A Characterization of Quasi-Decreasingness[*]

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1   Introduction

In 2010 Schernhammer and Gramlich [10] showed that quasi-decreasingness of a DCTRS $\mathcal{R}$ is equivalent to μ-termination of its context-sensitive unraveling $U_{\mathsf{CS}}(\mathcal{R})$ on original terms. While the direction that quasi-decreasingness of $\mathcal{R}$ implies μ-termination of $U_{\mathsf{CS}}(\mathcal{R})$ on original terms is shown directly; the converse – facilitating the use of context-sensitive termination tools like MU-TERM [1] and VMTL [9] – employs the additional notion of context-sensitive quasi-reductivity of $\mathcal{R}$. In the following, we give a direct proof of the fact that μ-termination of $U_{\mathsf{CS}}(\mathcal{R})$ on original terms implies quasi-decreasingness of $\mathcal{R}$. Moreover, we report our experimental findings on DCTRSs from the confluence problems database (Cops),[1] extending the experiments of Schernhammer and Gramlich.

Contribution.   A direct proof that μ-termination of a CSRS $U_{\mathsf{CS}}(\mathcal{R})$ on original terms implies quasi-decreasingness of the DCTRS $\mathcal{R}$. New experiments on a recent DCTRS collection.

2   Preliminaries

We assume familiarity with the basic notions of (conditional and context-sensitive) term rewriting [3, 6, 8], but shortly recapitulate terminology and notation that we use in the remainder. Given two arbitrary binary relations $\to_\alpha$ and $\to_\beta$, we write $_\alpha\!\!\leftarrow$, $\to_\alpha^+$, $\to_\alpha^*$ for the *inverse*, the *transitive closure*, and the *reflexive transitive closure* of $\to_\alpha$, respectively. The relation obtained by considering $\to_\alpha$ *relative to* $\to_\beta$, written $\to_{\alpha/\beta}$, is defined by $\to_\beta^* \cdot \to_\alpha \cdot \to_\beta^*$. We use $\mathcal{V}(\cdot)$ to denote the set of variables occurring in a given syntactic object, like a term, a pair of terms, a list of terms, etc. The set of terms $\mathcal{T}(\mathcal{F},\mathcal{V})$ over a given signature of function symbols $\mathcal{F}$ and set of variables $\mathcal{V}$ is defined inductively: $x \in \mathcal{T}(\mathcal{F},\mathcal{V})$ for all variables $x \in \mathcal{V}$, and for every n-ary function symbol $f \in \mathcal{F}$ and terms $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$ also $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})$. A *deterministic oriented 3-CTRS (DCTRS)* $\mathcal{R}$ is a set of conditional rewrite rules of the shape $\ell \to r \Leftarrow c$ where ℓ and r are terms and c is a possibly empty sequence of pairs of terms $s_1 \approx t_1,\ldots,s_n \approx t_n$. For all rules in $\mathcal{R}$ we have that $\ell \notin \mathcal{V}$, $\mathcal{V}(r) \subseteq \mathcal{V}(\ell,c)$, and $\mathcal{V}(s_i) \subseteq \mathcal{V}(\ell,t_1,\ldots,t_{i-1})$ for all $1 \leqslant i \leqslant n$. The rewrite relation induced by a DCTRS $\mathcal{R}$ is structured into levels. For each level i, a TRS $\mathcal{R}_i$ is defined recursively by $\mathcal{R}_0 = \varnothing$ and $\mathcal{R}_{i+1} = \{\ell\sigma \approx r\sigma \mid \ell \to r \Leftarrow c \in \mathcal{R} \land \forall s \approx t \in c.\ s\sigma \to_{\mathcal{R}_i}^* t\sigma\}$ where for a given TRS $\mathcal{S}$, $\to_{\mathcal{S}}$ denotes the induced rewrite relation (i.e., its closure under contexts and substitutions). Then the rewrite relation of $\mathcal{R}$ is $\to_{\mathcal{R}} = \bigcup_{i\geqslant 0} \to_{\mathcal{R}_i}$. We have $\mathcal{R} = \mathcal{R}_{\mathsf{c}} \uplus \mathcal{R}_{\mathsf{u}}$ where $\mathcal{R}_{\mathsf{c}}$ denotes the subset of rules with non-empty conditional part ($n > 0$) and $\mathcal{R}_{\mathsf{u}}$ the subset of unconditional rules ($n = 0$). A DCTRS $\mathcal{R}$ over signature $\mathcal{F}$ is *quasi-decreasing* if there is a well-founded order $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that $\succ = (\succ \cup \rhd)^+$, $\to_{\mathcal{R}} \subseteq \succ$, and for all rules $\ell \to r \Leftarrow s_1 \approx t_1,\ldots,s_n \approx t_n$ in $\mathcal{R}$, all substitutions $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$, and $0 \leqslant i < n$, if $s_j\sigma \to_{\mathcal{R}}^* t_j\sigma$ for all $1 \leqslant j \leqslant i$ then $\ell\sigma \succ s_{\mathrm{i+1}}\sigma$.

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Given a DCTRS $\mathcal{R}$ its *unraveling $U(\mathcal{R})$* (cf. [8, p. 212]) is defined as follows. For each conditional rule $\rho\colon \ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ (where $n > 0$) we introduce n fresh function symbols $U_1^\rho, \ldots, U_n^\rho$ and generate the set of $n + 1$ unconditional rules $U(\rho)$ as follows

$$\ell \to U_1^\rho(s_1, \mathsf{v}(\ell))$$
$$U_1^\rho(t_1, \mathsf{v}(\ell)) \to U_2^\rho(s_2, \mathsf{v}(\ell), \mathsf{ev}(t_1))$$
$$\vdots$$
$$U_n^\rho(t_n, \mathsf{v}(\ell), \mathsf{ev}(t_1, \ldots, t_{n-1})) \to r$$

where v and ev denote functions that yield the respective sequences of elements of $\mathcal{V}$ and $\mathcal{EV}$ in some arbitrary but fixed order, and $\mathcal{EV}(t_i) = \mathcal{V}(t_i) \setminus \mathcal{V}(\ell, t_1, \ldots, t_{i-1})$ denotes the *extra variables* of the right-hand side of the ith condition. Finally the unraveling of the DCTRS is $U(\mathcal{R}) = \mathcal{R}_{\mathsf{u}} \cup \bigcup_{\rho \in \mathcal{R}_{\mathsf{c}}} U(\rho)$.

A *context-sensitive rewrite system* (CSRS) is a TRS (over signature $\mathcal{F}$) together with a replacement map $\mu\colon \mathcal{F} \to 2^{\mathbb{N}}$ that restricts the argument positions of each function symbol in $\mathcal{F}$ at which we are allowed to rewrite. A position p is *active* in a term t if either $p = \epsilon$, or $p = iq$, $t = f(t_1, \ldots, t_n)$, $i \in \mu(f)$, and q is active in t_i. The set of active positions in a term t is denoted by $\mathcal{P}\mathsf{os}_\mu(t)$. Given a CSRS $\mathcal{R}$ a term s μ-rewrites to a term t, written $s \to_\mu t$, if $s \to_{\mathcal{R}} t$ at some position p and $p \in \mathcal{P}\mathsf{os}_\mu(s)$. A CSRS is called *$\mu$-terminating* if its context-sensitive rewrite relation is terminating. The (proper) subterm relation with respect to replacement map μ, written $\rhd_\mu$, restricts the ordinary subterm relation to active positions.

We conclude this section by recalling the notion of context-sensitive quasi-reductivity in an attempt to further appreciation for a proof without this notion.

▶ **Definition 1.** A CSRS $\mathcal{R}$ over signature $\mathcal{F}$ is *context-sensitively quasi-reductive* if there is an extended signature $\mathcal{F}' \supseteq \mathcal{F}$, a replacement map μ (with $\mu(f) = \{1, \ldots, n\}$ for every n-ary $f \in \mathcal{F}$), and a μ-monotonic, well-founded partial order $\succ_\mu$ on $\mathcal{T}(\mathcal{F}', \mathcal{V})$ such that for every rule $\ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_k \approx t_k$, every substitution $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, and every $0 \leqslant i \leqslant k - 1$:

- $\ell\sigma \ (\succ_\mu \cup \rhd_\mu)^+ \ s_{i+1}\sigma$ whenever $s_j\sigma \succeq_\mu t_j\sigma$ for every $1 \leqslant j \leqslant i$, and
- $\ell\sigma \succ_\mu r\sigma$ whenever $s_j\sigma \succeq_\mu t_j\sigma$ for every $1 \leqslant j \leqslant k$.

3   Characterization

In order to present our main result (the proof of Theorem 5 below) we first restate some definitions and theorems which we will use in the proof.

The usual unraveling is extended by a replacement map in order to restrict reductions in U-symbols to the first argument position [10, Definition 4].

▶ **Definition 2** (Unraveling $U_{\mathsf{CS}}(\mathcal{R})$). The *context-sensitive unraveling $U_{\mathsf{CS}}(\mathcal{R})$* is the unraveling $U(\mathcal{R})$ together with the replacement map μ such that $\mu(f) = \{1, \ldots, k\}$ if $f \in \mathcal{F}$ with arity k and $\mu(f) = \{1\}$ otherwise. We say that the resulting CSRS is *μ-terminating on original terms* [10, Definition 7], if there is no infinite $U_{\mathsf{CS}}(\mathcal{R})$-reduction starting from a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

Simulation completeness of $U_{\mathsf{CS}}(\mathcal{R})$ (i.e., that every $\mathcal{R}$-step can be simulated by a $U_{\mathsf{CS}}(\mathcal{R})$-reduction) can be shown by induction on the level of a conditional rewrite step [10, Theorem 1].

▶ **Theorem 3** (Simulation completeness). *For a DCTRS $\mathcal{R}$ we have $\to_{\mathcal{R}} \subseteq \to_{U_{\mathsf{CS}}(\mathcal{R})}^+$.*   ◀

Furthermore, we need the following auxiliary result.

▶ **Lemma 4.** *For any context-sensitive rewrite relation $\to_\mu$ induced by the replacement map μ, $\rhd_\mu$ commutes over $\to_\mu$, i.e., $\rhd_\mu \cdot \to_\mu \subseteq \to_\mu \cdot \rhd_\mu$.*

Proof. Assume $s \rhd_\mu t \to_\mu u$ for some terms s, t, and u. Then $s = C[t] \rhd_\mu t \to_\mu u$ for some nonempty context C. Thus we conclude by $C[t] \to_\mu C[u] \rhd_\mu u$. ◀

With this we are finally able to prove our main result.

▶ **Theorem 5.** *If the CSRS $U_{\mathsf{CS}}(\mathcal{R})$ is μ-terminating on original terms then the DCTRS $\mathcal{R}$ is quasi-decreasing.*

Proof. Assume that $U_{\mathsf{CS}}(\mathcal{R})$ is μ-terminating on original terms. We define an order $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$

$$\succ \stackrel{\text{def}}{=} (\to_{U_{\mathsf{CS}}(\mathcal{R})} \cup \rhd_\mu)^+ \cap (\mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})) \tag{$\star$}$$

and show that it satisfies the four properties from the definition of quasi-decreasingness:
1. We start by showing that $\succ$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Assume, to the contrary, that $\succ$ is not well-founded. Then we have an infinite sequence

   $$t_1 \succ t_2 \succ t_3 \succ \ldots \tag{$\dagger$}$$

   where all $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. By definition $\rhd_\mu$ is well-founded. Moreover, since $U_{\mathsf{CS}}(\mathcal{R})$ is μ-terminating on original terms, $\to_{U_{\mathsf{CS}}(\mathcal{R})}$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Further note that every $\to_{U_{\mathsf{CS}}(\mathcal{R})}$-terminating element (hence every term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$) is $\to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating, since by a repeated application of Lemma 4 every infinite reduction $t_1 \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} t_2 \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \cdots$ starting from a term $t_1 \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ can be transformed into an infinite $\to_{U_{\mathsf{CS}}(\mathcal{R})}$-reduction, contradicting well-foundedness of $\to_{U_{\mathsf{CS}}(\mathcal{R})}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We conclude by analyzing the following two cases:
   - Either ($\dagger$) contains $\to_{U_{\mathsf{CS}}(\mathcal{R})}$ only finitely often, contradicting well-foundedness of $\rhd_\mu$,
   - or there are infinitely many $\to_{U_{\mathsf{CS}}(\mathcal{R})}$-steps in ($\dagger$). But then we can construct a sequence $s_1 \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} s_2 \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} s_3 \to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu} \ldots$ with $s_1 = t_1$, contradicting the fact that all elements of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are $\to_{U_{\mathsf{CS}}(\mathcal{R})/\rhd_\mu}$-terminating.
2. Next we show $\succ = (\succ \cup \rhd)^+$. The direction $\succ \subseteq (\succ \cup \rhd)^+$ is obvious. For the other direction, $(\succ \cup \rhd)^+ \subseteq \succ$, assume we have $s (\succ \cup \rhd)^{n+1} t$. Then we proceed by induction on n. In the base case $s (\succ \cup \rhd) t$. If $s \succ t$ we are done. Otherwise, $s \rhd t$ and thus also $s \rhd_\mu t$ since $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and therefore $s \succ t$. In the step case $n = k + 1$ for some k, and $s (\succ \cup \rhd) u (\succ \cup \rhd)^k t$. Then we obtain $s \succ u$ by a similar case-analysis as in the base case. Moreover $u \succ t$ by induction hypothesis, and thus $s \succ t$.
3. Now we show that $\to_\mathcal{R} \subseteq \succ$. Assume $s \to_\mathcal{R} t$. Together with simulation completeness of $U_{\mathsf{CS}}(\mathcal{R})$, Theorem 3, we get $s \to^+_{U_{\mathsf{CS}}(\mathcal{R})} t$ which in turn implies $s \succ t$.
4. Finally, we show that if for all $\ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ in $\mathcal{R}$, substitutions $\sigma \colon \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, and $0 \leqslant i < n$, if $s_j\sigma \to^*_\mathcal{R} t_j\sigma$ for all $1 \leqslant j \leqslant i$ then $\ell\sigma \succ s_{i+1}\sigma$. We have the sequence

   $$\ell\sigma \to^+_{U_{\mathsf{CS}}(\mathcal{R})} U^\rho_{i+1}(s_{i+1}, \mathsf{v}(\ell), \mathsf{ev}(t_1, \ldots, t_i))\sigma \rhd_\mu s_{i+1}\sigma$$

   using the definition of $U_{\mathsf{CS}}(\mathcal{R})$ together with simulation completeness (Theorem 3). But then also $\ell\sigma \succ s_{i+1}\sigma$ as wanted because $\ell\sigma, s_{i+1}\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

Hence $\mathcal{R}$ is quasi-decreasing with the order $\succ$. ◀

Table 1 (Non-)quasi-decreasing DCTRSs out of 103 in Cops by transformation and tool.

	conditional $\mathcal{R}$			$U_{\mathsf{CS}}(\mathcal{R})$			$U(\mathcal{R})$					total
	AProVE	MU-TERM	VMTL	AProVE	MU-TERM	VMTL	AProVE	MU-TERM	NaTT	$\mathsf{T_TT_2}$	VMTL	
YES	80	78	80	78	78	79	81	78	77	78	78	84
NO	–	12	–	–	–	–	–	–	–	–	–	12

The converse of Theorem 5 has already been shown by Schernhammer and Gramlich [10, Theorem 4]:

▶ **Theorem 6.** *If a DCTRS $\mathcal{R}$ is quasi-decreasing then the CSRS $U_{\mathsf{CS}}(\mathcal{R})$ is μ-terminating on original terms.* ◀

Thus the desired equivalence follows as an easy corollary.

▶ **Corollary 7.** *Quasi-decreasingness of a DCTRS $\mathcal{R}$ is equivalent to μ-termination of the CSRS $U_{\mathsf{CS}}(\mathcal{R})$ on original terms.*

4 Experiments

In order to present up-to-date numbers for (non-)quasi-decreasingness we conducted experiments on the 103 DCTRSs contained in the confluence problems database using various automated termination tools. Of these, AProVE [4], MU-TERM 5.13 [1], and VMTL 1.3 [9] are able to directly show quasi-decreasingness and MU-TERM is the only tool that can show non-quasi-decreasingness [7]. AProVE, MU-TERM, and VMTL can also handle context-sensitive systems and we used them in combination with $U_{\mathsf{CS}}(\mathcal{R})$. Finally, we also ran the previous tools together with NaTT [11] and $\mathsf{T_TT_2}$ 1.16 [5] on $U(\mathcal{R})$. The results for a timeout of one minute are shown in Table 1. There are several points of notice. The most yes-instances (81) we get if we use AProVE together with $U(\mathcal{R})$. Interestingly, AProVE cannot show quasi-decreasingness of system 362 directly, although it succeeds (like all other tools besides NaTT) if provided with its unraveling. Moreover, systems 266, 278, and 279 can be shown to be quasi-decreasing by AProVE if we use $U(\mathcal{R})$ but not if we use $U_{\mathsf{CS}}(\mathcal{R})$ (even if we increase the timeout to 5 minutes). On system 363 only MU-TERM succeeds (in the direct approach). If we compare MU-TERM on conditional systems to MU-TERM with $U_{\mathsf{CS}}(\mathcal{R})$, the direct method succeeds on system 360 but not on system 329. Conversely, when using $U_{\mathsf{CS}}(\mathcal{R})$ it succeeds on system 329 but not on system 360. Moreover, MU-TERM seems to have some problems with systems 278 and 342, generating errors in the direct approach. With $U_{\mathsf{CS}}(\mathcal{R})$ VMTL succeeds on 79 systems, subsuming the results from AProVE and MU-TERM (78 each). On system 357 only VMTL together with $U_{\mathsf{CS}}(\mathcal{R})$ succeeds. With $U(\mathcal{R})$, NaTT succeeds on 77 systems, this is subsumed by $\mathsf{T_TT_2}$, succeeding on 78 systems, which in turn is subsumed by AProVE, succeeding, as mentioned above, on 81 systems. In total 84 systems are shown to be quasi-decreasing, 12 systems to be non-quasi-decreasing, and only 7 remain open. One of these, for example, is system 337 from Cops, for computing Bubble-sort [12]

$$x < 0 \to \mathsf{false} \qquad\qquad 0 < \mathsf{s}(y) \to \mathsf{true}$$
$$\mathsf{s}(x) < \mathsf{s}(y) \to x < y \qquad\qquad x : y : ys \to y : x : ys \Leftarrow x < y \approx \mathsf{true}$$

whose unraveling replaces the last (and only conditional) rule by the two rules:

$$x : y : ys \to \mathsf{U}(x < y, x, y, ys) \qquad \mathsf{U}(\mathsf{true}, x, y, ys) \to y : x : ys$$

5   Conclusion

We provide a direct proof for one direction of a previous characterization of quasi-decreasingness, i.e., that μ-termination of a CSRS $U_{\mathsf{CS}}(\mathcal{R})$ on original terms implies quasi-decreasingness of the DCTRS $\mathcal{R}$ without the need of a detour by using the notion of context-sensitive quasi-reductivity. We believe that our proof could easily be adapted to any other context-sensitive transformation as long as it is simulation complete. Moreover, we provide experimental results on a recent collection of DCTRSs. Knowing that a DCTRS is quasi-decreasing is, among other things, useful to show confluence with the Knuth-Bendix criterion for CTRSs [2].

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