Logical Methods in Computer Science
Volume 21, Issue 2, 2025, pp. 29:1–29:22
https://lmcs.episciences.org/

Submitted Mar. 06, 2024
Published Jun. 27, 2025

HYDRA BATTLES AND AC TERMINATION

NAO HIROKAWA [a] AND AART MIDDELDORP [b]

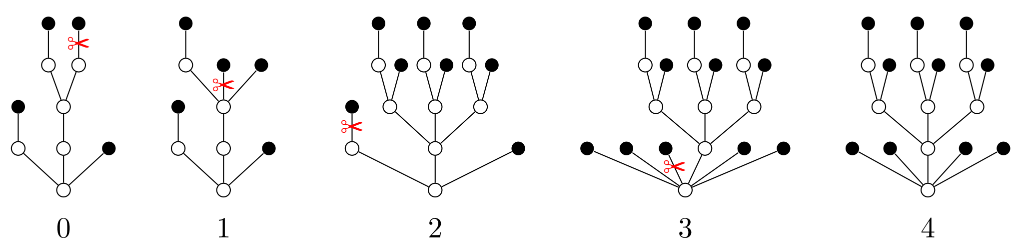
[a] School of Information Science, JAIST, Japan
e-mail address: hirokawa@jaist.ac.jp

[b] Department of Computer Science, University of Innsbruck, Austria
e-mail address: aart.middeldorp@uibk.ac.at

Abstract. We present a new encoding of the Battle of Hercules and Hydra as a rewrite system with AC symbols. Unlike earlier term rewriting encodings, it faithfully models any strategy of Hercules to beat Hydra. To prove the termination of our encoding, we employ type introduction in connection with many-sorted semantic labeling for AC rewriting and AC-MPO, a new AC compatible reduction order that can be seen as a much weakened version of AC-RPO.

1. Introduction

The mythological monster Hydra is a dragon-like creature with multiple heads. Whenever Hercules in his fight chops off a head, more and more new heads can grow instead, since the beast gets increasingly angry. Here we model a Hydra as an unordered tree. If Hercules cuts off a leaf corresponding to a head, the tree is modified in the following way: If the cut-off node h has a grandparent n, then the branch from n to the parent of h gets multiplied, where the number of copies corresponds to the number of decapitations so far. Hydra dies if there are no heads left, in that case Hercules wins. The following sequence shows an example fight:



Though the number of heads can grow considerably in one step, it turns out that the fight always terminates, and Hercules will win independent of his strategy. Proving termination

Key words and phrases: battle of Hercules and Hydra, term rewriting, AC termination.

The first author is supported by JSPS KAKENHI Grant Number JP22K11900. Part of this work was performed when the second author was employed at the Future Value Creation Research Center of Nagoya University, Japan.

DOI:10.46298/LMCS-21(2:29)2025

of the Battle is challenging since Kirby and Paris proved in their landmark paper [KP82] that termination for an arbitrary (computable) strategy is independent of Peano arithmetic. In [KP82] a termination argument based on ordinals is used.

Starting with [DJ90, p. 271], several TRS encodings of the Battle of Hercules and Hydra have been proposed and studied [Buc06, DM07, FZ96, Mos09, Tou98]. Touzet [Tou98] was the first to give a rigorous termination proof and in [ZWM15] the automation of ordinal interpretations is discussed. In this article we present yet another encoding. In contrast to earlier TRS encodings that model a specific strategy, it uses AC matching to represent *arbitrary* battles. To prove its termination, we adapt existing termination methods for AC rewriting.

The remainder of the article is organized as follows. After recalling some basic definitions in Section 2, we present our new encoding of the Battle in Section 3. We give a rigorous proof that our encoding faithfully represents the Battle. In Section 4 we present many-sorted semantic labeling for AC rewriting and apply it to our encoding. This results in an infinite AC rewrite system, which can be shown terminating by Rubio's AC-RPO [Rub02]. As a matter of fact, we do not need the full power of AC-RPO. Inspired by Steinbach's AC-KBO [Ste90], in Section 5 we introduce AC-MPO, a much weakened version of AC-RPO, and show that it is powerful enough for our purpose. Some of the properties of AC-MPO are proved in the appendix.

Related work is discussed in Section 6. In particular, we comment on earlier encodings of the Battle. We conclude in Section 7 with suggestions for future research.

A preliminary version of this article appeared in the proceedings of the 8th International Conference on Formal Structures for Computation and Deduction [HM23]. AC-MPO is a new result. New examples provide further illustration of the simulation of the Battle of Hercules and Hydra.

2. Preliminaries

Let $\mathcal{S}$ be a set of *sorts*. An *$\mathcal{S}$-sorted signature* $\mathcal{F}$ consists of function symbols f having a sort declaration $S_1 \times \cdots \times S_n \to S$. Here $S_1, \ldots, S_n$ and S are sorts in $\mathcal{S}$ and n is the *arity* of f. By $f^{(n)}$ we indicate that f has arity n. Let $\mathcal{V}$ be a countably infinite set of variables, where every variable has its own sort. We assume the existence of infinitely many variables of each sort. *Terms* of sort S are inductively defined as usual: Every variable of sort S is a term of sort S and if f has sort declaration $S_1 \times \cdots \times S_n \to S$ and t_i is a term of sort S_i for all $1 \leqslant i \leqslant n$ then $f(t_1, \ldots, t_n)$ is a term of sort S. *Ground terms* are terms without variables. The *root symbol* $\mathsf{root}(t)$ of a term t is t if it is a variable, and f if $t = f(t_1, \ldots, t_n)$. For every sort S we introduce a fresh constant $\square_S$, called the *hole*. A term over $\mathcal{F} \uplus \{\square_S \mid S \in \mathcal{S}\}$ is a *context* over $\mathcal{F}$ if it contains exactly one hole. Given a context C and a term t, we write $C[t]$ for the term resulting from replacing the hole in C by t. We write $s \trianglelefteq t$ if $t = C[s]$ for some context C. We write $s \triangleleft t$ if $s \trianglelefteq t$ and $s \neq t$. A mapping σ that associates each variable to a term of the same sort is a *substitution* if its domain $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. The application $t\sigma$ of σ to a term t is defined as $\sigma(t)$ if t is a variable and $f(t_1\sigma, \ldots, t_n\sigma)$ if $t = f(t_1, \ldots, t_n)$. A binary relation $\to$ on terms is *closed under substitutions* if $s\sigma \to t\sigma$ whenever $s \to t$, for all substitutions σ. It is *closed under contexts* if $C[s] \to C[t]$ whenever $s \to t$, for all contexts C. It has the *subterm property* if the inclusion $\triangleright \subseteq \to$ holds. Moreover, the relation $\to$ is said to be a *rewrite relation* if it is closed under contexts and substitutions. *Rewrite orders* are rewrite relations that are strict orders, and *reduction orders* are rewrite orders that are well-founded.

A *rewrite rule* $\ell \to r$ consists of two terms ℓ and r of the same sort such that all variables in r occur in ℓ. A (many-sorted) *term rewrite system* (TRS) is a set of rewrite rules. We denote by $\to_{\mathcal{R}}$ the smallest rewrite relation that contains the pairs of the TRS $\mathcal{R}$. A rule $\ell \to r$ is *non-collapsing* if r is not a variable. A TRS is called *non-collapsing* if all rules are non-collapsing. A TRS $\mathcal{R}$ is *terminating* if $\to_{\mathcal{R}}$ is well-founded.

Let $\mathcal{F}_{\mathsf{AC}}$ be a subset of the binary function symbols in $\mathcal{F}$ that have sort declarations of the form $S \times S \to S$. We denote by AC the set of equations

$$f(f(x,y),z) \approx f(x,f(y,z)) \qquad\qquad f(x,y) \approx f(y,x)$$

expressing the associativity and commutativity of each $f \in \mathcal{F}_{\mathsf{AC}}$. Since equations in AC are rewrite rules, we can view AC as a TRS. Using this fact, we define the relation $=_{\mathsf{AC}}$ as the reflexive, transitive, and symmetric closure of $\to_{\mathsf{AC}}$. Let $\mathcal{R}$ be a TRS. The relation $=_{\mathsf{AC}} \cdot \to_{\mathcal{R}} \cdot =_{\mathsf{AC}}$ is called AC rewriting and abbreviated by $\to_{\mathcal{R}/\mathsf{AC}}$. We say that $\mathcal{R}$ is *AC terminating* if $\to_{\mathcal{R}/\mathsf{AC}}$ is well-founded. A reduction order $>$ is *AC-compatible* if the inclusion $=_{\mathsf{AC}} \cdot > \cdot =_{\mathsf{AC}} \subseteq >$ holds. AC termination of a TRS $\mathcal{R}$ can be shown by finding an AC-compatible reduction order such that $\mathcal{R} \subseteq >$ holds.

The above definitions specialize to the usual unsorted setting when the set of sorts is a singleton set.

Finally, we recall two order extensions. Let $>$ be a strict order on a set A. The *lexicographic extension* $>^{\mathsf{lex}}$ of $>$ is defined on tuples over A as follows: $(a_1,\ldots,a_m) >^{\mathsf{lex}} (b_1,\ldots,b_n)$ if $n = m$ and there exists an index $1 \leqslant k \leqslant n$ such that $a_k > b_k$ and $a_i = b_i$ for all $i < k$. The *multiset extension* $>^{\mathsf{mul}}$ of $>$ is defined on multisets over A as follows: $M >^{\mathsf{mul}} N$ if there exist multisets X and Y such that $N = (M - X) \uplus Y$, $\varnothing \neq X \subseteq M$, and every $b \in Y$ admits an element $a \in X$ with $a > b$.

3. Encoding

First we give a formal account of the Hydra Battle.

Definition 3.1. To represent Hydras, we use a signature containing a constant symbol h representing a head, a binary symbol $|$ for siblings, and a unary function symbol i representing the internal nodes. We use infix notation for $|$ and declare it to be an AC symbol. We write $\mathcal{T}_{\mathcal{H}}$ for the set of ground terms over $\{\mathsf{h},\mathsf{i},|\}$. *Encodings* of Hydras are terms t in $\mathcal{T}_{\mathcal{H}}$ with $\mathsf{root}(t) \in \{\mathsf{h},\mathsf{i}\}$.

To improve readability we omit parentheses in terms with nested $|$ symbols in examples.

Example 3.2. The Hydras in the above example fight are represented by the terms

$$\begin{aligned}
H_0 &= \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{h}) \\
H_1 &= \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h} \mid \mathsf{h})) \mid \mathsf{h}) \\
H_2 &= \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \mid \mathsf{h}) \\
H_3 &= \mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \mid \mathsf{h} \mid \mathsf{h}) \\
H_4 &= \mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \mid \mathsf{h} \mid \mathsf{h})
\end{aligned}$$

and they are encodings of Hydras. The term $\mathsf{h} \mid \mathsf{h}$ is included in $\mathcal{T}_{\mathcal{H}}$ but not regarded as an encoding of a Hydra.

Definition 3.3. Let n be a natural number. The TRS $\mathcal{R}_n$ operates on encodings of Hydras and consists of the following four rules:

$$\begin{aligned} \mathsf{i}(\mathsf{i}(\mathsf{h})) &\xrightarrow{1} \mathsf{i}(\mathsf{h}^{n+2}) & \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid y) &\xrightarrow{3} \mathsf{i}(\mathsf{h}^{n+2} \mid y) \\ \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x)) &\xrightarrow{2} \mathsf{i}(\mathsf{i}(x)^{n+2}) & \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x) \mid y) &\xrightarrow{4} \mathsf{i}(\mathsf{i}(x)^{n+2} \mid y) \end{aligned}$$

Here t^k for $k \geqslant 1$ is defined inductively as follows:

$$t^k = \begin{cases} t & \text{if } k = 1 \\ t^{k-1} \mid t & \text{if } k > 1 \end{cases}$$

The transition relation $\Rightarrow_n$ on encodings of Hydras is defined as follows: $H \Rightarrow_n H'$ if

(1) $H = \mathsf{i}(\mathsf{h})$ and $H' = \mathsf{h}$, or
(2) $H =_{\mathsf{AC}} \mathsf{i}(\mathsf{h} \mid t)$ and $H' = \mathsf{i}(t)$ for some term t, or
(3) $H \to_{\mathcal{R}_n/\mathsf{AC}} H'$.

That H and H' are encodings of successive Hydras at stages n and $n+1$ in a battle is expressed as $H \Rightarrow_n H'$. So fights with Hydras are represented by finite or infinite sequences of the form $H_0 \Rightarrow_0 H_1 \Rightarrow_1 H_2 \Rightarrow_2 \cdots$.

Example 3.4 (continued from Example 3.2)**.** We have the following sequence:

$$H_0 \Rightarrow_0 H_1 \Rightarrow_1 \cdots \Rightarrow_3 H_4$$

For instance, the first step is verified as follows: Let $\ell \to r$ be the third rule in $\mathcal{R}_0$, and let $C = \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\Box) \mid \mathsf{h})$ and $\sigma = \{y \mapsto \mathsf{i}(\mathsf{h})\}$. Since $H_0 = C[\ell\sigma]$ and $H_1 =_{\mathsf{AC}} C[r\sigma]$ hold, we have $H_0 \to_{\mathcal{R}_0/\mathsf{AC}} H_1$ and so $H_0 \Rightarrow_0 H_1$ is obtained.

Now we present our TRS encoding of the Hydra Battle. We represent natural numbers n by $\mathsf{s}^n(\mathsf{0})$, which is abbreviated to n.

Definition 3.5. Let $\mathcal{F}$ be the signature consisting of the constant $\mathsf{0}$, the unary symbol s, the five binary symbols A – E as well as the three symbols h, i, and $\mid$ in Definition 3.1. The TRS $\mathcal{H}$ over $\mathcal{F}$ consists of the following 14 rewrite rules:

$$\begin{aligned} \mathsf{A}(n, \mathsf{i}(\mathsf{h})) &\xrightarrow{1} \mathsf{A}(\mathsf{s}(n), \mathsf{h}) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x))) &\xrightarrow{8} \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x))) \\ \mathsf{A}(n, \mathsf{i}(\mathsf{h} \mid x)) &\xrightarrow{2} \mathsf{A}(\mathsf{s}(n), \mathsf{i}(x)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x) \mid y)) &\xrightarrow{9} \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)) \mid y) \\ \mathsf{A}(n, \mathsf{i}(x)) &\xrightarrow{3} \mathsf{B}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x))) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x) \mid y)) &\xrightarrow{10} \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x)) \mid y) \\ \mathsf{C}(\mathsf{0}, x) &\xrightarrow{4} \mathsf{E}(x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x))) &\xrightarrow{11} \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x))) \\ \mathsf{C}(\mathsf{s}(n), x) &\xrightarrow{5} x \mid \mathsf{C}(n, x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid y)) &\xrightarrow{12} \mathsf{i}(\mathsf{C}(n, \mathsf{h}) \mid y) \\ \mathsf{i}(\mathsf{E}(x) \mid y) &\xrightarrow{6} \mathsf{E}(\mathsf{i}(x \mid y)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}))) &\xrightarrow{13} \mathsf{i}(\mathsf{C}(n, \mathsf{h})) \\ \mathsf{i}(\mathsf{E}(x)) &\xrightarrow{7} \mathsf{E}(\mathsf{i}(x)) & \mathsf{B}(n, \mathsf{E}(x)) &\xrightarrow{14} \mathsf{A}(\mathsf{s}(n), x) \end{aligned}$$

The Battle is started with the term $\mathsf{A}(\mathsf{0}, H)$ where H is the encoding of the initial Hydra. Rule 1 takes care of the dying Hydra ○—●. An application of the rule ends the battle with a term of the form $\mathsf{A}(n, \mathsf{h})$. The h denotes here a dead Hydra; a Hydra with only one head is represented by the term $\mathsf{i}(\mathsf{h})$. Rule 2 cuts a head without grandparent node, and so no copying takes place. Due to the power of AC matching, the removed head need not be the

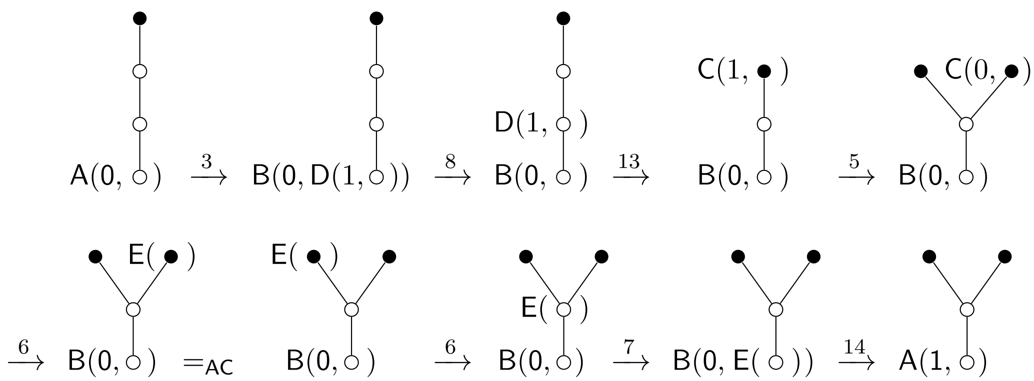


Figure 1: Rewriting from $\mathsf{A}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}))))$ to $\mathsf{A}(\mathsf{1},\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h})))$.

leftmost one. With rule 3, the search for locating a head with grandparent node starts. The search is performed with the auxiliary symbol D and involves rules 8–13. When the head to be cut is located (in rules 10–13), copying begins with the auxiliary symbol C and rules 4 and 5. The end of the copying phase is signaled with E, which travels upwards with rules 6 and 7. Finally, rule 14 creates the next stage of the Battle. Note that we make extensive use of AC matching to simplify the search process.

Theorem 3.6. *Let n be a natural number. If $H \Rightarrow_n H'$ then* $\mathsf{A}(\mathsf{n}, H) \to^{+}_{\mathcal{H}/\mathsf{AC}} \mathsf{A}(\mathsf{s}(\mathsf{n}), H')$.

Before presenting the proof, we illustrate how AC rewriting of $\mathcal{H}$ simulates fights with Hydras.

Example 3.7. Consider a fight with the Hydra of shape $\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h})))$. The fight starts with the transition from $\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h})))$ to $\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h}))$. This is simulated by the rewrite sequence

$$\begin{array}{llll}
& \mathsf{A}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h})))) & \xrightarrow{3} \mathsf{B}(\mathsf{0},\mathsf{D}(\mathsf{s}(\mathsf{0}),\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h}))))) & \xrightarrow{8} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{D}(\mathsf{s}(\mathsf{0}),\mathsf{i}(\mathsf{i}(\mathsf{h}))))) \\
\xrightarrow{13} & \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{0}),\mathsf{h})))) & \xrightarrow{5} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(\mathsf{0},\mathsf{h})))) & \xrightarrow{4} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h})))) \\
=_{\mathsf{AC}} & \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{h}) \mid \mathsf{h}))) & \xrightarrow{6} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h})))) & \xrightarrow{7} \mathsf{B}(\mathsf{0},\mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h})))) \\
\xrightarrow{14} & \mathsf{A}(\mathsf{s}(\mathsf{0}),\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h}))) & &
\end{array}$$

which is visualized in Figure 1. Rules 9–12 are variations of 8 and 13, which are used for handling nodes that have siblings. To illustrate these, consider the first step $H_0 \Rightarrow_0 H_1$ in the example fight in the introduction. The step is simulated by the following rewrite sequence:

$$\begin{array}{rl}
\mathsf{A}(\mathsf{0}, H_0) & \xrightarrow{3} \mathsf{B}(\mathsf{0},\mathsf{D}(\mathsf{s}(\mathsf{0}), H_0)) \\
=_{\mathsf{AC}} \cdot & \xrightarrow{9} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{D}(\mathsf{s}(\mathsf{0}),\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
& \xrightarrow{8} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(\mathsf{0}),\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
& \xrightarrow{12} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{0}),\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
& \xrightarrow{5} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(\mathsf{0},\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\
& \xrightarrow{4} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))
\end{array}$$

$$=_{\mathsf{AC}} \cdot \xrightarrow{6} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))$$
$$\xrightarrow{7} \mathsf{B}(\mathsf{0},\mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))$$
$$\xrightarrow{6} \mathsf{B}(\mathsf{0},\mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})))$$
$$\xrightarrow{14} \mathsf{A}(\mathsf{s}(\mathsf{0}),\mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h} \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) =_{\mathsf{AC}} \mathsf{A}(\mathsf{s}(\mathsf{0}),H_1)$$

It is important to note that the TRS $\mathcal{H}$ defined above is *unsorted* and we establish in this article the result that it is AC terminating on all terms. When simulating a battle, like in the statement of the Theorem 3.6, we deal with well-behaved terms adhering to the sort discipline introduced shortly. The restriction to sorted terms is crucial for our termination proof, but entails no loss of generality. This is due to the following result, which is a special case of [MO00, Corollary 3.9].

Theorem 3.8. *A non-collapsing* TRS *over a many-sorted signature is* AC *terminating if and only if the corresponding* TRS *over the unsorted version of the signature is* AC *terminating.* □

The idea of using sorts to simplify termination proof goes back to Zantema [Zan94]. The TRS $\mathcal{H}$ can be seen as a TRS over the many-sorted signature $\mathcal{F}'$:

$\mathsf{h} : \mathsf{O}$	$\mathsf{i},\mathsf{E} : \mathsf{O} \to \mathsf{O}$	$\mid\, : \mathsf{O} \times \mathsf{O} \to \mathsf{O}$	$\mathsf{A},\mathsf{B} : \mathsf{N} \times \mathsf{O} \to \mathsf{S}$
$\mathsf{0} : \mathsf{N}$	$\mathsf{s} : \mathsf{N} \to \mathsf{N}$		$\mathsf{C},\mathsf{D} : \mathsf{N} \times \mathsf{O} \to \mathsf{O}$

where N, O and S are sort symbols. Since $\mathcal{H}$ is non-collapsing, Theorem 3.8 guarantees that AC termination of $\mathcal{H}$ follows from AC termination of well-sorted terms over $\mathcal{F}'$.

In the remainder of this section we present a proof of Theorem 3.6 and its converse.

Lemma 3.9. *If $n > 0$ then $\mathsf{C}(\mathsf{n},t) \to^*_{\mathcal{H}/\mathsf{AC}} t^n \mid \mathsf{E}(t)$ for all terms t.*

Proof. We use induction on n. If $n = 1$ then

$$\mathsf{C}(\mathsf{n},t) \xrightarrow{5} t \mid \mathsf{C}(\mathsf{0},t) \xrightarrow{6} t \mid \mathsf{E}(t) = t^n \mid \mathsf{E}(t)$$

Suppose the result holds for $n \geqslant 1$ and consider $n+1$. The induction hypothesis yields $\mathsf{C}(\mathsf{n},t) \to^*_{\mathcal{H}/\mathsf{AC}} t^n \mid \mathsf{E}(t)$. Hence

$$\mathsf{C}(\mathsf{s}(\mathsf{n}),t) \xrightarrow{5} t \mid \mathsf{C}(\mathsf{n},t) \to^*_{\mathcal{H}/\mathsf{AC}} t \mid (t^n \mid \mathsf{E}(t)) =_{\mathsf{AC}} t^{n+1} \mid \mathsf{E}(t)$$ □

Lemma 3.10. *Let n be a natural number. If $H \Rightarrow_n H'$ then $\mathsf{D}(\mathsf{s}(\mathsf{n}),H) \to^*_{\mathcal{H}/\mathsf{AC}} \mathsf{E}(H')$.*

Proof. We use structural induction on H and consider the following two cases.

- First suppose $H \to_{\mathcal{R}_n/\mathsf{AC}} H'$ is a root step. If the first rule of $\mathcal{R}_n$ is used then $H = \mathsf{i}(\mathsf{i}(\mathsf{h}))$ and $H' =_{\mathsf{AC}} \mathsf{i}(h^{n+2})$. We have $\mathsf{D}(\mathsf{s}(\mathsf{n}),H) \xrightarrow{13} \mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}),\mathsf{h}))$. Using Lemma 3.9 we obtain

  $$\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}),\mathsf{h})) \to^*_{\mathcal{H}/\mathsf{AC}} \mathsf{i}(\mathsf{h}^{n+1} \mid \mathsf{E}(\mathsf{h})) =_{\mathsf{AC}} \cdot \xrightarrow{6} \mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h}^{n+1})) =_{\mathsf{AC}} \mathsf{E}(H')$$

  If the second rule of $\mathcal{R}_n$ is used then $H =_{\mathsf{AC}} \mathsf{i}(\mathsf{i}(\mathsf{h} \mid t))$ and $H' =_{\mathsf{AC}} \mathsf{i}(\mathsf{i}(t)^{n+2})$ for some term t. We have $\mathsf{D}(\mathsf{s}(\mathsf{n}),H) =_{\mathsf{AC}} \cdot \xrightarrow{11} \mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}),\mathsf{i}(t)))$. Using Lemma 3.9 we obtain

  $$\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}),\mathsf{i}(t))) \to^*_{\mathcal{H}/\mathsf{AC}} \mathsf{i}(\mathsf{i}(t)^{n+1} \mid \mathsf{E}(\mathsf{i}(t))) =_{\mathsf{AC}} \cdot \xrightarrow{6} \mathsf{E}(\mathsf{i}(\mathsf{i}(t) \mid \mathsf{i}(t)^{n+1})) =_{\mathsf{AC}} \mathsf{E}(H')$$

  If the third rule of $\mathcal{R}_n$ is used then $H =_{\mathsf{AC}} \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid t)$ and $H' =_{\mathsf{AC}} \mathsf{i}(\mathsf{h}^{n+2} \mid t)$ for some term t. We have $\mathsf{D}(\mathsf{s}(\mathsf{n}),H) =_{\mathsf{AC}} \cdot \xrightarrow{12} \mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}),\mathsf{h}) \mid t)$. The remaining argument is the same as in the preceding cases. If the fourth rule of $\mathcal{R}_n$ is used then $H =_{\mathsf{AC}} \mathsf{i}(\mathsf{i}(\mathsf{h} \mid s) \mid t)$ and $H' =_{\mathsf{AC}} \mathsf{i}(\mathsf{i}(s)^{n+2} \mid t)$ for some terms s and t. Using Lemma 3.9 we obtain

$$\begin{aligned}\mathsf{D}(\mathsf{s}(\mathsf{n}), H) &=_{\mathsf{AC}} \cdot \xrightarrow{10} \mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}), \mathsf{i}(s)) \mid t) \rightarrow^*_{\mathcal{H}/\mathsf{AC}} \mathsf{i}((\mathsf{i}(s)^{n+1} \mid \mathsf{E}(\mathsf{i}(s))) \mid t) \\ &=_{\mathsf{AC}} \cdot \xrightarrow{6} \mathsf{E}(\mathsf{i}(\mathsf{i}(s) \mid (\mathsf{i}(s)^{n+1} \mid t))) =_{\mathsf{AC}} \mathsf{E}(H')\end{aligned}$$

- Otherwise, $H =_{\mathsf{AC}} \mathsf{i}(H_1 \mid H_2 \mid \cdots \mid H_m)$ and $H' =_{\mathsf{AC}} \mathsf{i}(H_1' \mid H_2 \mid \cdots \mid H_m)$ for some $m \geqslant 1$ and Hydras $H_1, \dots, H_m, H_1'$ with $H_1 \rightarrow_{\mathcal{R}_n/\mathsf{AC}} H_1'$. We obtain $\mathsf{D}(\mathsf{s}(\mathsf{n}), H_1) \rightarrow^*_{\mathcal{H}/\mathsf{AC}} \mathsf{E}(H_1')$ from the induction hypothesis. Note that $\mathsf{root}(H_1) = \mathsf{i}$. If $m = 1$ then

$$\begin{aligned}\mathsf{D}(\mathsf{s}(\mathsf{n}), H) &=_{\mathsf{AC}} \mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(H_1)) \xrightarrow{8} \mathsf{i}(\mathsf{D}(\mathsf{s}(\mathsf{n}), H_1)) \rightarrow^*_{\mathcal{H}/\mathsf{AC}} \mathsf{i}(\mathsf{E}(H_1')) \xrightarrow{7} \mathsf{E}(\mathsf{i}(H_1')) \\ &=_{\mathsf{AC}} \mathsf{E}(H')\end{aligned}$$

and if $m > 1$ we reach the same conclusion using rules 9 and 6 instead of 8 and 7. □

Proof of Theorem 3.6. Our task is to show

$$\mathsf{A}(\mathsf{n}, H) \rightarrow^*_{\mathcal{H}/\mathsf{AC}} \mathsf{A}(\mathsf{s}(\mathsf{n}), H')$$

If $H \Rightarrow_n H'$ is derived from condition (1) or (2) in Definition 3.3, the claim is immediate by rules 1 and 2 of $\mathcal{H}$. Otherwise, $H \rightarrow_{\mathcal{R}_n/\mathsf{AC}} H'$. This implies $\mathsf{root}(H) = \mathsf{i}$. Using rules 3 and 14 together with Lemma 3.10 yields

$$\mathsf{A}(\mathsf{n}, H) \xrightarrow{3} \mathsf{B}(\mathsf{n}, \mathsf{D}(\mathsf{s}(\mathsf{n}), H)) \rightarrow^*_{\mathcal{H}/\mathsf{AC}} \mathsf{B}(\mathsf{n}, \mathsf{E}(H')) \xrightarrow{14} \mathsf{A}(\mathsf{s}(\mathsf{n}), H')$$ □

In the remaining part of this section we prove the converse of Theorem 3.6.

Theorem 3.11. *Let $H, H' \in \mathcal{T}_{\mathcal{H}}$ be encodings of Hydras and let n be a natural number. If $\mathsf{A}(\mathsf{n}, H) \rightarrow^*_{\mathcal{H}/\mathsf{AC}} \mathsf{A}(\mathsf{s}(\mathsf{n}), H')$ then $H \Rightarrow_n H'$.*

In order to show the claim we need a few auxiliary lemmata. Let $\mathcal{C}_{\mathcal{H}}$ be the set of ground contexts over $\{\mathsf{h}, \mathsf{i}, \mid\}$.

Definition 3.12. We define U as the set consisting of all terms of the forms $\mathsf{A}(\mathsf{n}, t)$, $\mathsf{B}(\mathsf{n}, C[\mathsf{C}(\mathsf{m}, t)])$, $\mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), t)])$, and $\mathsf{B}(\mathsf{n}, C[\mathsf{E}(t)])$, where $n, m \in \mathbb{N}$, $t \in \mathcal{T}_{\mathcal{H}}$, and $C \in \mathcal{C}_{\mathcal{H}}$.

The set U contains all terms reachable from $\mathsf{A}(\mathsf{n}, H)$.

Lemma 3.13. *If $t \in U$ and $t \rightarrow^*_{\mathcal{H} \cup \mathsf{AC}} u$ then $u \in U$.*

Proof. The claim is easily shown by induction on the length of $t \rightarrow^*_{\mathcal{H} \cup \mathsf{AC}} u$. □

In order to analyze the rewrite sequence $\mathsf{A}(\mathsf{n}, H) \rightarrow^*_{\mathcal{H}/\mathsf{AC}} \mathsf{A}(\mathsf{s}(\mathsf{n}), H')$ we define three subsets of $\mathcal{H}$: $\mathcal{H}_1 = \{1, 2\}$, $\mathcal{H}_2 = \{3\text{–}9, 14\}$, and $\mathcal{H}_3 = \{10\text{–}13\}$. The second rewrite sequence in Example 3.7 can then be described as follows:

$$\begin{aligned}\mathsf{A}(0, H_0) &\rightarrow^*_{\mathcal{H}_2/\mathsf{AC}} \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\ &\rightarrow_{\mathcal{H}_3/\mathsf{AC}} \mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h})) \\ &\rightarrow^*_{\mathcal{H}_2/\mathsf{AC}} \mathsf{A}(1, H_1)\end{aligned}$$

Definition 3.14. We define V as the extension of U with $\mathcal{T}_\mathcal{H}$ and all terms of the forms $C[\mathsf{C}(\mathsf{n},t)]$ $C[\mathsf{D}(\mathsf{n},t)]$, and $C[\mathsf{E}(t)]$ where $n \in \mathbb{N}$, $t \in \mathcal{T}_\mathcal{H}$, and $C \in \mathcal{C}_\mathcal{H}$. The mapping $\pi\colon V \to \mathcal{T}_\mathcal{H}$ is defined as follows:

$$\pi(t) = \begin{cases} \mathsf{h} & \text{if } t = \mathsf{h} \\ \mathsf{i}(\pi(u)) & \text{if } t = \mathsf{i}(u) \\ \pi(u) \mid \pi(v) & \text{if } t = u \mid v \\ u & \text{if } t = \mathsf{A}(\mathsf{n},u) \text{ or } t = \mathsf{D}(\mathsf{n},u) \text{ or } t = \mathsf{E}(u) \\ \pi(u) & \text{if } t = \mathsf{B}(\mathsf{n},u) \\ u^{n+1} & \text{if } t = \mathsf{C}(\mathsf{n},u) \end{cases}$$

Taking the role of C into account, the mapping π computes the Hydra in a given term. Applying π to the terms in the above rewrite sequence of $\mathcal{H}_2/\mathsf{AC}$ and $\mathcal{H}_3/\mathsf{AC}$, we obtain

$$\begin{aligned} H_0 = \pi(\mathsf{A}(0, H_0)) &=_{\mathsf{AC}} \pi(\mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{h})))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))) \\ &\to_{\mathcal{R}_0/\mathsf{AC}} \pi(\mathsf{B}(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}) \mid \mathsf{i}(\mathsf{h}))) \mid \mathsf{i}(\mathsf{h}) \mid \mathsf{h}))) \\ &=_{\mathsf{AC}} \pi(\mathsf{A}(1, H_1)) = H_1 \end{aligned}$$

This verifies that H_1 is a successor of H_0.

Lemma 3.15. *The following properties hold.*

(1) $\pi(t) = t$ *for all terms* $t \in \mathcal{T}_\mathcal{H}$*,*
(2) $\pi(C[t]) = C[\pi(t)]$ *for all terms* $t \in V$ *and contexts* $C \in \mathcal{C}_\mathcal{H}$*,*
(3) $\pi(C[t]) =_{\mathsf{AC}} \pi(D[u])$ *for all terms* $t, u \in \mathcal{T}_\mathcal{H}$ *and contexts* $C, D \in \mathcal{C}_\mathcal{H}$ *with* $t =_{\mathsf{AC}} u$ *and* $C =_{\mathsf{AC}} D$*.*

Proof. The first statement is proved by induction on $t \in \mathcal{T}_\mathcal{H}$. If $t = \mathsf{h}$ then $\pi(t) = \mathsf{h} = t$. If $t = \mathsf{i}(u)$ with $u \in \mathcal{T}_\mathcal{H}$ then $\pi(t) = \mathsf{i}(\pi(u)) = \mathsf{i}(u) = t$. If $t = u \mid v$ with $u, v \in \mathcal{T}_\mathcal{H}$ then $\pi(t) = \pi(u) \mid \pi(v) = u \mid v = t$. For the second statement we use induction on the context $C \in \mathcal{C}_\mathcal{H}$. If $C = \square$ then $\pi(C[t]) = \pi(t) = C[\pi(t)]$. If $C = \mathsf{i}(D)$ then $\pi(C[t]) = \mathsf{i}(\pi(D[t])) = \mathsf{i}(D[\pi(t)]) = C[\pi(t)]$. If $C = D \mid u$ then $D \in \mathcal{C}_\mathcal{H}$ and $u \in \mathcal{T}_\mathcal{H}$ and thus $\pi(C[t]) = \pi(D[t]) | \pi(u) = D[\pi(t)] | u = C[\pi(t)]$. If $C = u|D$ then $D \in \mathcal{C}_\mathcal{H}$ and $u \in \mathcal{T}_\mathcal{H}$ and thus $\pi(C[t]) = \pi(u) \mid \pi(D[t]) = u \mid D[\pi(t)] = C[\pi(t)]$. The third statement follows from statements (1) and (2): $\pi(C[t]) = C[\pi(t)] = C[t] =_{\mathsf{AC}} D[t] =_{\mathsf{AC}} D[u] = D[\pi(u)] = \pi(D[u])$. ☐

The following lemma relates AC rewriting of $\mathcal{H}$ to rewriting of Hydras according to Definition 3.3.

Lemma 3.16. *The following statements hold for all terms* $s, t \in U$*.*

(1) *If* $s =_{\mathsf{AC}} t$ *then* $\pi(s) =_{\mathsf{AC}} \pi(t)$*.*
(2) *If* $s \to_{\mathcal{H}_2} t$ *then* $\pi(s) =_{\mathsf{AC}} \pi(t)$*.*
(3) *If* $s \to_{\mathcal{H}_3} t$ *then* $\pi(s) \to_{\mathcal{R}_n/\mathsf{AC}} \pi(t)$ *with* $s = \mathsf{B}(\mathsf{n}, s')$ *for some* $n \geqslant 0$*.*

Proof. Let $s, t \in U$.

(1) If $s = \mathsf{A}(\mathsf{n}, u)$ with $u \in \mathcal{T}_\mathcal{H}$ then $t = \mathsf{A}(\mathsf{n}, v)$ for some term $v \in \mathcal{T}_\mathcal{H}$ with $u =_{\mathsf{AC}} v$. Since $\pi(s) = u$ and $\pi(t) = v$, $\pi(s) =_{\mathsf{AC}} \pi(t)$ follows. If $s = \mathsf{B}(\mathsf{n}, C[\mathsf{C}(\mathsf{m}, u)])$ with $n, m \in \mathbb{N}$, $C \in \mathcal{C}_\mathcal{H}$ and $u \in \mathcal{T}_\mathcal{H}$ then $t = \mathsf{B}(\mathsf{n}, D[\mathsf{C}(\mathsf{m}, v)])$ with $C =_{\mathsf{AC}} D$ and $u =_{\mathsf{AC}} v$. Using Lemma 3.15(1,2) we obtain $\pi(s) = \pi(C[\mathsf{C}(\mathsf{m}, u)]) = C[\pi(\mathsf{C}(\mathsf{m}, u))] = C[u^{m+1}]$ and $\pi(t) = D[v^{m+1}]$. From $u =_{\mathsf{AC}} v$ we infer $u^{m+1} =_{\mathsf{AC}} v^{m+1}$ and thus $\pi(s) =_{\mathsf{AC}} \pi(t)$ by

Lemma 3.15(3). The cases $s = \mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), u)])$ and $s = \mathsf{B}(\mathsf{n}, C[\mathsf{E}(u)])$ are treated in the same way.

(2) For the second statement we make a case analysis based on the employed rule in $\mathcal{H}_2$.
- If $s \xrightarrow{3} t$ then $s = \mathsf{A}(\mathsf{n}, \mathsf{i}(u))$ and $t = \mathsf{B}(\mathsf{n}, \mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(u)))$ for some $n \geqslant 0$ and $u \in \mathcal{T}_\mathcal{H}$. We have $\pi(s) = \mathsf{i}(u) = \pi(\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(u))) = \pi(t)$ by the definition of t.
- If $s \xrightarrow{4} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{C}(0, u)])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{E}(u)])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u \in \mathcal{T}_\mathcal{H}$. We have $\pi(s) = \pi(C[\mathsf{C}(0, u)]) = C[u^1] = C[u] = \pi(C[u]) = \pi(t)$.
- If $s \xrightarrow{5} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{C}(\mathsf{s}(\mathsf{m}), u)])$ and $t = \mathsf{B}(\mathsf{n}, C[u \mid \mathsf{C}(\mathsf{m}, u)])$ for some $n \geqslant 0$, $m \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u \in \mathcal{T}_\mathcal{H}$. We have $\pi(s) = C[u^{m+2}] =_{\mathsf{AC}} C[u \mid u^{m+1}] = C[\pi(u \mid u^{m+1})] = \pi(C[u \mid \mathsf{C}(\mathsf{m}, u)]) = \pi(t)$.
- If $s \xrightarrow{6} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{E}(u)|v)])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{E}(\mathsf{i}(u|v))])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u, v \in \mathcal{T}_\mathcal{H}$. We have $\pi(s) = \pi(C[\mathsf{i}(\mathsf{E}(u) \mid v)]) = C[\mathsf{i}(u \mid v)] = \pi(C[\mathsf{E}(\mathsf{i}(u \mid v))]) = \pi(t)$.
- If $s \xrightarrow{7} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{E}(u))])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{E}(\mathsf{i}(u))])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u \in \mathcal{T}_\mathcal{H}$. We have $\pi(s) = \pi(C[\mathsf{i}(\mathsf{E}(u))]) = C[\mathsf{i}(u)] = \pi(C[\mathsf{E}(\mathsf{i}(u))]) = \pi(t)$.
- If $s \xrightarrow{8} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(\mathsf{i}(u)))])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(u)))])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u \in \mathcal{T}_\mathcal{H}$. We have $\pi(s) = C[\mathsf{i}(\mathsf{i}(u))] = \pi(t)$.
- If $s \xrightarrow{9} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(\mathsf{i}(u) \mid v))])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(u)) \mid v)])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u, v \in \mathcal{T}_\mathcal{H}$. In this case we obtain $\pi(s) = C[\mathsf{i}(\mathsf{i}(u) \mid v)] = \pi(t)$.
- If $s \xrightarrow{14} t$ then $s = \mathsf{B}(\mathsf{n}, \mathsf{E}(u))$ and $t = \mathsf{A}(\mathsf{s}(\mathsf{n}), u)$ for some $n \geqslant 0$ and $u \in \mathcal{T}_\mathcal{H}$. In this case we have $\pi(s) = \pi(\mathsf{E}(u)) = u = \pi(t)$.

(3) Again we make a case analysis on the applied rewrite rule.
- If $s \xrightarrow{10} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(\mathsf{i}(\mathsf{h} \mid u) \mid v))])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}), \mathsf{i}(u)) \mid v)])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u, v \in \mathcal{T}_\mathcal{H}$. We obtain $\pi(s) = C[\mathsf{i}(\mathsf{i}(\mathsf{h} \mid u) \mid v)]$ and $\pi(t) = C[\mathsf{i}(\mathsf{i}(u)^{n+2} \mid v)]$. Hence $\pi(s) \to_{\mathcal{R}_n} \pi(t)$ by applying rule 4 of $\mathcal{R}_n$.
- If $s \xrightarrow{11} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(\mathsf{i}(\mathsf{h} \mid u)))])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}), \mathsf{i}(u)))])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $u, v \in \mathcal{T}_\mathcal{H}$. We obtain $\pi(s) = C[\mathsf{i}(\mathsf{i}(\mathsf{h} \mid u))]$ and $\pi(t) = C[\mathsf{i}(\mathsf{i}(u)^{n+2})]$. Hence $\pi(s) \to_{\mathcal{R}_n} \pi(t)$ by applying rule 2 of $\mathcal{R}_n$.
- If $s \xrightarrow{12} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid v))])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}), \mathsf{h}) \mid v)])$ for some $n \geqslant 0$, $C \in \mathcal{C}_\mathcal{H}$ and $v \in \mathcal{T}_\mathcal{H}$. We obtain $\pi(s) = C[\mathsf{i}(\mathsf{i}(\mathsf{h}) \mid v)]$ and $\pi(t) = C[\mathsf{i}(\mathsf{h}^{n+2} \mid v)]$. Hence $\pi(s) \to_{\mathcal{R}_n} \pi(t)$ by applying rule 3 of $\mathcal{R}_n$.
- If $s \xrightarrow{13} t$ then $s = \mathsf{B}(\mathsf{n}, C[\mathsf{D}(\mathsf{s}(\mathsf{n}), \mathsf{i}(\mathsf{i}(\mathsf{h})))])$ and $t = \mathsf{B}(\mathsf{n}, C[\mathsf{i}(\mathsf{C}(\mathsf{s}(\mathsf{n}), \mathsf{h}))])$ for some $n \geqslant 0$ and $C \in \mathcal{C}_\mathcal{H}$. We obtain $\pi(s) = C[\mathsf{i}(\mathsf{i}(\mathsf{h}))]$ and $\pi(t) = C[\mathsf{i}(\mathsf{h}^{n+2})]$. Hence $\pi(s) \to_{\mathcal{R}_n} \pi(t)$ by applying rule 1 of $\mathcal{R}_n$. □

So we are ready to prove the main claim.

Proof of Theorem 3.11. Suppose $s = \mathsf{A}(\mathsf{n}, H) \to^+_{\mathcal{H}/\mathsf{AC}} \mathsf{A}(\mathsf{s}(\mathsf{n}), H') = t$. Inspection of $\mathcal{H}$ reveals that one of the following two cases holds:

(a) $s \to_{\mathcal{H}_1/\mathsf{AC}} t$, or

(b) $s \to^*_{\mathcal{H}_2/\mathsf{AC}} \cdot \to_{\mathcal{H}_3/\mathsf{AC}} \cdot \to^*_{\mathcal{H}_2/\mathsf{AC}} t$.

We first consider (a). If $s \to_{\mathcal{H}_1/\mathsf{AC}} t$ is a root step using rule 1 then $H = \mathsf{i}(\mathsf{h})$ and $H' = \mathsf{h}$. If $s \to_{\mathcal{H}_1/\mathsf{AC}} t$ is a root step using rule 2 then $H =_{\mathsf{AC}} \mathsf{i}(\mathsf{h} \mid u)$ and $H' =_{\mathsf{AC}} \mathsf{i}(u)$ for some term u. Next we consider (b). We have $s \to^*_{\mathcal{H}_2/\mathsf{AC}} s' \to_{\mathcal{H}_3/\mathsf{AC}} t' \to^*_{\mathcal{H}_2/\mathsf{AC}} t$ for some s' and t'. From

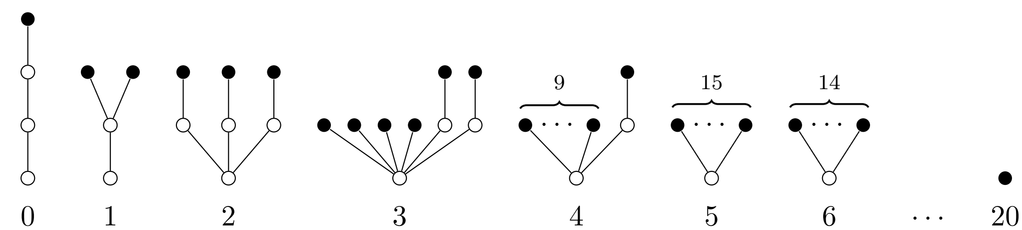
Lemma 3.13 we obtain $s, s', t', t \in U$. Hence

$$H = \pi(s) =_{\mathsf{AC}} \pi(s') \to_{\mathcal{R}_n/\mathsf{AC}} \pi(t') =_{\mathsf{AC}} \pi(t) = H'$$

is obtained by Lemma 3.16 and thus $H \to_{\mathcal{R}_n/\mathsf{AC}} H'$. Hence, we conclude $H \Rightarrow_n H'$. □

4. Many-Sorted Semantic Labeling modulo AC

Kirby and Paris [KP82] proved the termination of the Hydra Battle by associating ordinal numbers to Hydras (see [KM76, DM07] for notions and notations for ordinal numbers). Consider, for example, the following fight with the Hydra in Example 3.7:



By interpreting h, i, and $|$ as 1, the power of ω, and natural addition on ordinals, respectively, the sequence of Hydras turns into the decreasing sequence of ordinals:

$$\omega^{\omega^{\omega}} > \omega^{\omega^{2}} > \omega^{\omega\cdot 3} > \omega^{\omega\cdot 2+4} > \omega^{\omega+9} > \omega^{15} > \omega^{14} > \cdots > 1$$

One can verify that in general every transition reduces the ordinal interpretation of the Hydra. Because the order $>$ on ordinals is well-founded, the termination is concluded.

In the case of the term rewriting encoding, the mutual dependence between the function symbols A and B in rules 3 and 14 of $\mathcal{H}$ makes proving termination of $\mathcal{H}/\mathsf{AC}$ a non-trivial task. We use the technique of semantic labeling (Zantema [Zan95]) to resolve the dependence by labeling both A and B by the ordinal value of the Hydra encoded in their second arguments. Semantic labeling for rewriting modulo has been investigated in [OMG00]. We need, however, a version for many-sorted rewriting since the distinction between ordinals and natural numbers is essential for the effectiveness of semantic labeling for $\mathcal{H}/\mathsf{AC}$.

Before introducing semantic labeling, we recall some basic semantic definitions. An *algebra* $\mathcal{A}$ for an $\mathcal{S}$-sorted signature $\mathcal{F}$ is a pair $(\{S_{\mathcal{A}}\}_{S\in\mathcal{S}}, \{f_{\mathcal{A}}\}_{f\in\mathcal{F}})$, where each $S_{\mathcal{A}}$ is a non-empty set, called the *carrier of sort* S, and each $f_{\mathcal{A}}$ is a function of type $f : (S_1)_{\mathcal{A}} \times \cdots \times (S_n)_{\mathcal{A}} \to S_{\mathcal{A}}$, called the *interpretation function* of $f : S_1 \times \cdots \times S_n \to S$. A mapping that associates each variable of sort S to an element in $S_{\mathcal{A}}$ is called an *assignment*. We write $\mathcal{A}^{\mathcal{V}}$ for the set of all assignments. Given an assignment $\alpha \in \mathcal{A}^{\mathcal{V}}$, the *interpretation* of a term t is inductively defined as follows:

$$[\alpha]_{\mathcal{A}}(t) = \begin{cases} \alpha(t) & \text{if } t \text{ is a variable} \\ f_{\mathcal{A}}([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Let $\mathcal{A} = (\{S_{\mathcal{A}}\}_{S\in\mathcal{S}}, \{f_{\mathcal{A}}\}_{f\in\mathcal{F}})$ be an $\mathcal{S}$-sorted $\mathcal{F}$-algebra. We assume that each carrier set $S_{\mathcal{A}}$ is equipped with a well-founded order $>_S$ such that the interpretation functions are weakly monotone. Here a function ϕ of type $A_1 \times \cdots \times A_n \to B$ is *weakly monotone* if $\phi(a_1, \ldots, a_i, \ldots, a_n) \geqslant_B \phi(a_1, \ldots, b, \ldots, a_n)$ whenever $a_i \geqslant_{A_i} b$. We call $(\mathcal{A}, \{>_S\}_{S\in\mathcal{S}})$ a weakly monotone many-sorted algebra. Given terms s and t of sort S, we write $s \geqslant_{\mathcal{A}} t$ ($s =_{\mathcal{A}} t$) if $[\alpha]_{\mathcal{A}}(s) \geqslant_S [\alpha]_{\mathcal{A}}(t)$ ($[\alpha]_{\mathcal{A}}(s) =_S [\alpha]_{\mathcal{A}}(t)$) holds for all $\alpha \in \mathcal{A}^{\mathcal{V}}$.

A labeling L for $\mathcal{F}$ consists of sets of labels $L_f \subseteq S_\mathcal{A}$ for every $f : S_1 \times \cdots \times S_n \to S$. The labeled signature $\mathcal{F}_\mathsf{lab}$ consists of function symbols $f_a : S_1 \times \cdots \times S_n \to S$ for every function symbol $f : S_1 \times \cdots \times S_n \to S$ in $\mathcal{F}$ and label $a \in L_f$ together with all function symbols $f \in \mathcal{F}$ such that $L_f = \varnothing$. A *labeling* (L, lab) for $(\mathcal{A}, \{>_S\}_{S \in \mathcal{S}})$ consists of a labeling L for the signature $\mathcal{F}$ together with a mapping $\mathsf{lab}_f \colon (S_1)_\mathcal{A} \times \cdots \times (S_n)_\mathcal{A} \to L_f$ for every function symbol $f : S_1 \times \cdots \times S_n \to S$ in $\mathcal{F}$ with $L_f \neq \varnothing$. We call (L, lab) *weakly monotone* if all its labeling functions lab_f are weakly monotone. The mapping lab_f determines the label of the root symbol f of a term $f(t_1, \ldots, t_n)$, based on the values of its arguments $t_1, \ldots, t_n$. Formally, for every assignment $\alpha \in \mathcal{A}^\mathcal{V}$ we define a mapping lab_α inductively as follows:

$$\mathsf{lab}_\alpha(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing \\ f_a(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

where $a = \mathsf{lab}_f([\alpha]_\mathcal{A}(t_1), \ldots, [\alpha]_\mathcal{A}(t_n))$. Note that $\mathsf{lab}_\alpha(t)$ and t have the same sort. Given a TRS $\mathcal{R}$ over a (many-sorted) signature $\mathcal{F}$, we define the *labeled* TRS $\mathcal{R}_\mathsf{lab}$ over the signature $\mathcal{F}_\mathsf{lab}$ as follows:

$$\mathcal{R}_\mathsf{lab} = \{\mathsf{lab}_\alpha(\ell) \to \mathsf{lab}_\alpha(r) \mid \ell \to r \in \mathcal{R} \text{ and } \alpha \in A^\mathcal{V}\}$$

Since there is no need to label the AC symbol $|$ in the encoding of the Hydra Battle, we assume for simplicity that $L_f = \varnothing$ for every AC symbol $f \in \mathcal{F}$. The TRS $\mathcal{D}\mathsf{ec}$ consists of all rewrite rules

$$f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n)$$

with $f : S_1 \times \cdots \times S_n \to S$ a function symbol in $\mathcal{F}$, $a, b \in L_f$ such that $a >_S b$, and pairwise different variables $x_1, \ldots, x_n$. A weakly monotone algebra $(\mathcal{A}, >)$ is a *quasi-model* of $\mathcal{R}/\mathsf{AC}$ if $\ell \geqslant_\mathcal{A} r$ for all rewrite rules $\ell \to r$ in $\mathcal{R}$ and $\ell =_\mathcal{A} r$ for all equations $\ell \approx r$ in AC. So in a quasi-model, AC symbols are interpreted as associative and commutative functions.

Theorem 4.1. *Let $\mathcal{R}/\mathsf{AC}$ be a* TRS *over a many-sorted signature $\mathcal{F}$, $(\mathcal{A}, \{>_S\}_{S \in \mathcal{S}})$ a quasi-model of $\mathcal{R}/\mathsf{AC}$ with a weakly monotone labeling (L, lab). If $(\mathcal{R}_\mathsf{lab} \cup \mathcal{D}\mathsf{ec})/\mathsf{AC}$ is terminating then $\mathcal{R}/\mathsf{AC}$ is terminating.*

Proof. We show

(1) if $t \to_\mathcal{R} u$ then $\mathsf{lab}_\alpha(t) \to^*_{\mathcal{D}\mathsf{ec}} \cdot \to_{\mathcal{R}_\mathsf{lab}} \mathsf{lab}_\alpha(u)$

(2) if $t \to_\mathsf{AC} u$ then $\mathsf{lab}_\alpha(t) =_\mathsf{AC} \mathsf{lab}_\alpha(u)$

(3) if $t =_\mathsf{AC} u$ then $\mathsf{lab}_\alpha(t) =_\mathsf{AC} \mathsf{lab}_\alpha(u)$

for all sorts S, terms $t, u \in \mathcal{T}_S(\mathcal{F}, \mathcal{V})$, and assignments $\alpha \in \mathcal{A}^\mathcal{V}$. The claim follows from the first and third statements. First suppose $t \to_\mathcal{R} u$ is a root step using the rewrite rule $\ell \to r$. So $t = \ell\sigma$ and $u = r\sigma$ for some substitution σ. Define the assignment $\beta = [\alpha]_\mathcal{A} \circ \sigma$ and the (labeled) substitution $\tau = \mathsf{lab}_\alpha \circ \sigma$. An easy induction proof yields $\mathsf{lab}_\alpha(s\sigma) = \mathsf{lab}_\beta(s)\tau$ for all terms s. By definition $\mathsf{lab}_\beta(\ell) \to \mathsf{lab}_\beta(r) \in \mathcal{R}_\mathsf{lab}$. Hence $\mathsf{lab}_\alpha(t) = \mathsf{lab}_\beta(\ell)\tau \to_{\mathcal{R}_\mathsf{lab}} \mathsf{lab}_\beta(r)\tau = \mathsf{lab}_\alpha(u)$. Next suppose $t \to_\mathcal{R} u$ takes place below the root. So $t = f(t_1, \ldots, t_i, \ldots t_n)$ and $u = f(t_1, \ldots, u_i, \ldots t_n)$ with $t_i \to_\mathcal{R} u_i$. Let $S_1 \times \cdots \times S_n \to S$ be the sort declaration of f. The induction hypothesis yields $\mathsf{lab}_\alpha(t_i) \to^*_{\mathcal{D}\mathsf{ec}} \cdot \to_{\mathcal{R}_\mathsf{lab}} \mathsf{lab}_\alpha(u_i)$.

We obtain $[\alpha]_\mathcal{A}(t_i) \geqslant_{S_i} [\alpha]_\mathcal{A}(u_i)$ from the quasi-model assumption. If $L_f = \varnothing$ then

$$\begin{aligned}\mathsf{lab}_\alpha(t) &= f(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_i), \dots, \mathsf{lab}_\alpha(t_n)) \to^*_{\mathcal{D}\mathsf{ec}} \cdot \to_{\mathcal{R}_\mathsf{lab}} \\ &\quad f(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(u_i), \dots, \mathsf{lab}_\alpha(t_n)) = \mathsf{lab}_\alpha(u)\end{aligned}$$

Suppose $L_f \neq \varnothing$ and let

$$\begin{aligned}a &= \mathsf{lab}_f([\alpha]_\mathcal{A}(t_1), \dots, [\alpha]_\mathcal{A}(t_i), \dots, [\alpha]_\mathcal{A}(t_n)) \\ b &= \mathsf{lab}_f([\alpha]_\mathcal{A}(t_1), \dots, [\alpha]_\mathcal{A}(u_i), \dots, [\alpha]_\mathcal{A}(t_n))\end{aligned}$$

We obtain $a \geqslant_S b$ from the weak monotonicity of the labeling function lab_f. Therefore, the following rewrite sequence is constructed:

$$\begin{aligned}\mathsf{lab}_\alpha(t) &= f_a(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_i), \dots, \mathsf{lab}_\alpha(t_n)) \to^*_{\mathcal{D}\mathsf{ec}} \\ &\quad f_b(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_i), \dots, \mathsf{lab}_\alpha(t_n)) \to^*_{\mathcal{D}\mathsf{ec}} \cdot \to_{\mathcal{R}_\mathsf{lab}} \\ &\quad f_b(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(u_i), \dots, \mathsf{lab}_\alpha(t_n)) = \mathsf{lab}_\alpha(u)\end{aligned}$$

This concludes the proof of the first statement. The second statement is shown in the same way, but since AC symbols are not labeled the rules of $\mathcal{D}\mathsf{ec}$ do not come into play. The third statement is obtained from the the second statement together with the fact that $t =_\mathsf{AC} u$ implies $t \to^*_\mathsf{AC} u$. □

After these preliminaries, we are ready to put many-sorted semantic labeling to the test. Consider the many-sorted algebra $\mathcal{A}$ with carriers $\mathbb{N}$ for sort N and $\mathbb{O}$, the set of ordinal numbers smaller than ϵ_0, for sorts O and S and the following interpretation functions:

$$\begin{aligned}&\mathsf{0}_\mathcal{A} = \mathsf{h}_\mathcal{A} = 1 && \mathsf{s}_\mathcal{A}(n) = n + 1 && \mathsf{i}_\mathcal{A}(x) = \omega^x \\ &x \mathbin{|_\mathcal{A}} y = x \oplus y && \mathsf{E}_\mathcal{A}(x) = x + 1 && \mathsf{C}_\mathcal{A}(n, x) = x \otimes n + 1 \\ &\mathsf{A}_\mathcal{A}(n, x) = \mathsf{B}_\mathcal{A}(n, x) = \mathsf{D}_\mathcal{A}(n, x) = x\end{aligned}$$

Here $\oplus$ denotes natural addition on ordinals and $\otimes$ denotes natural product characterized by $x \otimes 0 = 0$ and $x \otimes (n + 1) = (x \otimes n) \oplus x$. Both satisfy strict monotonicity.

Lemma 4.2. *The algebra $(\mathcal{A}, \{>_\mathsf{O}, >_\mathsf{N}\})$ is a quasi-model of $\mathcal{H}/\mathsf{AC}$.*

Proof. First note that the interpretation functions are weakly monotone. The rewrite rules in $\mathcal{H}$ are oriented by $\geqslant_\mathsf{O}$:

$$\begin{aligned}
\mathsf{A}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(\mathsf{h}_\mathcal{A})) = \omega &>_\mathsf{O} 1 = \mathsf{A}_\mathcal{A}(\mathsf{s}_\mathcal{A}(n), \mathsf{h}_\mathcal{A}) && (1)\\
\mathsf{A}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(\mathsf{h}_\mathcal{A} \mathbin{|_\mathcal{A}} x)) = \omega^{x+1} &>_\mathsf{O} \omega^x = \mathsf{A}_\mathcal{A}(\mathsf{s}_\mathcal{A}(n), \mathsf{i}_\mathcal{A}(x)) && (2)\\
\mathsf{A}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(x)) = \omega^x &=_\mathsf{O} \omega^x = \mathsf{B}_\mathcal{A}(n, \mathsf{D}_\mathcal{A}(\mathsf{s}_\mathcal{A}(n), \mathsf{i}_\mathcal{A}(x))) && (3)\\
\mathsf{C}_\mathcal{A}(\mathsf{0}_\mathcal{A}, x) = x + 1 &=_\mathsf{O} x + 1 = \mathsf{E}_\mathcal{A}(x) && (4)\\
\mathsf{C}_\mathcal{A}(\mathsf{s}_\mathcal{A}(n), x) = (x \otimes n) \oplus x + 1 &=_\mathsf{O} (x \otimes n) \oplus x + 1 = x \mathbin{|_\mathcal{A}} \mathsf{C}_\mathcal{A}(n, x) && (5)\\
\mathsf{i}_\mathcal{A}(\mathsf{E}_\mathcal{A}(x) \mathbin{|_\mathcal{A}} y) = \omega^{x \oplus y + 1} &>_\mathsf{O} \omega^{x \oplus y} + 1 = \mathsf{E}_\mathcal{A}(\mathsf{i}_\mathcal{A}(x \mathbin{|_\mathcal{A}} y)) && (6)\\
\mathsf{i}_\mathcal{A}(\mathsf{E}_\mathcal{A}(x)) = \omega^{x+1} &>_\mathsf{O} \omega^x + 1 = \mathsf{E}_\mathcal{A}(\mathsf{i}_\mathcal{A}(x)) && (7)\\
\mathsf{D}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(\mathsf{i}_\mathcal{A}(x))) = \omega^{\omega^x} &=_\mathsf{O} \omega^{\omega^x} = \mathsf{i}_\mathcal{A}(\mathsf{D}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(x))) && (8)\\
\mathsf{D}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(\mathsf{i}_\mathcal{A}(x) \mathbin{|_\mathcal{A}} y)) = \omega^{\omega^x \oplus y} &=_\mathsf{O} \omega^{\omega^x \oplus y} = \mathsf{i}_\mathcal{A}(\mathsf{D}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(x)) \mathbin{|_\mathcal{A}} y) && (9)\\
\mathsf{D}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(\mathsf{i}_\mathcal{A}(\mathsf{h}_\mathcal{A} \mathbin{|_\mathcal{A}} x) \mathbin{|_\mathcal{A}} y)) = \omega^{\omega^{x+1} \oplus y} &>_\mathsf{O} \omega^{(\omega^x \otimes n) \oplus y + 1} = \mathsf{i}_\mathcal{A}(\mathsf{C}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(x)) \mathbin{|_\mathcal{A}} y) && (10)\\
\mathsf{D}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(\mathsf{i}_\mathcal{A}(\mathsf{h}_\mathcal{A} \mathbin{|_\mathcal{A}} x))) = \omega^{\omega^{x+1}} &>_\mathsf{O} \omega^{(\omega^x \otimes n) + 1} = \mathsf{i}_\mathcal{A}(\mathsf{C}_\mathcal{A}(n, \mathsf{i}_\mathcal{A}(x))) && (11)
\end{aligned}$$

$$\mathsf{D}_{\mathcal{A}}(n, \mathsf{i}_{\mathcal{A}}(\mathsf{i}_{\mathcal{A}}(\mathsf{h}_{\mathcal{A}}) \,|_{\mathcal{A}}\, y)) = \omega^{\omega \oplus y} \;>_{\mathsf{O}}\; \omega^{(n+1) \oplus y} = \mathsf{i}_{\mathcal{A}}(\mathsf{C}_{\mathcal{A}}(n, \mathsf{h}_{\mathcal{A}}) \,|_{\mathcal{A}}\, y) \tag{12}$$

$$\mathsf{D}_{\mathcal{A}}(n, \mathsf{i}_{\mathcal{A}}(\mathsf{i}_{\mathcal{A}}(\mathsf{h}_{\mathcal{A}}))) = \omega^{\omega} \;>_{\mathsf{O}}\; \omega^{n+1} = \mathsf{i}_{\mathcal{A}}(\mathsf{C}_{\mathcal{A}}(n, \mathsf{h}_{\mathcal{A}})) \tag{13}$$

$$\mathsf{B}_{\mathcal{A}}(n, \mathsf{E}_{\mathcal{A}}(x)) = x + 1 \;>_{\mathsf{O}}\; x = \mathsf{A}_{\mathcal{A}}(\mathsf{s}_{\mathcal{A}}(n), x) \tag{14}$$

Note that inequalities (10)—(13) use the fact that $\omega >_{\mathsf{O}} n$ holds for $n \in \mathbb{N}$. The compatibility of $\mathcal{A}$ with AC follows from the associativity and the commutativity of $\oplus$:

$$(x \,|_{\mathcal{A}}\, y) \,|_{\mathcal{A}}\, z = (x \oplus y) \oplus z \;=_{\mathsf{O}}\; x \oplus (y \oplus z) = x \,|_{\mathcal{A}}\, (y \,|_{\mathcal{A}}\, z)$$

$$x \,|_{\mathcal{A}}\, y = x \oplus y \;=_{\mathsf{O}}\; y \oplus x = x \,|_{\mathcal{A}}\, y$$

Therefore, $\mathcal{A}$ is a quasi-model of $\mathcal{H}/\mathsf{AC}$. □

We now label A and B by the value of their second argument. Let $L_{\mathsf{A}} = L_{\mathsf{B}} = \mathbb{O}$ and $L_f = \varnothing$ for the other function symbols f, and define lab as follows:

$$\mathsf{lab}_{\mathsf{A}}(n, x) = \mathsf{lab}_{\mathsf{B}}(n, x) = x$$

The labeling (L, lab) results in the infinite rewrite system $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$ with $\mathcal{H}_{\mathsf{lab}}$ consisting of the rewrite rules

$$\begin{aligned}
\mathsf{A}_{\omega}(n, \mathsf{i}(\mathsf{h})) &\xrightarrow{1} \mathsf{A}_1(\mathsf{s}(n), \mathsf{h}) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x))) &\xrightarrow{8} \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)))\\
\mathsf{A}_{\omega^{v+1}}(n, \mathsf{i}(\mathsf{h} \mid x)) &\xrightarrow{2} \mathsf{A}_{\omega^v}(\mathsf{s}(n), \mathsf{i}(x)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(x) \mid y)) &\xrightarrow{9} \mathsf{i}(\mathsf{D}(n, \mathsf{i}(x)) \mid y)\\
\mathsf{A}_{\omega^v}(n, \mathsf{i}(x)) &\xrightarrow{3} \mathsf{B}_{\omega^v}(n, \mathsf{D}(\mathsf{s}(n), \mathsf{i}(x))) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x) \mid y)) &\xrightarrow{10} \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x)) \mid y)\\
\mathsf{C}(0, x) &\xrightarrow{4} \mathsf{E}(x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid x))) &\xrightarrow{11} \mathsf{i}(\mathsf{C}(n, \mathsf{i}(x)))\\
\mathsf{C}(\mathsf{s}(n), x) &\xrightarrow{5} x \mid \mathsf{C}(n, x) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}) \mid y)) &\xrightarrow{12} \mathsf{i}(\mathsf{C}(n, \mathsf{h}) \mid y)\\
\mathsf{i}(\mathsf{E}(x) \mid y) &\xrightarrow{6} \mathsf{E}(\mathsf{i}(x \mid y)) & \mathsf{D}(n, \mathsf{i}(\mathsf{i}(\mathsf{h}))) &\xrightarrow{13} \mathsf{i}(\mathsf{C}(n, \mathsf{h}))\\
\mathsf{i}(\mathsf{E}(x)) &\xrightarrow{7} \mathsf{E}(\mathsf{i}(x)) & \mathsf{B}_{v+1}(n, \mathsf{E}(x)) &\xrightarrow{14} \mathsf{A}_v(\mathsf{s}(n), x)
\end{aligned}$$

for all $v \in \mathbb{O}$ and $\mathcal{D}\mathsf{ec}$ consisting of the rewrite rules

$$\mathsf{A}_v(n, x) \to \mathsf{A}_w(n, x) \qquad\qquad \mathsf{B}_v(n, x) \to \mathsf{B}_w(n, x)$$

for all $v, w \in \mathbb{O}$ with $v > w$.

Example 4.3. The first rewrite sequence in Example 3.7 is simulated as follows:

$$\begin{aligned}
\mathsf{A}_u(0, \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h})))) &\xrightarrow{3} \mathsf{B}_u(0, \mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{i}(\mathsf{h})))))\\
&\xrightarrow{8} \mathsf{B}_u(0, \mathsf{i}(\mathsf{D}(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h})))))\\
&\xrightarrow{13} \mathsf{B}_u(0, \mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h})))) \to_{\mathcal{D}\mathsf{ec}} \mathsf{B}_{v+1}(0, \mathsf{i}(\mathsf{i}(\mathsf{C}(\mathsf{s}(0), \mathsf{h}))))\\
&\xrightarrow{5} \mathsf{B}_{v+1}(0, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{C}(0, \mathsf{h}))))\\
&\xrightarrow{4} \mathsf{B}_{v+1}(0, \mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{E}(\mathsf{h})))) =_{\mathsf{AC}} \mathsf{B}_{v+1}(0, \mathsf{i}(\mathsf{i}(\mathsf{E}(\mathsf{h}) \mid \mathsf{h})))\\
&\xrightarrow{6} \mathsf{B}_{v+1}(0, \mathsf{i}(\mathsf{E}(\mathsf{i}(\mathsf{h} \mid \mathsf{h}))))\\
&\xrightarrow{7} \mathsf{B}_{v+1}(0, \mathsf{E}(\mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h}))))\\
&\xrightarrow{14} \mathsf{A}_v(\mathsf{s}(0), \mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h})))
\end{aligned}$$

Here $u = \omega^{\omega^\omega}$ and $v = \omega^{\omega^2}$.

According to Theorem 4.1, the AC termination of $\mathcal{H}$ on many-sorted terms follows from the AC termination of $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$.

Corollary 4.4. *If $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$ is* AC *terminating, $\mathcal{H}$ is* AC *terminating on sorted terms.* □

5. AC-MPO

In order to show AC termination of $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$ we use a simplified version of AC-RPO.

Definition 5.1. Let $\mathcal{F}_{\mathsf{AC}}$ be the set of AC symbols in $\mathcal{F}$. Given a non-variable term $t = f(t_1, \dots, t_n)$, the multiset $\triangledown(t)$ is defined inductively as follows:

$$\triangledown(t) = \triangledown_f(t_1) \uplus \cdots \uplus \triangledown_f(t_n)$$

$$\triangledown_f(t) = \begin{cases} \triangledown_f(t_1) \uplus \triangledown_f(t_2) & \text{if } t = f(t_1, t_2) \text{ and } f \in \mathcal{F}_{\mathsf{AC}} \\ \{t\} & \text{otherwise} \end{cases}$$

For example, if $+$ is an AC symbol, we have $\triangledown_+(\mathsf{a} + (\mathsf{b} + x)) = \{\mathsf{a}, \mathsf{b}, x\}$. If f is a non-AC symbol, we have $\triangledown(f(t_1, \dots, t_n)) = \{t_1, \dots, t_n\}$.

The multiset extension $=_{\mathsf{AC}}^{\mathsf{mul}}$ of the equivalence relation $=_{\mathsf{AC}}$ is inductively defined as follows: $\varnothing =_{\mathsf{AC}}^{\mathsf{mul}} \varnothing$ and $\{s\} \uplus M =_{\mathsf{AC}}^{\mathsf{mul}} \{t\} \uplus N$ if $s =_{\mathsf{AC}} t$ and $M =_{\mathsf{AC}}^{\mathsf{mul}} N$. It is not difficult to see that $=_{\mathsf{AC}}^{\mathsf{mul}}$ is an equivalence relation. We have $\triangledown(s) =_{\mathsf{AC}}^{\mathsf{mul}} \triangledown(t)$ whenever $s =_{\mathsf{AC}} t$.

Definition 5.2. *Precedences* are strict orders on function symbols. Let $>$ be a precedence. We define $>_{\mathsf{acmpo}}$ inductively as follows: $s >_{\mathsf{acmpo}} t$ if $s \notin \mathcal{V}$ and one of the following conditions holds:

(1) $\triangledown(s) \geqslant_{\mathsf{acmpo}}^{\mathsf{mul}} \{t\}$,

(2) $\mathsf{root}(s) > \mathsf{root}(t)$ and $\{s\} >_{\mathsf{acmpo}}^{\mathsf{mul}} \triangledown(t)$,

(3) $\mathsf{root}(s) = \mathsf{root}(t)$ and $\triangledown(s) >_{\mathsf{acmpo}}^{\mathsf{mul}} \triangledown(t)$.

In the third condition $=_{\mathsf{AC}}$ is used instead of $=$ in the definition of multiset extension. We write $\geqslant_{\mathsf{acmpo}}$ for the union of $>_{\mathsf{acmpo}}$ and $=_{\mathsf{AC}}$.

The first condition is equivalent to $s' \geqslant_{\mathsf{acmpo}} t$ for some $s' \in \triangledown(s)$ and the second condition is equivalent to the conjunction of $\mathsf{root}(s) > \mathsf{root}(t)$ and $s >_{\mathsf{acmpo}} t'$ for all $t' \in \triangledown(t)$. These equivalences will be used freely in the sequel. The multiset comparison in the third condition is spelled out as follows: $\triangledown(s) >_{\mathsf{acmpo}}^{\mathsf{mul}} \triangledown(t)$ if there exist multisets S_1, S_2, T_1 and T_2 such that $\triangledown(s) = S_1 \uplus S_2$, $\triangledown(t) = T_1 \uplus T_2$, $S_1 =_{\mathsf{AC}}^{\mathsf{mul}} T_1$, $S_2 \neq \varnothing$, and for every $t' \in T_2$ there exists a term $s' \in S_2$ such that $s' >_{\mathsf{acmpo}} t'$.

Note that if there are no AC symbols, the above definition reduces to the original recursive path order of Dershowitz [Der82], nowadays known as the *multiset path order*. Moreover, if AC symbols are minimal in a precedence, AC-RPO with the multiset status reduces to AC-MPO. Hence the simplified AC-RPO will be called AC-MPO.

The proof of the following result can be found in the appendix. *Incrementality* of AC-MPO means that for precedences $>$ and $\sqsupset$ the inclusion $>_{\mathsf{acmpo}} \subseteq \sqsupset_{\mathsf{acmpo}}$ holds whenever $> \subseteq \sqsupset$.

Theorem 5.3. *If* AC *symbols are minimal in the precedence $>$ then $>_{\mathsf{acmpo}}$ is an incremental* AC*-compatible rewrite order with the subterm property.* □

As a consequence, $>_{\mathsf{acmpo}}$ is an AC-compatible reduction order when the underlying signature is finite. This also holds for infinite signatures, provided the precedence $>$ is well-founded and there are only finitely many AC symbols. This extension is important because the signature of $\mathcal{H}_{\mathsf{lab}}$ is infinite. Below, we will formally prove the correctness of the extension, by adopting the approach of [MZ97].

A strict order $>$ on a set A is a *partial well-order* if for every infinite sequence $a_0, a_1, \ldots$ of elements in A there exist indices i and j such that $i < j$ and $a_i \leqslant a_j$. Well-founded total orders (*well-orders*) are partial well-orders. Given a partial well-order $>$ on $\mathcal{F}$, the *embedding* TRS $\mathcal{E}\mathsf{mb}(\mathcal{F}, >)$ consists of the rules $f(x_1, \ldots, x_n) \to x_i$ for every n-ary function symbol and $1 \leqslant i \leqslant n$, together with the rules $f(x_1, \ldots, x_n) \to g(x_{i_1}, \ldots, x_{i_m})$ for all function symbols f and g with arities m and n such that $f > g$, and indices $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$. Here $x_1, \ldots, x_n$ are pairwise distinct variables.

Theorem 5.4 [MZ97, Theorem 5.3]**.** *A rewrite order $>$ is well-founded if $\mathcal{E}\mathsf{mb}(\mathcal{F}, \sqsupset) \subseteq >$ for some partial well-order $\sqsupset$.* □

Theorem 5.5. *Consider a signature $\mathcal{F}$ with only finitely many* AC *symbols that are minimal in a given well-founded precedence $>$. The relation $>_{\mathsf{acmpo}}$ is an* AC*-compatible reduction order.*

Proof. We only need to show well-foundedness of $>_{\mathsf{acmpo}}$ because the other properties follow by Theorem 5.3. Let $\sqsupset$ be an arbitrary partial well-order that contains $>$ and in which AC symbols are minimal. The inclusion $\mathcal{E}\mathsf{mb}(\mathcal{F}, \sqsupset) \subseteq \sqsupset_{\mathsf{acmpo}}$ is easily verified. Hence the well-foundedness of $\sqsupset_{\mathsf{acmpo}}$ is obtained from Theorem 5.4. Since $> \subseteq \sqsupset$, the incrementality of AC-MPO yields $>_{\mathsf{acmpo}} \subseteq \sqsupset_{\mathsf{acmpo}}$. It follows that $>_{\mathsf{acmpo}}$ is well-founded. □

We show the termination of $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$ by AC-MPO. To this end, we consider the following precedence $>$ on the labeled signature:

$$
\begin{aligned}
\mathsf{A}_v &> \mathsf{A}_w && \text{for all } v, w \in \mathbb{O} \text{ with } v > w \\
\mathsf{B}_v &> \mathsf{B}_w && \text{for all } v, w \in \mathbb{O} \text{ with } v > w \\
\mathsf{B}_{v+1} > \mathsf{A}_v &> \mathsf{B}_v && \text{for all } v \in \mathbb{O} \\
\mathsf{B}_0 &> \mathsf{s} > \mathsf{D} > \mathsf{C} > \mathsf{i} > \mathsf{E} > \mid
\end{aligned}
$$

Note that $>$ is well-founded and the only AC symbol $\mid$ is minimal. In order to ease the compatibility verification we employ the following simple criterion.

Lemma 5.6. *Let $\ell \to r$ be a rewrite rule and let $>$ be a precedence. If $\mathsf{root}(\ell) > g$ for all function symbols g in r then $\ell >_{\mathsf{acmpo}} r$.* □

Theorem 5.7. $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec} \subseteq >_{\mathsf{acmpo}}$

Proof. Lemma 5.6 applies to all rules of $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$, except 5–9. We consider rule 6 here; the other rewrite rules are handled in a similar fashion. Since case (1) of Definition 5.2 yields $\mathsf{E}(x) >_{\mathsf{acmpo}} x$, we have $\nabla(\mathsf{E}(x) \mid y) = \{\mathsf{E}(x), y\} >_{\mathsf{acmpo}}^{\mathsf{mul}} \{x, y\} = \nabla(x \mid y)$. Thus $\mathsf{E}(x)|y >_{\mathsf{acmpo}} x|y$ follows by case (3). Using case (3) again, we obtain $\mathsf{i}(\mathsf{E}(x)|y) >_{\mathsf{acmpo}} \mathsf{i}(x|y)$. Because of $\mathsf{i} > \mathsf{E}$, the desired orientation $\mathsf{i}(\mathsf{E}(x) \mid y) >_{\mathsf{acmpo}} \mathsf{i}(x \mid y)$ is concluded by case (2). □

Theorem 5.8. *The* TRS $\mathcal{H}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}$ *is* AC *terminating.* □

From Theorems 3.6 and 5.7 we conclude that Hercules eventually beats Hydra in any battle. Theorems 5.7 and 3.8 in connection with Corollary 4.4 yield the AC termination of $\mathcal{H}$ on arbitrary terms.

6. Related Work

In an influential survey paper, Dershowitz and Jouannaud [DJ90, p. 270] introduced a 5-rule rewrite system to simulate the Hydra Battle. The proposed rewrite system was later shown to be erroneous. A corrected version together with a detailed termination analysis has been given by Dershowitz and Moser [DM07], see also Moser [Mos09]. Earlier, Touzet [Tou98] presented an 11-rule rewrite system that encodes a specific battle with weakened Hydras (whose height is bounded by 4) and proved total termination by a semantic termination method. It is worth noting that our rewrite system $\mathcal{H}$ is not even simply terminating on unsorted terms. In fact, we have the following cyclic sequence with respect to $\mathcal{H} \cup \mathcal{E}\mathsf{mb}(\mathcal{F}, \varnothing)$:

$$\begin{aligned}\mathsf{A}(\mathsf{E}(\mathsf{i}(x)), \mathsf{i}(x)) &\xrightarrow{3} \mathsf{B}(\mathsf{E}(\mathsf{i}(x)), \mathsf{D}(\mathsf{s}(\mathsf{E}(\mathsf{i}(x))), \mathsf{i}(x))) \to^*_{\mathcal{E}\mathsf{mb}(\mathcal{F},\varnothing)} \mathsf{B}(\mathsf{E}(\mathsf{i}(x)), \mathsf{i}(x)) \\ &\xrightarrow{14} \mathsf{A}(\mathsf{s}(\mathsf{E}(\mathsf{i}(x))), \mathsf{i}(x)) \to_{\mathcal{E}\mathsf{mb}(\mathcal{F},\varnothing)} \mathsf{A}(\mathsf{E}(\mathsf{i}(x)), \mathsf{i}(x))\end{aligned}$$

So the TRS $\mathcal{H}$ is not simply terminating (see [MZ97, Lemma 4.6]). This is the reason that our termination proof employs semantic labeling.

The rewrite systems referred to above model the so-called *standard* battle, which corresponds to a specific strategy for Hercules. In this regard it is interesting to quote Kirby and Paris [KP82], who introduced the battle as an accessible example of an independence result for Peano arithmetic (P):

> A *strategy* is a function which determines for Hercules which head to chop off at each stage of any battle. It is not hard to find a reasonably fast *winning strategy* (i.e. a strategy which ensures that Hercules wins against any hydra). More surprisingly, Hercules cannot help winning:
>
> Theorem 2. (i) *Every strategy is a winning strategy.*
>
> [...]
>
> Theorem 2. (ii) *The statement "every recursive strategy is a winning strategy" is not provable from* P.

In a recent paper [EKO21, Section 6], rules are presented to slay Hydras, independent of the strategy. These rules do not constitute a term rewrite system in the usual sense (they operate on terms with *sequence variables*). More importantly, the infinitely many rules do not faithfully represent the battle. Earlier, Ferreira and Zantema [FZ96, Section 10] presented an infinite rewrite system to model the standard strategy and gave a direct ordinal interpretation to conclude its termination. In neither of the latter two papers stages of the battle are modeled.

7. Conclusion

We presented a new TRS encoding of the Battle of Hydra and Hercules. Unlike earlier encodings, it makes use of AC symbols. This allows us to faithfully model any strategy of Hercules, as envisaged in the paper by Kirby and Paris [KP82] in which the Battle was first presented. To prove the termination of the encoding we employed many-sorted rewriting modulo AC and we extended semantic labeling modulo AC to many-sorted TRSs. The infinite TRS produced by semantic labeling was proved terminating by suitably instantiating AC-RPO.

One of the reviewers for this article pointed out that the Hydra battle can still be simulated even if rule 6 in $\mathcal{H}$ is replaced by the simpler $\mathsf{E}(x) \mid y \to \mathsf{E}(x \mid y)$. While we expect that this variant also has the termination property, the presented termination methods are not applicable. In fact, the rule cannot be ordered by AC-MPO and the AC symbol $\mid$ cannot be labeled. The reviewer also suggested an alternative encoding of Hydras that omits i from $\mathsf{i}(t_1 \mid \cdots \mid t_n)$. For instance, H_0 in Example 3.2 is written as $\mathsf{i}(\mathsf{h}) \mid \mathsf{i}(\mathsf{i}(\mathsf{h} \mid \mathsf{h})) \mid \mathsf{h}$ in this encoding. While this simplifies representations of Hydras, it seems difficult to construct an ordinal interpretation of $\mid$ for semantic labeling. Further investigations of AC termination techniques are required.

The finite TRS $\mathcal{H}$ poses an interesting challenge for automatic termination tools. None of the tools (AProVE [GAB+17], muterm [AGLNM11]) competing in the "TRS Equational" category of the Termination Competition 2024[1] succeeds on $\mathcal{H}/\mathsf{AC}$. This is not really surprising since most methods implemented in termination tool come with a multiple recursive upper bound on the derivation height (e.g. [Hof92, Lep01, MS11]). The tools even fail to prove termination of $\mathcal{H}$ without AC. The tool $\mathsf{T_TT_2}$ [KSZM09] has support for ordinal interpretations [ZWM15] but also fails on $\mathcal{H}$.

Formalizing the techniques used in this article in a proof assistant is an important task to ensure the correctness of the results. Interestingly, the informal paper [HM22] in which we announced our encoding also presents a termination proof, essentially extending a semantic method of Touzet [Tou98] and Zantema [Zan01] to AC rewriting. Although we believe the non-trivial extension to be correct, its use in proving the AC termination of $\mathcal{H}$ has a critical mistake, which we recently discovered.

Another topic for future research is to investigate the scope of many-sorted semantic labeling. Can the termination of earlier encodings of the battle be established with many-sorted semantic labeling followed by some standard simplification order? Variants of the battle by Buchholz [Buc87] and Lepper [Lep04] are also of interest here.

Acknowledgements

We are grateful to the anonymous reviewers for their pertinent comments, which helped to improve the presentation. We thank Teppei Saito for his thorough feedback on the proofs for AC-MPO.

References

[AGLNM11] Beatriz Alarcón, Raúl Gutiérrez, Salvador Lucas, and Rafael Navarro-Marset. Proving termination properties with MU-TERM. In *Proc. 13th Algebraic Methodology and Software Technology*, volume 6486 of *Lecture Notes in Computer Science*, pages 201–208, 2011. doi:10.1007/978-3-642-17796-5_12.

[Buc87] Wilfried Buchholz. An independence result for $(\pi^1_1 - \mathrm{CA}) + \mathrm{BI}$. *Annals of Pure and Applied Logic*, 33:131–155, 1987. doi:10.1016/0168-0072(87)90078-9.

[Buc06] Wilfried Buchholz. Another rewrite system for the standard Hydra battle. In *Proc. Mini-Workshop: Logic, Combinatorics and Independence Results*, volume 3(4) of *Oberwolfach Reports*, pages 3099–3102. European Mathematical Society, 2006.

[1]https://termcomp.github.io/Y2024/

[Der82] Nachum Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, 17:279–301, 1982. doi:10.1016/0304-3975(82)90026-3.

[DJ90] Nachum Dershowitz and Jean-Pierre Jouannaud. *Rewrite Systems. Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, pages 243–320. Elsevier, 1990.

[DM07] Nachum Dershowitz and Georg Moser. The Hydra battle revisited. In *Rewriting, Computation and Proof, Essays Dedicated to Jean-Pierre Jouannaud on the Occasion of his 60th Birthday*, volume 4600 of *Lecture Notes in Computer Science*, pages 1–27, 2007. doi:10.1007/978-3-540-73147-4_1.

[EKO21] Jörg Endrullis, Jan Willem Klop, and Roy Overbeek. Star games and Hydras. *Logical Methods in Computer Science*, 17(2):20:1–20:32, 2021. doi:10.23638/LMCS-17(2:20)2021.

[FZ96] Maria C. F. Ferreira and Hans Zantema. Total termination of term rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 7(2):133–162, 1996. doi:10.1007/BF01191381.

[GAB+17] Jürgen Giesl, Cornelius Aschermann, Marc Brockschmidt, Fabian Emmes, Florian Frohn, Carsten Fuhs, Jera Hensel, Carsten Otto, Martin Plücker, Peter Schneider-Kamp, Thomas Ströder, Stephanie Swiderski, and René Thiemann. Analyzing program termination and complexity automatically with AProVE. *Journal of Automated Reasoning*, 58(1):3–31, 2017. doi:10.1007/s10817-016-9388-y.

[HM22] Nao Hirokawa and Aart Middeldorp. Hydra battles and AC termination. In *Proc. 18th International Workshop on Termination*, pages 21–25, 2022.

[HM23] Nao Hirokawa and Aart Middeldorp. Hydra battles and AC termination. In *Proc. 8th Formal Structures for Computation and Deduction*, volume 260 of *Leibniz International Proceedings in Informatics*, pages 12:1–12:16, 2023. doi:10.4230/LIPIcs.FSCD.2023.12.

[Hof92] Dieter Hofbauer. Termination proofs by multiset path orderings imply primitive recursive derivation lengths. *Theoretical Computer Science*, 105(1):129–140, 1992. doi:10.1016/0304-3975(92)90289-R.

[KM76] Kazimierz Kuratowski and Andrzej Mostowski. *Set Theory — With an Introduction to Descriptive Set Theory (second edition)*, volume 86 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1976.

[KP82] Laurence Kirby and Jeff Paris. Accessible independence results for Peano arithmetic. *Bulletin of the London Mathematical Society*, 14:285–325, 1982. doi:10.1112/blms/14.4.285.

[KSZM09] Martin Korp, Christian Sternagel, Harald Zankl, and Aart Middeldorp. Tyrolean Termination Tool 2. In *Proc. 20th International Conference on Rewriting Techniques and Applications*, volume 5595 of *Lecture Notes in Computer Science*, pages 295–304, 2009. doi:10.1007/978-3-642-02348-4_21.

[Lep01] Ingo Lepper. Derivation lengths and order types of Knuth–Bendix orders. *Theoretical Computer Science*, 269(1-2):433–450, 2001. doi:10.1016/S0304-3975(01)00015-9.

[Lep04] Ingo Lepper. Simply terminating rewrite systems with long derivations. *Archive for Mathematical Logic*, 43(1):1–18, 2004. doi:10.1007/s00153-003-0190-2.

[MO00] Aart Middeldorp and Hitoshi Ohsaki. Type introduction for equational rewriting. *Acta Informatica*, 36(12):1007–1029, 2000. doi:10.1007/PL00013300.

[Mos09] Georg Moser. The Hydra battle and Cichon's principle. *Applicable Algebra in Engineering, Communication and Computing*, 20(2):133–158, 2009. doi:10.1007/s00200-009-0094-4.

[MS11] Georg Moser and Andreas Schnabl. The derivational complexity induced by the dependency pair method. *Logical Methods in Computer Science*, 7(3), 2011. doi:10.2168/LMCS-7(3:1)2011.

[MZ97] Aart Middeldorp and Hans Zantema. Simple termination of rewrite systems. *Theoretical Computer Science*, 175(1):127–158, 1997. doi:10.1016/S0304-3975(96)00172-7.

[OMG00] Hitoshi Ohsaki, Aart Middeldorp, and Jürgen Giesl. Equational termination by semantic labelling. In *Proc. 14th EACSL Annual Conference on Computer Science Logic*, volume 1862 of *Lecture Notes in Computer Science*, pages 457–471, 2000. doi:10.1007/3-540-44622-2_31.

[Rub02] Albert Rubio. A fully syntactic AC-RPO. *Information and Computation*, 178(2):515–533, 2002. doi:10.1006/inco.2002.3158.

[Ste90] Joachim Steinbach. AC-termination of rewrite systems: A modified Knuth–Bendix ordering. In *Proc. 2nd International Conference on Algebraic and Logic Programming*, volume 463 of *Lecture Notes in Computer Science*, pages 372–386, 1990. doi:10.1007/3-540-53162-9_52.

[Tou98] Hélène Touzet. Encoding the Hydra battle as a rewrite system. In *Proc. 23rd International Symposium on Mathematical Foundations of Computer Science*, volume 1450 of *Lecture Notes in Computer Science*, pages 267–276, 1998. doi:10.1007/BFb0055776.
[Zan94] Hans Zantema. Termination of term rewriting: Interpretation and type elimination. *Journal of Symbolic Computation*, 17(1):23–50, 1994. doi:10.1006/jsco.1994.1003.
[Zan95] Hans Zantema. Termination of term rewriting by semantic labelling. *Fundamenta Informaticae*, 24:89–105, 1995. doi:10.3233/FI-1995-24124.
[Zan01] Hans Zantema. The termination hierarchy for term rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 12(1-2):3–19, 2001. doi:10.1007/s002000100061.
[ZWM15] Harald Zankl, Sarah Winkler, and Aart Middeldorp. Beyond polynomials and Peano arithmetic—Automation of elementary and ordinal interpretations. *Journal of Symbolic Computation*, 69:129–158, 2015. doi:10.1016/j.jsc.2014.09.033.

Appendix A. Proof of Theorem 5.3

We first show the AC-compatibility of AC-MPO.

Lemma A.1. *The relation* $>_{\mathsf{acmpo}}$ *is* AC-*compatible.*

Proof. First assume $s >_{\mathsf{acmpo}} t =_{\mathsf{AC}} u$. By induction on $|s| + |t|$ we show $s >_{\mathsf{acmpo}} u$. We distinguish three cases, according to Definition 5.2.

(1) If $s' \geqslant_{\mathsf{acmpo}} t$ for some $s' \in \nabla(s)$ then also $s' \geqslant_{\mathsf{acmpo}} u$, either by the induction hypothesis or by the transitivity of $=_{\mathsf{AC}}$. Therefore $s >_{\mathsf{acmpo}} u$ by case (1).
(2) Suppose $\mathsf{root}(s) > \mathsf{root}(t)$ and $s >_{\mathsf{acmpo}} t'$ for all $t' \in \nabla(t)$. From $t =_{\mathsf{AC}} u$ we derive $\mathsf{root}(t) = \mathsf{root}(u)$ and $\nabla(t) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla(u)$. So for every $u' \in \nabla(u)$ there exists a term $t' \in \nabla(t)$ with $t' =_{\mathsf{AC}} u'$. Because $s >_{\mathsf{acmpo}} t' =_{\mathsf{AC}} u'$ and $|t| > |t'|$, the induction hypothesis yields $s >_{\mathsf{acmpo}} u'$. Hence $\{s\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$ and thus $s >_{\mathsf{acmpo}} u$ by case (2).
(3) Suppose $\mathsf{root}(s) = \mathsf{root}(t)$ and $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$. From $t =_{\mathsf{AC}} u$ we derive $\mathsf{root}(t) = \mathsf{root}(u)$ and $\nabla(t) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla(u)$. As $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$, there exist multisets S_1, S_2, T_1, and T_2 such that $\nabla(s) = S_1 \uplus S_2$, $\nabla(t) = T_1 \uplus T_2$, $S_1 =^{\mathsf{mul}}_{\mathsf{AC}} T_1$, $S_2 \neq \varnothing$, and for every $t' \in T_2$ there exists a term $s' \in S_2$ with $s' >_{\mathsf{acmpo}} t'$. As $\nabla(t) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla(u)$, we may write $\nabla(u) = U_1 \uplus U_2$ with $T_1 =^{\mathsf{mul}}_{\mathsf{AC}} U_1$ and $T_2 =^{\mathsf{mul}}_{\mathsf{AC}} U_2$. As $S_1 =^{\mathsf{mul}}_{\mathsf{AC}} T_1 =^{\mathsf{mul}}_{\mathsf{AC}} U_1$, we obtain $S_1 =^{\mathsf{mul}}_{\mathsf{AC}} U_1$ from the transitivity of $=_{\mathsf{AC}}$. For every $u' \in U_2$ there exists a term $t' \in T_2$ with $t' =_{\mathsf{AC}} u'$. Moreover, there exists a term $s' \in S_2$ with $s' >_{\mathsf{acmpo}} t'$. Since $s' >_{\mathsf{acmpo}} t' =_{\mathsf{AC}} u'$ and $|s| + |t| > |s'| + |t'|$, the induction hypothesis yields $s' >_{\mathsf{acmpo}} u'$. Consequently, $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$. Hence $s >_{\mathsf{acmpo}} u$ by case (3).

Next assume $s =_{\mathsf{AC}} t >_{\mathsf{acmpo}} u$. By induction on $|t| + |u|$ we show $s >_{\mathsf{acmpo}} u$. From $s =_{\mathsf{AC}} t$ we infer $\mathsf{root}(s) = \mathsf{root}(t)$ and $\nabla(s) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla(t)$. We distinguish three cases for $t >_{\mathsf{acmpo}} u$.

(1) Suppose $\nabla(t) \geqslant_{\mathsf{acmpo}} \{u\}$. Since $\nabla(s) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla(t)$, we obtain $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \{u\}$ by the induction hypothesis or the transitivity of $=_{\mathsf{AC}}$. Hence $s >_{\mathsf{acmpo}} u$ by case (1).
(2) Suppose $\mathsf{root}(t) > \mathsf{root}(u)$ and $t >_{\mathsf{acmpo}} u'$ for all $u' \in \nabla(u)$. The induction hypothesis yields $s >_{\mathsf{acmpo}} u'$ for all $u' \in \nabla(u)$. Since also $\mathsf{root}(s) > \mathsf{root}(u)$, $s >_{\mathsf{acmpo}} u$ by case (2).
(3) Suppose $\mathsf{root}(t) = \mathsf{root}(u)$ and $\nabla(t) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$. Since $\nabla(s) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla(t)$, we obtain $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$ by the induction hypothesis and the transitivity of $=_{\mathsf{AC}}$. Hence $s >_{\mathsf{acmpo}} u$ by case (3). □

Next we show transitivity.

Lemma A.2. *The relation* $>_{\mathsf{acmpo}}$ *is transitive.*

Proof. Suppose $s >_{\mathsf{acmpo}} t >_{\mathsf{acmpo}} u$. We show $s >_{\mathsf{acmpo}} u$ by induction on $|s| + |t| + |u|$. We do a case analysis on $s >_{\mathsf{acmpo}} t$.

(1) If $s' \geqslant_{\mathsf{acmpo}} t$ for some $s' \in \nabla(s)$ then $s' >_{\mathsf{acmpo}} u$ by the induction hypothesis or the AC-compatibility of $>_{\mathsf{acmpo}}$ (Lemma A.1).
(2) Suppose $\mathsf{root}(s) > \mathsf{root}(t)$ and $\{s\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$. We perform a second case analysis on $t >_{\mathsf{acmpo}} u$.
   - If $\nabla(t) \geqslant_{\mathsf{acmpo}} \{u\}$ then we obtain $s >_{\mathsf{acmpo}} u$ by the induction hypothesis or the AC-compatibility of $>_{\mathsf{acmpo}}$.
   - If $\mathsf{root}(t) > \mathsf{root}(u)$ and $\{t\} >_{\mathsf{acmpo}} \nabla(u)$ then $s >_{\mathsf{acmpo}} t >_{\mathsf{acmpo}} v$ for all $v \in \nabla(u)$ and thus $\{s\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$ by the induction hypothesis. Hence $s >_{\mathsf{acmpo}} u$ by case (2).
   - Suppose $\mathsf{root}(t) = \mathsf{root}(u)$ and $\nabla(t) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$. We obtain $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$ from the induction hypothesis and the AC-compatibility of $>_{\mathsf{acmpo}}$. Thus, $s >_{\mathsf{acmpo}} u$ follows by case (3).
(3) Suppose $\mathsf{root}(s) = \mathsf{root}(t)$ and $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$. Also in this case we perform an additional case analysis on $t >_{\mathsf{acmpo}} u$.
   - If $\nabla(t) \geqslant_{\mathsf{acmpo}} \{u\}$ then we obtain $\nabla(s) >_{\mathsf{acmpo}} \{u\}$ by the induction hypothesis or the AC-compatibility of $>_{\mathsf{acmpo}}$.
   - Suppose $\mathsf{root}(t) > \mathsf{root}(u)$ and $\{t\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$. We have $\mathsf{root}(s) > \mathsf{root}(u)$. For every $v \in \nabla(u)$ we have $s >_{\mathsf{acmpo}} t >_{\mathsf{acmpo}} v$, and thus $s >_{\mathsf{acmpo}} v$ by the induction hypothesis. Hence $\{s\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$ and thus $s >_{\mathsf{acmpo}} u$ by case (2).
   - Suppose $\mathsf{root}(t) = \mathsf{root}(u)$ and $\nabla(t) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$. From $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$ we infer $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(u)$ by the induction hypothesis, the AC-compatibility of $>_{\mathsf{acmpo}}$, and the transitivity of $=_{\mathsf{AC}}$. Hence $s >_{\mathsf{acmpo}} u$ by case (3). □

The subterm property is next.

Lemma A.3. *The relation* $>_{\mathsf{acmpo}}$ *has the subterm property.*

Proof. Let $t = f(t_1, \dots, t_n)$. Fix $i \in \{1, \dots, n\}$. We show $t >_{\mathsf{acmpo}} t_i$. The subterm property is then obtained by induction and the transitivity of $>_{\mathsf{acmpo}}$ (Lemma A.2). We distinguish two cases.

(1) If $t_i \in \nabla(t)$ then $\nabla(t) \geqslant_{\mathsf{acmpo}} \{t_i\}$ and thus $t >_{\mathsf{acmpo}} t_i$ by case (1).
(2) If $t_i \notin \nabla(t)$ then f is an AC symbol, $n = 2$ and $\mathsf{root}(t_i) = f$. Since $\nabla(t_i) \subsetneq \nabla(t)$, $\nabla(t) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t_i)$ holds and thus $t >_{\mathsf{acmpo}} t_i$ by case (3). □

The preceding lemmata are used to prove irreflexivity.

Lemma A.4. *The relation* $>_{\mathsf{acmpo}}$ *is irreflexive.*

Proof. Assume to the contrary $t >_{\mathsf{acmpo}} t$. We derive a contradiction by induction on t. We distinguish three cases.

(1) Suppose $t' \geqslant_{\mathsf{acmpo}} t$ for some $t' \in \nabla(t)$. Since $t \rhd t'$, the subterm property (Lemma A.3) yields $t >_{\mathsf{acmpo}} t$. So $t' \geqslant_{\mathsf{acmpo}} t >_{\mathsf{acmpo}} t'$, and thus $t' >_{\mathsf{acmpo}} t'$ is obtained by the transitivity (Lemma A.2) or AC compatibility (Lemma A.1) of $>_{\mathsf{acmpo}}$. The induction hypothesis yields the desired contraction.
(2) If $t >_{\mathsf{acmpo}} t$ is derived by case (2) then $\mathsf{root}(t) > \mathsf{root}(t)$, which contradicts the irreflexivity of the precedence $>$.

(3) Suppose $\nabla(t) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$. Let U be the set of all proper subterms of t, and $\succ$ the restriction of $>_{\mathsf{acmpo}}$ to $U \times U$. The multiset extension $\succ^{\mathsf{mul}}$ coincides with the restriction of $>^{\mathsf{mul}}_{\mathsf{acmpo}}$ to finite multisets over U. Hence $\nabla(t) \succ^{\mathsf{mul}} \nabla(t)$ follows from $\nabla(t) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$. The relation $\succ$ is irreflexive according to the induction hypothesis. Moreover, $\succ$ inherits transitivity from $>_{\mathsf{acmpo}}$ (Lemma A.2). Hence $\succ$ is a strict order and thus so is its multiset extension $\succ^{\mathsf{mul}}$. Since $\nabla(t)$ is a finite multiset over U, $\nabla(t) \succ^{\mathsf{mul}} \nabla(t)$ cannot hold, yielding the desired contradiction. □

In the proof of closure under substitutions we use the fact that for an f-rooted term t and a substitution σ the multiset $\nabla(t\sigma)$ is the multiset sum of $\nabla_f(t'\sigma)$ for all $t' \in \nabla(t)$.

Lemma A.5. *The relation $>_{\mathsf{acmpo}}$ is closed under substitutions.*

Proof. Suppose $s >_{\mathsf{acmpo}} t$ and let σ be a substitution. We show $s\sigma >_{\mathsf{acmpo}} t\sigma$ by induction on $|s| + |t|$. We distinguish three cases.

(1) Suppose $s' \geqslant_{\mathsf{acmpo}} t$ for some $s' \in \nabla(s)$. If $s' >_{\mathsf{acmpo}} t$ then we obtain $s'\sigma >_{\mathsf{acmpo}} t\sigma$ from the induction hypothesis. If $s' =_{\mathsf{AC}} t$ then we obtain $s'\sigma =_{\mathsf{AC}} t\sigma$ from the closure under substitutions of $=_{\mathsf{AC}}$. So in both cases we have $s'\sigma \geqslant_{\mathsf{acmpo}} t\sigma$. If $s'\sigma \in \nabla(s\sigma)$ then $\nabla(s\sigma) >^{\mathsf{mul}}_{\mathsf{acmpo}} \{t\sigma\}$ and thus $s\sigma >_{\mathsf{acmpo}} t\sigma$ by case (1). If $s'\sigma \notin \nabla(s\sigma)$ then $s' \in \mathcal{V}$ and thus $s' = t$ follows from $s' \geqslant_{\mathsf{acmpo}} t$. Hence $s'\sigma = t\sigma$. Since $s'\sigma \lhd s\sigma$ we obtain $s\sigma >_{\mathsf{acmpo}} t\sigma$ from the subterm property (Lemma A.3).
(2) Suppose $\mathsf{root}(s) > \mathsf{root}(t)$ and $\{s\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$. Clearly $\mathsf{root}(s\sigma) = \mathsf{root}(s) > \mathsf{root}(t) = \mathsf{root}(t\sigma)$. Consider an arbitrary term $u \in \nabla(t\sigma)$. Let $f = \mathsf{root}(t)$. There exists a term $t' \in \nabla(t)$ such that $u \in \nabla_f(t'\sigma)$. The induction hypothesis yields $s\sigma >_{\mathsf{acmpo}} t'\sigma$. Since $u \in \nabla_f(t'\sigma)$ satisfies $u \trianglelefteq t'\sigma$, we have $t'\sigma \geqslant_{\mathsf{acmpo}} u$ by the subterm property (Lemma A.3) and thus $s\sigma >_{\mathsf{acmpo}} u$ by transitivity (Lemma A.2). Hence $\{s\sigma\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t\sigma)$ and thus $s\sigma >_{\mathsf{acmpo}} t\sigma$ by case (2).
(3) Suppose $\mathsf{root}(s) = \mathsf{root}(t)$ and $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$. We write f for $\mathsf{root}(s)$. Let $s' \in \nabla(s)$ and $t' \in \nabla(t)$ such that $s' >_{\mathsf{acmpo}} t'$ or $s' =_{\mathsf{AC}} t'$ is used in $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$.
   - We first consider $s' >_{\mathsf{acmpo}} t'$. Since $s' \lhd s$ and $t' \lhd t$, the induction hypothesis yields $s'\sigma >_{\mathsf{acmpo}} t'\sigma$. From $s' >_{\mathsf{acmpo}} t'$ we infer $s' \notin \mathcal{V}$ and thus $s'\sigma \notin \mathcal{V}$. Because of $s' \in \nabla(s)$, we have $\mathsf{root}(s') \neq f$ and thus $\mathsf{root}(s'\sigma) \neq f$. Hence $\nabla_f(s'\sigma) = \{s'\sigma\}$. Consider a term $u \in \nabla_f(t'\sigma)$. Since $u \trianglelefteq t'\sigma$, we obtain $t'\sigma \geqslant_{\mathsf{acmpo}} u$ by the subterm property (Lemma A.3) and thus $s'\sigma >_{\mathsf{acmpo}} u$ by transitivity (Lemma A.2). Hence $\nabla_f(s'\sigma) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla_f(t'\sigma)$ follows.
   - Suppose $s' =_{\mathsf{AC}} t'$. Since $=_{\mathsf{AC}}$ is closed under substitutions, we have $s'\sigma =_{\mathsf{AC}} t'\sigma$ and thus $\nabla_f(s'\sigma) =^{\mathsf{mul}}_{\mathsf{AC}} \nabla_f(t'\sigma)$.

   It follows that the derivation of $\nabla(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t)$ can be simulated, resulting in $\nabla(s\sigma) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla(t\sigma)$. Hence $s\sigma >_{\mathsf{acmpo}} t\sigma$ by case (3). □

The following technical result is used in the proof that AC-MPO is closed under contexts (if AC symbols are minimal in the precedence).

Lemma A.6. *If $f \in \mathcal{F}_{\mathsf{AC}}$ is minimal in $>$ and $s = f(s_1, s_2) >_{\mathsf{acmpo}} t$ then $\nabla_f(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \nabla_f(t)$.*

Proof. We have $\nabla(s) = \nabla_f(s) = \nabla_f(s_1) \uplus \nabla_f(s_2)$. We distinguish three cases.

(1) Suppose $s' \geqslant_{\mathsf{acmpo}} t$ for some $s' \in \triangledown(s)$. We obtain $\mathsf{root}(s') \neq f$ from $s' \in \triangledown(s) = \triangledown_f(s)$ and thus $\triangledown_f(s') = \{s'\}$. If also $\mathsf{root}(t) \neq f$, then $\triangledown_f(t) = \{t\}$ and thus $s' \geqslant_{\mathsf{acmpo}} t$ leads to $\triangledown_f(s) \supsetneq \triangledown_f(s') \geqslant^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown_f(t)$ and hence $\triangledown_f(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown_f(t)$. Suppose $\mathsf{root}(t) = f$. Let $v \in \triangledown_f(t)$. We have $t \rhd v$ and thus $t >_{\mathsf{acmpo}} v$. As $s' \geqslant_{\mathsf{acmpo}} t >_{\mathsf{acmpo}} v$, we obtain $s' >_{\mathsf{acmpo}} v$ by transitivity or AC-compatibility. Hence $\triangledown_f(s') >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown_f(t)$ and therefore $\triangledown_f(s) \supsetneq \triangledown_f(s') >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown_f(t)$ and $\triangledown_f(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown_f(t)$.
(2) Since f is minimal in the precedence, $s >_{\mathsf{acmpo}} t$ cannot be obtained by case (2).
(3) Suppose $\mathsf{root}(s) = \mathsf{root}(t) = f$ and $\triangledown(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown(t)$. Since $\triangledown_f(s) = \triangledown(s)$ and $\triangledown_f(t) = \triangledown(t)$, the claim holds. □

Lemma A.7. *The relation $>_{\mathsf{acmpo}}$ is closed under contexts if* AC *symbols are minimal in the precedence $>$.*

Proof. Suppose $s >_{\mathsf{acmpo}} t$ and consider a context of the form $C = f(\dots,\Box,\dots)$. If $f \notin \mathcal{F}_{\mathsf{AC}}$ then $\triangledown(C[s]) - \triangledown(C[t]) = \{s\}$ and $\triangledown(C[t]) - \triangledown(C[s]) = \{t\}$ by the irreflexivity of $>_{\mathsf{acmpo}}$ (Lemma A.4), and thus $\triangledown(C[s]) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown(C[t])$ follows from $s >_{\mathsf{acmpo}} t$. Hence $C[s] >_{\mathsf{acmpo}} C[t]$ by case (3). Suppose $f \in \mathcal{F}_{\mathsf{AC}}$. We have $C = f(\Box, u)$ or $C = f(u, \Box)$ and distinguish two cases.

- If $f = \mathsf{root}(s)$ then $\triangledown_f(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown_f(t)$ by Lemma A.6. So the inequality
$$\triangledown(C[s]) - \triangledown(C[t]) = \triangledown_f(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown_f(t) = \triangledown(C[t]) - \triangledown(C[s])$$
holds. Therefore, we obtain $C[s] >_{\mathsf{acmpo}} C[t]$ by case (3).
- If $f \neq \mathsf{root}(s)$ then $\triangledown(C[s]) = \{s\} \uplus \triangledown_f(u)$ and $\triangledown(C[t]) = \triangledown_f(t) \uplus \triangledown_f(u)$. According to case (3), it is enough to show $s >_{\mathsf{acmpo}} t'$ for all $t' \in \triangledown_f(t)$. Let $t' \in \triangledown_f(t)$. We have $t \trianglerighteq t'$ and thus $t \geqslant_{\mathsf{acmpo}} t'$ by the subterm property (Lemma A.3). As $s >_{\mathsf{acmpo}} t \geqslant_{\mathsf{acmpo}} t'$, we obtain $s >_{\mathsf{acmpo}} t'$ by transitivity or AC-compatibility. □

Incrementality is the final property in Theorem 5.3.

Lemma A.8. *The relation $>_{\mathsf{acmpo}}$ is incremental.*

Proof. Let $>$ and $\sqsupset$ be precedences with $> \subseteq \sqsupset$. Suppose $s >_{\mathsf{acmpo}} t$. We show $s \sqsupset_{\mathsf{acmpo}} t$ by induction on $|s| + |t|$. We distinguish three cases.
(1) If $\triangledown(s) \geqslant^{\mathsf{mul}}_{\mathsf{acmpo}} \{t\}$ then $s' =_{\mathsf{AC}} t$ or $s' >_{\mathsf{acmpo}} t$ for some $s' \in \triangledown(s)$. In the latter case, the induction hypothesis yields $s' \sqsupset_{\mathsf{acmpo}} t$. Hence in both cases $s \sqsupset_{\mathsf{acmpo}} t$ by case (1).
(2) If $\mathsf{root}(s) > \mathsf{root}(t)$ and $\{s\} >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown(t)$ then $\{s\} \sqsupset_{\mathsf{acmpo}} \triangledown(t)$ by the induction hypothesis. Hence, $s \sqsupset_{\mathsf{acmpo}} t$ is obtained by case (2).
(3) Suppose $\mathsf{root}(s) = \mathsf{root}(t)$ and $\triangledown(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown(t)$. Let $s' \in \triangledown(s)$ and $t' \in \triangledown(s)$ be a term pair such that $s' >_{\mathsf{acmpo}} t'$ is used in $\triangledown(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown(t)$. Since $s' \lhd s$ and $t' \lhd t$, the induction hypothesis yields $s' \sqsupset_{\mathsf{acmpo}} t'$. Thus, the derivation of $\triangledown(s) >^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown(t)$ can be simulated by $\sqsupset_{\mathsf{acmpo}}$. Therefore, $\triangledown(s) \sqsupset^{\mathsf{mul}}_{\mathsf{acmpo}} \triangledown(t)$, and hence $s \sqsupset_{\mathsf{acmpo}} t$ is obtained by case (3). □