}(y_j) >_\delta a_j$, and $e_j = r_{j\,\mathcal{D}}(y_j) >_\delta b_j$. Since the rule is oriented for all assignments of the variables $y_1, \ldots, y_9 \in D_{\geqslant 0}$ we also know that

$$\begin{aligned} &\deg(\mathsf{f}_\mathcal{D}(c_1, \ldots, c_{i-1}, \ell_{i\,\mathcal{D}}(x), c_{i+1}, \ldots c_9)) \\ &\geqslant \deg(\mathsf{f}_\mathcal{D}(e_1, \ldots, e_{i-1}, r_{i\,\mathcal{D}}(x), e_{i+1}, \ldots e_9)) \end{aligned}$$

This simplifies to $d \cdot \deg(\ell_{i\,\mathcal{D}}) \geqslant d \cdot \deg(r_{i\,\mathcal{D}})$ and further to $\deg(\ell_{i\,\mathcal{D}}) \geqslant \deg(r_{i\,\mathcal{D}})$.

For $i = 1$ this is $\deg(\mathsf{q}_\mathcal{D}(\mathsf{h}_\mathcal{D}(y_2))) \geqslant \deg(\mathsf{h}_\mathcal{D}(\mathsf{h}_\mathcal{D}(\mathsf{q}_\mathcal{D}(y_2))))$ which implies $\deg(\mathsf{h}_\mathcal{D}) = 1$. Together with Lemma 6.6 the inequalities for $i = 2$ and 3 then imply $\deg(\mathsf{g}_\mathcal{D}) = \deg(\mathsf{f}_\mathcal{D}) = 1$. Since $\mathsf{f}_\mathcal{D}$ is linear, compatibility with $\mathcal{Q}$ means

$$f_0 + \sum_{i=1}^{9} f_i \cdot \ell_{i\,\mathcal{D}}(y_i) >_\delta f_0 + \sum_{i=1}^{9} f_i \cdot r_{i\,\mathcal{D}}(y_i)$$

for all $y_1, \ldots, y_9 \in D_{\geqslant 0}$. By setting y_j to 0 for all $j \neq i$ we obtain inequalities of the shape $f_i \cdot \ell_{i\,\mathcal{D}}(y_i) + A_i >_\delta f_i \cdot r_{i\,\mathcal{D}}(y_i) + B_i$ for some $A_i, B_i \in D_{\geqslant 0}$. Dividing these by f_i we obtain inequalities of the form

$$\ell_{i\,\mathcal{D}}(y_i) \geqslant r_{i\,\mathcal{D}}(y_i) + C_i \tag{2}$$

for some $C_i \in D$.

So far we know that $h_\mathcal{D}(x) = h_1x + h_0$ and $g_\mathcal{D}(x, y) = g_2x + g_1y + g_0$ for some $g_2, g_1, h_1 \geqslant 1$ and $g_0, h_0 \geqslant 0$. From (2) for $i = 2$ we get

$$h_1y_2 + h_0 \geqslant (g_2 + g_1)y_2 + g_0 + C_2$$

for all $y_2 \in D_{\geqslant 0}$. Since this holds for all values of y_2, the leading coefficient on the left must be at least as large as the one on the right: $h_1 \geqslant g_2 + g_1 \geqslant 2$.

Looking at (2) for $i = 1$ we now can infer that $q_\mathcal{D}$ is at least quadratic, since if it were linear we would obtain the inequality

$$q_1h_1 \cdot x + q_1h_0 + q_0 \geqslant q_1h_1^2 \cdot x + h_1^2q_0 + h_1h_0 + h_0 + C_1$$

for all $x \in D_{\geqslant 0}$. This inequality implies $q_1h_1 \geqslant q_1h_1^2$, which in turn implies $h_1 \leqslant 1$, contradicting $h_1 \geqslant 2$.

We next infer $\deg(q_\mathcal{D}) = 2$ via Lemma 6.7. The preconditions of the lemma are shown as follows. By using (2) with $i = 4$, we obtain an inequality for all $y_4 \geqslant 0$ that can be rearranged to

$$\underbrace{g_1 \cdot [\mathsf{h}(\mathsf{h}(\mathsf{h}(y_4)))]_\mathcal{D} - C_4}_{=:\, p(y_4)} \geqslant q_\mathcal{D}(g_2y_4 + \underbrace{g_1[t]_\mathcal{D} + g_0}_{=:\, c}) - g_2q_\mathcal{D}(y_4)$$

with a linear polynomial p. The other non-trivial precondition we have to ensure for the application of Lemma 6.7 is $c > 0$, where we perform a proof by contradiction, so assume $c \leqslant 0$. Using $g_1, g_2 \geqslant 1$ and $g_0 \geqslant 0$ we then obtain $[t]_\mathcal{D} = 0$. Since $g_\mathcal{D}$ is linear, by the definition of t we next conclude $g_0 = [\mathsf{z}]_\mathcal{D} = [\mathsf{f}(\mathsf{z}, \ldots, \mathsf{z})]_\mathcal{D} =$

$[\mathsf{q}(\mathsf{z})]_{\mathcal{D}} = \cdots = [\mathsf{v}_n(\mathsf{z})]_{\mathcal{D}} = 0$. This implies $f_{\mathcal{D}}(0, \ldots, 0) = 0$ for each $f \in \mathcal{F}$, and thus, $[s]_{\mathcal{D}}(0, \ldots, 0) = 0$ for every term s. The latter fact is a contradiction to the strict orientation of the rule in $\mathcal{Q}$. This completes the proof by contradiction and also the proof of $\deg(q_{\mathcal{D}}) = 2$.

Having established $\deg(q_{\mathcal{D}}) = 2$, together with $\deg(\ell_{5\mathcal{D}}) \geqslant \deg(r_{5\mathcal{D}})$, $\deg(\ell_{6\mathcal{D}}) \geqslant \deg(r_{6\mathcal{D}})$, we obtain $\deg(\mathsf{a}_{\mathcal{D}}(y, y)) = 2$. Applying Lemma 6.6 we further have $\deg(\mathsf{a}_{\mathcal{D}}(x, y)) = 2$ with the shape $\mathsf{a}_{\mathcal{D}}(x, y) = a_5x^2 + a_4y^2 + a_3xy + a_2x + a_1y + a_0$. From (2) with $i = 7$ we know that

$$y_7 \geqslant a_5\mathsf{z}_{\mathcal{D}}^2 + a_4y_7^2 + a_3\mathsf{z}_{\mathcal{D}}y_7 + a_2\mathsf{z}_{\mathcal{D}} + a_1y_7 + a_0 + C_7$$

for all $y_7 \in D_{\geqslant 0}$, which is only possible for $a_4 = 0$. The same reasoning applied to (2) with $i = 8$ results in $a_5 = 0$. Therefore $\mathsf{a}_{\mathcal{D}}(x, y) = a_3xy + a_2x + a_1y + a_0$ and Lemma 6.2 is applicable, resulting in $\deg([\ulcorner P \urcorner]_{\mathcal{D}}) = P(\deg(\mathsf{v}_{1\mathcal{D}}), \ldots, \deg(\mathsf{v}_{n\mathcal{D}}))$ and $\deg([\ulcorner Q \urcorner]_{\mathcal{D}}) = Q(\deg(\mathsf{v}_{1\mathcal{D}}), \ldots, \deg(\mathsf{v}_{n\mathcal{D}}))$. Finally $\deg(\ell_{9\mathcal{D}}) \geqslant \deg(r_{9\mathcal{D}})$ implies

$$P(\deg(\mathsf{v}_{1\mathcal{D}}), \ldots, \deg(\mathsf{v}_{n\mathcal{D}})) \geqslant Q(\deg(\mathsf{v}_{1\mathcal{D}}), \ldots, \deg(\mathsf{v}_{n\mathcal{D}})) \qquad \square$$

Corollary 6.9. *Polynomial termination over $\mathbb{Q}$ and $\mathbb{R}$ is undecidable for one-rule* TRS*s.* □

7 FORMALIZATION

The sources of the formalization are available in the Archive of Formal Proofs (AFP) [37]. A browsable version is provided at

http://cl-informatik.uibk.ac.at/experiments/linear_undecidable/

which has been generated with Isabelle 2024. That website provides links to all theorems in this paper that are marked by ( ). Note that each Isabelle symbol at a theorem statement in this paper is a hyperlink that directly jumps to the correct point in the formalization.

At the start of the formalization task we actually had to choose which Isabelle library on polynomials to utilize. We briefly mention some available alternatives, both from the Isabelle distribution and from the AFP.

- The Isabelle distribution contains a type `'a poly` ( ) that represents univariate polynomials with coefficients of type `'a`.
- The AFP provides a type `'a mpoly` ( ) that captures multivariate polynomials [35]. Here the variables are always natural numbers.
- The same AFP entry also provides a type `('v,'a) tpoly` ( ) that captures multivariate polynomials with variables of type `'v`, coefficients of type `'a`.

The advantage of `'a poly` is that many results for this type of polynomials are available, and that one can nicely perform algebraic reasoning, but also syntactical reasoning: a polynomial of type `'a poly` can be converted into a list of coefficients (and vice versa), and lists are easy to manipulate algorithmically. Although `'a poly` is not sufficient enough to express the results of this paper, we frequently utilize the type `'a poly` whenever the reasoning is done on univariate polynomials.

Such a switch to `'a poly` is conveniently possible for the `'a mpoly` type; there are conversion functions between the two representations under the condition that a polynomial of type `'a mpoly` contains at most one variable, or by partially plugging in values in a multivariate polynomial for all but one variable. To this end, we also utilize further AFP entries, that provide more results about such conversions [11, 12]. Note that there are several facts for the type `'a mpoly` available, but the library is not as extensive as for `'a poly`. One other disadvantage is the fact that `'a mpoly` does not offer direct support for syntactic reasoning, i.e., recursion on the structure of a polynomial is not supported.

Such a syntactic reasoning is available however for the type `('v,'a) tpoly`, which is defined as an algebraic datatype with constructors for variables, constants, addition, and multiplication. However, algebraic reasoning is not well supported for `('v,'a) tpoly`, and one also would have to include normalization in the reasoning. For instance the polynomials $2 \cdot x$ and $x \cdot 2$ as elements of type `('v,'a) tpoly` differ, and they are not automatically normalized to the same representation.

Due to the pros and cons of the various representations, we decided to use the types `'a poly` and `'a mpoly` in our formalization.

The first problem that we tackled is the lacking ability to provide function definitions via structural recursion on `'a mpoly`. Such a mechanism is heavily required in particular in the encoding of polynomials as terms, cf. Definition 3.1. Internally, `'a mpoly` is represented as a dictionary from monomials to their coefficients, and similarly monomials are dictionaries from variables to their exponents. Technically, we build our construction on Isabelle's `sorted_list_of_set` ( ) function that converts any finite set into a list, provided that there is a linear order on the elements of the set. So, we first define a function `var_list` ( ) that given a monomial m returns a list of variables with their exponents: it is defined by applying `sorted_list_of_set` on the keys of m (then we have a list of variables of the monomial), and then look up all the exponents in the dictionary m. Hence, we can convert y^2x^3 into the list $[(x, 3), (y, 2)]$, assuming that the linear order is defined such that $x < y$. In the same way, `monom_list` ( ) converts a polynomial into a list of pairs of monomials with their coefficients.

Via `var_list` and `monom_list`, we obtain a syntactic representation of a polynomial, and can thus define the encoding of polynomials into terms in Definition 3.1 via this list-based representation ( ).

```
definition encode_num x n = ((λ t. a_t (Var x) t)^^n) z_t
definition encode_monom x m c = rec_list (encode_num x c)
 (λ (i,e) _. (λ t. v_t i t)^^e) (var_list m)
definition encode_poly x r = rec_list z_t
 (λ (m,c) _ t. a_t (encode_monom x m c) t) (monom_list r)
```

Here, `rec_list` is a recursor on lists, i.e., fold. Moreover, `a_t` t_1 t_2, `v_t` i t and `z_t` construct the terms $\mathsf{a}(t_1, t_2)$, $\mathsf{v}_i(t)$ and z, respectively.

Based on these functions, it is now easy to define the TRSs $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{Q}$ within Isabelle, and we do this within a context `poly_input` that fixes the two polynomials P and Q (as constants `p` and `q`), and assumes that these polynomials only have positive coefficients.

Within the `poly_input` context, all further main theorems are stated, e.g., the last corollary of Section 6 is stated as follows, where `Q` is the TRS $\mathcal{Q}$, `F_Q` is the signature $\mathcal{F}$ of $\mathcal{Q}$, and the type-variable `'a` is some arbitrary linearly ordered field that additionally provides a floor- and a ceiling-operation, i.e., it represents the choice of D ( ).

corollary `positive_poly_problem p q ⟷ termination_by_delta_poly_interpretation (TYPE('a :: floor_ceiling)) F_Q Q`

To arrive at this theorem, we actually not only had to formally verify all the proofs in this paper, but also had to significantly extend the library about polynomials, namely for properties that are usually taken for granted on paper. For instance, the proof that a polynomial in two variables of degree at most two must have the shape $a_5x^2 + a_4y^2 + a_3xy + a_2x + a_1y + a_0$ was a tedious endeavor, taking roughly 100 lines of Isabelle code (). The problem is that only very few facts are available that are connected to degrees of type `'a mpoly`. Overall, the formalization contains 1500 lines that solely extend the existing library on polynomials. Example properties would be the extensionality result of multivariate polynomials in rings of characteristic 0 (); the connection between inequalities such as $\forall x \geqslant b.p(x) \leqslant q(x)$ and the degrees of p and q (); and also even more basic properties, e.g., the definition of a degree function for monomials, and lemmas that state the relationships between monomial degrees and polynomial degrees.

After having proved all these preliminaries, the formalization actually closely mimics the definitions and proofs in this paper. There is only one deviation that is worth mentioning: for KBO we import the existing formalization of the AFP [34]. Here the formal version slightly deviates from what is presented in Section 2: the formal version also permits quasi-precedences and a few minor other extensions. However, this difference is negligible, as it was shown that the precedence plays no role for the undecidability result, and this is also true for the other extensions.

To summarize, the formalization covers all major results of this paper, closely follows the proofs in this paper, but required a significant extension of the library on polynomials. For all undecidability proofs we just prove the soundness part of the reductions, i.e., in particular we do not formally verify that the reductions are computable.

Technically, we heavily rely on the automation provided by Sledgehammer [5, 6], in particular to search and combine facts in the existing library in order to show obvious statements about polynomials.

We conclude this section with an overview on the size of the formalization:

- preliminaries on polynomials 1,589 lines
- Section 2 465 lines
- Section 3 1,485 lines
- Section 4 341 lines
- Section 5 1,545 lines
- Section 6 2,093 lines
- total size 7,518 lines

8 CONCLUSION

In this paper we have shown that the following properties of *one-rule* TRSs are undecidable:

(1) linear termination for polynomial terminating TRSs over $\mathbb{N}$,
(2) polynomial termination over $\mathbb{N}$ for ω-terminating TRSs,
(3) polynomial termination over $\mathbb{Q}$ and $\mathbb{R}$,
(4) KBO termination with subterm coefficients.

All results have been formally verified in Isabelle/HOL.

Contrary to polynomial termination over $\mathbb{N}$, we cannot limit the interpretations to linear polynomials and still remain undecidable for $\mathbb{R}$. In fact for any fixed upper bound on the degree of the interpretations, we can decide polynomial termination using Tarski's quantifier elimination procedure [36] or Collins' cylindrical algebraic decomposition [8]. For $\mathbb{Q}$ the question of decidability for linear (or otherwise bounded) interpretations is still open.

Polynomial interpretations can be used in an incremental fashion, extending their termination proving power. The idea, which goes back to Lankford [23], is that in a first step a polynomial interpretation over $\mathbb{N}$ is used that orients all rewrite rules of a given TRS $\mathcal{R}$ weakly and at least one rule strictly. After removing the rules that are strictly oriented, the process is repeated. When no rule remains, the incremental termination proof succeeds. In this case, $\mathcal{R}$ is called *incremental polynomially terminating* over $\mathbb{N}$ (*IPT* for short). The first two authors [31, Corollary 22] proved that PT is undecidable for systems that are IPT. Moreover, they showed that IPT is an undecidable property of terminating TRSs [31, Corollary 36]. Since PT and IPT coincide for one-rule TRSs, the latter result is an immediate consequence of (2). In addition, from (1) we obtain that ILT is an undecidable property of IPT systems. (Here ILT denotes IPT with the polynomial interpretations restricted to linear ones.)

It is unknown whether our results can be strengthened to one-rule string rewrite systems. In the introduction we already remarked that termination of one-rule string rewrite systems is an open problem. Matrix [13] and arctic [20] interpretations, which are heavily used in termination tools, can be viewed as linear interpretations (over different well-founded domains). Exploring these methods from a decidability viewpoint is an interesting topic for future research.

Regarding the formalization, it would also be interesting to connect our development with the existing formalization of Matiyasevich's proof of the DPRM theorem [3, 4] on the undecidability of Hilbert's 10th problem. This is not immediate since polynomials in that AFP-entry include a monus-operation on natural numbers.

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